1.2: ordered Field Axioms

★ Postulate 1: Field Axioms : $\forall$ a, b, c there are functions + and $\cdot$ on $\mathbb{R}^2$ defined with the properties:

1. (closure property): $a+b$, $a.b \in \mathbb{R}$

2. (Associative property): $a+(b+c) = (a+b)+c$, $a.(b.c) = (a.b).c$

3. (commutative property): $a+b = b+a$, $a.b = b.a$

4. (Distributive law): $a.(b+c) = ab + ac$.

5. (Existence of the Additive Identity): $\exists!\ 0 \in \mathbb{R}$ s.t $0+a = a$, $\forall a \in \mathbb{R}$.

6. (Existence of the multiplication Identity): $\exists!\ 1 \in \mathbb{R}$ s.t $1 \neq 0$ and $1.a = a$, $\forall a \in \mathbb{R}$.

7. (Existence of Additive Inverse): $\forall a \in \mathbb{R}$, $\exists\ -a \in \mathbb{R}$ s.t $a + -a = 0$.

8. (Existence of multiplication Inverse): $\forall a \in \mathbb{R} \setminus \{0\}$, $\exists!\ a^{-1} \in \mathbb{R}$ s.t $aa^{-1} = 1$.

★ postulate 2: order Axiom: $\exists$ a relation $<$ on $\mathbb{R} \times \mathbb{R}$ s.t:

(i) $\forall a, b \in \mathbb{R}$, exactly one of the following is true: $a<b$, $a>b$ or $a=b$. (Trichotomy).

(ii) $\forall a, b, c \in \mathbb{R}$, $a<b$ and $b<c \Rightarrow a<c$ (Transitive)

(iii) $\forall a, b, c \in \mathbb{R}$, $a<b$ and $c \in \mathbb{R}$ then $a+c < b+c$. (Additive).

(iv) $\forall a, b, c \in \mathbb{R}$,
- $a<b$, $c>0 \rightarrow ac < bc$
- $a<b$, $c<0 \rightarrow ac > bc$ (multiplicative).

RMK : - $a \le b$ means $a < b$ or $a = b$.

- $a < b < c$ means $a < b$ and $b < c$.

- $a \in \mathbb{R}$ nonnegative if $a \ge 0$
  $a \in \mathbb{R}$ positive if $a > 0$

RMK :

- Natural number : $\mathbb{N} := \{1, 2, 3, 4, \dots\}$.
- Integer number : $\mathbb{Z} := \{0, \pm 1, \pm 2, \dots\}$.
- Rationals number : $\mathbb{Q} := \{ \frac{a}{b} , b \neq 0 , a, b \in \mathbb{Z} \}$. $\frac{m}{n} = \frac{p}{q} \to mq = np$
- IRRational number : $\mathbb{Q}^c := \mathbb{R} \setminus \mathbb{Q}$.

• $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$.

RMK : $\mathbb{N}$ and $\mathbb{Z}$ satisfy the following property :

(i) $n, m \in \mathbb{Z}$ then $n+m, n-m, m \cdot n \in \mathbb{Z}$.

(ii) if $n \in \mathbb{Z}$ then $n \in \mathbb{N} \Leftrightarrow n \ge 1$.

(iii) There is no $n \in \mathbb{Z}$ that satisfy $0 < n < 1$.

Def : The absolute value of $a \in \mathbb{R}$ is $|a| := \begin{cases} a , & a \ge 0 \\ -a , & a < 0 \end{cases}$

RMK : The absolute value is multiplicative : $|ab| = |a||b| \quad \forall a, b \in \mathbb{R}$.

Thm 1 : let $a \in \mathbb{R}$ and $M \geq 0$ Then $|a| \leq M \Leftrightarrow -M \leq a \leq M$.

Thm 2 :

(i) (Positive definite) : $\forall a \in \mathbb{R}$, $|a| \geq 0$ with $|a| = 0 \Leftrightarrow a = 0$.

(ii) (symmetric) : $\forall a,b \in \mathbb{R}$, $|a-b| = |b-a|$.

(iii) (triangle inequality) : $|a+b| \leq |a| + |b|$ and $\big||a| - |b|\big| \leq |a-b|$.

Note : $b < c \not\Rightarrow |a+b| < |a+c|$.

Thm 3 : let $x, y, a \in \mathbb{R}$, Then

(i) $x < y + \varepsilon$, $\forall \varepsilon > 0 \Leftrightarrow x \leq y$.

(ii) $x > y - \varepsilon$, $\forall \varepsilon > 0 \Leftrightarrow x \geq y$.

(iii) $|a| < \varepsilon$, $\forall \varepsilon > 0 \Leftrightarrow a = 0$

Def : let $a, b \in \mathbb{R}$, A closed interval is of the form:

$[a,b] := \{x : a \leq x \leq b\}$. $\quad (-\infty, b] := \{x : x \leq b\}$.

$[a,\infty) := \{x : x \geq a\}$ $\quad (-\infty, \infty) := \{x : x \in \mathbb{R}\}$.

★ open interval :

$(a,b) := \{x \in \mathbb{R} : a < x < b\}$ $\quad (-\infty, b) := \{x \in \mathbb{R} : x < b\}$

$(a,\infty) := \{x \in \mathbb{R} : x > a\}$ $\quad (-\infty, \infty) := \{x \in \mathbb{R}\}$.

→ $[a,b) := \{x \in \mathbb{R} : a \leq x < b\}$

$(a,b] := \{x \in \mathbb{R} : a < x \leq b\}$.

- An interval I is bounded iff it has the form $[a,b]$, (a,b), $[a,b)$, $(a,b]$ for $-\infty < a \leq b < \infty$.

- If $a=b$ then an Interval I is degenerate and nondegenerate if $a<b$.

- The length of a bounded interval I with endpts a and b is $|I| = |b-a|$.

RMK by Thm1: $|a| \leq M \iff a \in [-M, M]$.

1.3 : Completeness Axiom

Def : let $E \subseteq \mathbb{R}$ be anonempty set , Then

(i) E is bounded above iff $\exists$ an $M \in \mathbb{R}$ s.t $x \leq M$, $\forall x \in E$ (M is an upperbound of E) .

(ii) A number β is called a supremum of E iff β is an upperbound and $\beta \leq M$, for all upperbounds M of E .

we say E has a finite supremum and we write $\sup E = \beta$.

Note : every supremum its upperbound , The oppisite is not true .

RMK :

(1) $\sup E$ (if it exists) is the smallest (least) upperbound.

(2) to prove $\sup E = \beta$ we prove two things :

(i) β is an upper bound of E (i.e , $x \leq \beta$, $\forall x \in E$) .

(ii) If M is any an upperbound of E then $\beta \leq M$.

RMK :

1. If a set has one upperbound , it has infinitely many upperbounds .

2. If $\sup E = \beta$ exists then it is unique .

Thm1 : Approximation property for supremum .

If $\sup E = \beta < +\infty$ and $\varepsilon > 0$ then $\exists$ a point $x \in E$ s.t $\underline{\beta - \varepsilon < x \leq \beta}$.

Thm 2: If $E \subset \mathbb{Z}$ has a supremum then $\sup E \in \mathbb{Z}$

Inparticular If the sup of a set which contains only integers exists, then that sup must be an integers.

★ Postulate 3: Completeness Axiom.

If E is anonempty subset of $\mathbb{R}$ that is bdd above then E has finite supremum.

Thm 3: Archimedean property.

given $a, b \in \mathbb{R}$ with $a > 0$, $\exists$ an integer $n \in \mathbb{N}$ s.t $b < na$.

Remark: $\sup E$ is not always belong to E.

Thm 4: Density of Rationals:

The rational number $\mathbb{Q}$ are dense in $\mathbb{R}$, That is

$\forall a, b \in \mathbb{R}$ with $a < b$ $\exists q \in \mathbb{Q} : a < q < b$.

Df: let $E \subset \mathbb{R}$ be anonempty

(i) The set E is said to be bounded below iff $\exists m \in \mathbb{R}$ s.t $m \leq x$ $\forall x \in E$. in this case m is said to be a lower bound of E.

(ii) A number α is called an infimum of the set E iff α is a lowerbound of E and $\alpha \geq \gamma$ for all lower bounds γ of E.

In this case we say that E has an infimum α and write $\inf E = \alpha$.

(iii) E is said to be bounded iff it is bounded above and below.

That is $\exists m, M$ s.t $m \leq x \leq M$ $\forall x \in E$. OR $\exists M > 0$ s.t $|x| \leq M$ $\forall x \in E$.

RMK : When a set E is contains its supremum we write $\max E = \sup E$.
similarly if $\inf E \in E$ we write $\inf E = \min E$.

Thm 5 : Reflection principle : let $E \subseteq \mathbb{R}$ be nonempty
(i) E has a supremum iff $-E$ has an infimum, in which case $\inf(-E) = -\sup(E)$.

(ii) E has a infimum iff $-E$ has a supremum, in which case $\sup(-E) = -\inf(E)$.

Thm 6 : Monotone property : suppose that $A \subseteq B$ are nonempty sets of $\mathbb{R}$:
(i) If B has a supremum then $\sup A \leq \sup B$.

(ii) If B has a infimum then $\inf A \geq \inf B$.

Thm 7 : Approximation property for infimum :
If a set $E \subseteq \mathbb{R}$ has afinite infimum α and $\varepsilon > 0$ is any positive number then there is a point $x \in E$ s.t $\alpha + \varepsilon > x \geq \alpha$.

* completeness property for Infimum :
If $E \subseteq \mathbb{R}$ is anonempty and bdd below then E has a finite infimum.

* The extended real numbers :
$\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\} = [-\infty, \infty]$. Thus x is an extended real number iff $x \in \mathbb{R}$, $x = \infty$ or $x = -\infty$.

- $\emptyset \neq E \subseteq \mathbb{R}$ is unbdd above if it has no upperbound and unbdd below if it has no lowerbound.

- $\emptyset \neq E \subseteq \mathbb{R}$ we define $\sup E = \infty$ if E is unbdd above and $\inf E = -\infty$ if E is unbdd below.

- we define $\sup \emptyset = -\infty$, $\inf \emptyset = \infty$.