chapter 5: permutation groups.

الهدف من الشابتر انه أعمل group فيه functions مع عملية compositions

$f: X \to X$

$f^{-1}: X \to X$

$G \to f, \underline{o}$ composition

لازم f يكون 1-1 (onto) ← bijection

$G: \{f: X \to X \; ; \; f \text{ is bijection}\}$

$\langle G, \underline{o} \rangle$ is a group.

X : finite on this chapter.

if X finite and $X \to X$ then function 1-1 → onto

find set

exp: $X = \{a, b, c, d\}$

$f: X \to X$. كم عدد Functions الي بقدر أعملها؟

$|f| = 4! = 24$

يعني 24 function بقدر أعمل

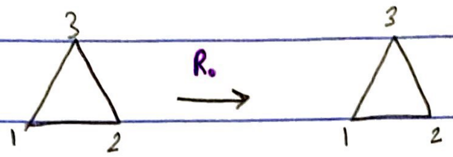
- $X = \{a_1, a_2, \dots a_n\}$

  $f: X \to X$.

  $|f| = n!$

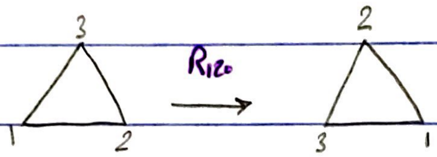
- S_n: the set of all 1-1 functions on $\{1, 2, \dots n\}$ since $X = \{1, 2, \dots, n\}$

→ In $D_3 = \{R_0, R_{120}, R_{240}, F_1, F_2, F_3\}$



Identity

دوران بـ 120°

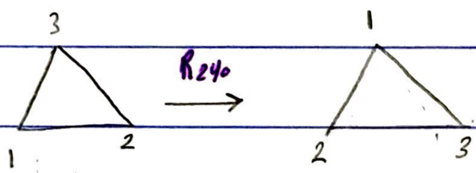


$f(1) = 3$

$f(2) = 1$ → $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$

$f(3) = 2$

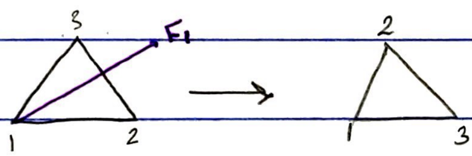
دوران بـ 240°



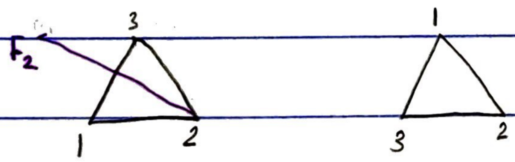
$f(1) = 2$

$f(2) = 3$ $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$

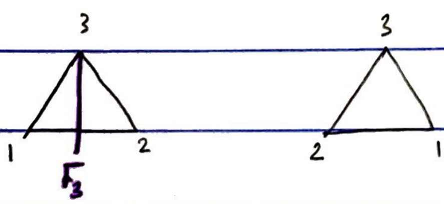
$f(3) = 1$



$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix}$



$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$



$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix}$

$S_3 = G = \{e, (3\,1\,2), (2\,3\,1), \underset{=(2\,3)}{(3\,2)}, \underset{=(1\,3)}{(3\,1)}, \underset{=(1\,2)}{(2\,1)}\}$

$D_3 = S_3$

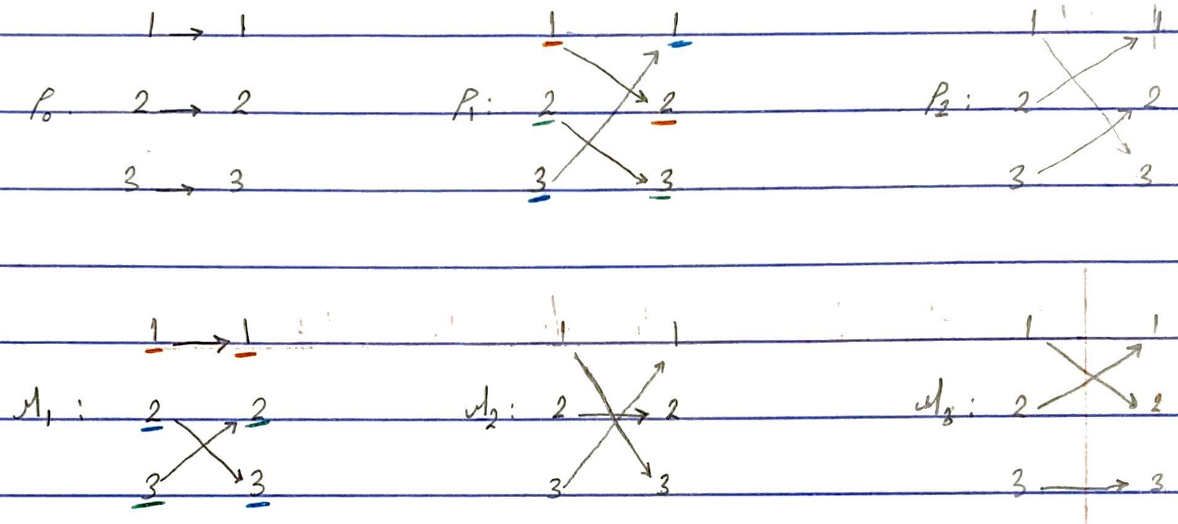
بس كل واحد إله نظرة مختلفة

Def:

let A be a set, then permutation on a set A is a bijection mapping from A to A.

$|A| = 3! = 6$ **exp:** let $A = \{1,2,3\}$ Then the set of all bijection mapping from A to A are

كلهم 1-1

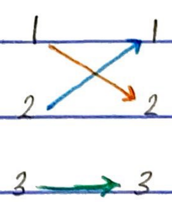


So if we denoted the above set of all permutations by S_3 called symmetric group of $A = \{1,2,3\}$ Then $S_3 = \{P_0, P_1, P_2, \mu_1, \mu_2, \mu_3\}$ then the group S_3 under the operation defined by $f * g = f \circ g$ represented by the table.

P_0 العنصر المحايد فيعطل زي ماهو لما اجي اعمل العملية

	P_0	P_1	P_2	μ_1	μ_2	μ_3
P_0	P_0	P_1	P_2	μ_1	μ_2	μ_3
P_1	P_1	P_2	P_0	$\boxed{\mu_3}$	μ_1	μ_2
P_2	P_2	P_0	P_1	μ_2	μ_3	μ_1
μ_1	μ_1	μ_2	μ_3	P_0	P_1	P_2
μ_2	μ_2	μ_3	μ_1	P_2	P_0	P_1
μ_3	μ_3	μ_1	μ_2	P_1	P_2	P_0

to find $P_1 \circ \mu_1 \rightarrow$ بلشرحها



هسا بدور فوق شو هاي أي وحدة منهم

وبطلع μ_3 يعني $P_1 \circ \mu_1 = \mu_3$

→ Not commutitive. **Note $P_1 \circ \mu_1 \neq \mu_1 \circ P_1$ so not abelian.**

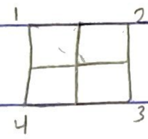
(هسا بوجد كم وحدة هيك والباقي بجيبه بدون هاي الشغلة (يعني منهن 2 زي بعض يكون جنب أو فوق بعض))

exp: symmetric group S_n

let $A = \{1, 2, \dots n\}$. The set of all permutations of A is called the symmetric group of degree n and is denoted by S_n.

بالكتاب

exp: $|S_4| = 24$.



$$\text{let } D_4 = \{\underset{}{\rho_0}, \underset{\uparrow\,90^\circ}{\rho_1}, \underset{\uparrow\,180^\circ}{\rho_2}, \underset{\uparrow\,270^\circ}{\rho_3}, \underbrace{d_1, d_2}_{\text{diagonal}}, \underbrace{L_1, L_2}_{\text{عبر الراس}}\}$$

	ρ_0	ρ_1	ρ_2	ρ_3	d_1	d_2	L_1	L_2
ρ_0	ρ_0	ρ_1	ρ_2	ρ_3	d_1	d_2	L_1	L_2
ρ_1	ρ_1	ρ_2	ρ_3	ρ_0				
ρ_2	ρ_2	ρ_3	ρ_0	ρ_1				
ρ_3	ρ_3	ρ_0	ρ_1	ρ_2				
d_1	d_1							
d_2	d_2		ونكمل					
L_1	L_1							
L_2	L_2							

ch2

$$\rho_1 \circ d_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}}_{\text{نبلش من هاي}} = \underbrace{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}}_{} = L_1$$

exp: let $A = \{1, 2, 3, 4\}$, let $\alpha : A \to A$ be given as $\alpha(1) = 3$, $\alpha(2) = 1$, $\alpha(3) = 4$, $\alpha(4) = 2$. let $\beta : A \to A$ be given as $\beta(1) = 1$, $\beta(2) = 3$, $\beta(3) = 4$, $\beta(4) = 2$.

- clearly α and β are permutations on A.

  $\rightarrow$ both α and β are 1-1 and onto.

$$\rightarrow \alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}$$

are abeian ?

$$\alpha \circ \beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}$$ بلش

$$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}$$

$$\beta \circ \alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$$

$\alpha\beta \neq \beta\alpha$ So Not Abelian.

→ Cycle Notation.

let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 4 & 3 & 2 & 6 & 1 & 8 & 7 \end{pmatrix} \in S_8$

$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 4 & 6 & 1 & 2 & 3 & 7 & 8 \end{pmatrix}$

طريقة كتابة جديدة

$\alpha = (156)(24)(3)(78)$ → cycle Notation

↓ هاي نفسها

والباقي بروح لحال

$\underset{f}{\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 2 & 3 & 4 & 6 & 1 & 7 & 8 \end{pmatrix}} \underset{g}{\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 4 & 3 & 2 & 5 & 6 & 7 & 8 \end{pmatrix}} \underset{h}{\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{pmatrix}} \underset{k}{\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 8 & 7 \end{pmatrix}}$

$f \circ g \circ h \circ k = \alpha$ $\qquad (24), (78)$: transposition

أي طولهم 2
فيهم عنصرين

(3)
لأنها لحالها
ما بنكتبها

$\alpha = (156)(24)(78)$

→ $\sigma = (1524)(36)(7)(8)$

$\sigma = (1524)(36)$ → disjoint cycle.

$\alpha \circ \underline{\sigma} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 2 & 1 & 5 & 4 & 3 & 8 & 7 \end{pmatrix} = (163)(2)(45)(78)$

الطريقة العادية

$= (163)(45)(78).$

$\alpha \circ \sigma = (156)(24)(78) \circ (1524)(36)$

$= (163)(2)(45)(78).$

Thm 5.1 : products of disjoint cycles.

every permutation of a finite set can be written as a cycle or as product of disjoint cycles.

exp:

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 5 & 4 & 6 & 1 & 9 & 7 & 3 & 8 \end{pmatrix} \in S_9 .$$

ممكن تكون من $\in S_{20}$
وكل الأرقام بعد 9 بتروح لحالها

$$\alpha = (125)(34698)(7) = \underbrace{(125)}_{3}\underbrace{(34698)}_{5}$$

order → 3 5 .

Thm 5.2: Disjoint cycles Commute

If the pair of cycles $\alpha = (a_1, a_2, \ldots a_m)$ and $\beta = (b_1, b_2, \ldots b_n)$ have no entries in common, then $\alpha\beta = \beta\alpha$.

exp.

ما في بينهم أي مشترك

$\alpha = (1452)$, $\beta = (378)$ $\in S_8$.

1→1 2→2 6→6
4→4 3→7
5→5 7→8

$$\alpha\beta = (1452)(378)(6) = (1452)(378)$$

$$\beta\alpha = (1452)(378)(6) = (1452)(378)$$

→ $\alpha\beta = \beta\alpha$

Thm 5.3 : order of permutation.

The order of a permutation of a finite set written in disjoint cycle from is the least common multiple of the lengths of the cycles.

المضاعف المشترك الأصغر

exp Thm 1 :

→ 3 6 9 12 $\boxed{15}$ 18 ....

→ 5 10 $\boxed{15}$ 20 25 30 ....

$$|\alpha| = L.C.M(3,5) = 15$$

exp: let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 5 & 1 & 8 & 4 & 3 & 6 & 7 \end{pmatrix}$. Find $|\alpha|$. كتبتها على cycle

$\alpha = (12548763) \rightarrow |\alpha| = 8$ لأنه cycle واحدة

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exp: let $\beta \circ \alpha = (2461)(574) = (124576)(3) = (124576)$

$|\beta\circ\alpha| = 6$

Thm 5.4: products of 2-cycles.

Every permutation in S_n, $n>1$, is a product of 2-cycles.

قصده الطول يكون 2 في cycles.

exp: $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 5 & 4 & 6 & 1 & 9 & 7 & 3 & 8 \end{pmatrix} =$

$= (125)(34698)$

$= (15)(12)(38)(39)(36)(34)$

exp: $\bullet\ (2\ 4\ 7\ 9) = (29)(27)(24)$

$\bullet\ (1\ 4\ 3\ 2\ 8) = (18)(12)(13)(14)$.

Thm 5.5 : Always even or Always odd.

In a permutation α can be expressed as a product of an even (odd) number of 2-cycles, then every decomposition of α into a product of 2 cycles must have an even (odd) number of 2-cycles. In symbols

If $\alpha = \beta_1\beta_2 \dots \beta_r$ and $\alpha = \sigma_1\sigma_2 \dots \sigma_s$

where the β's and σ's are 2-cycles, then r and s are both even or odd.

exp:

$\underbrace{(2479)} = \underbrace{(29)(27)(24)}$ $\qquad$ $(14328) = (18)(12)(13)(14)$.

$\hookrightarrow$ order $= \underset{\text{even}}{4} \rightarrow 4-1 = 3$ odd $\qquad$ $\underset{\text{odd}}{5}$ $\qquad$ even

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Definition : even and odd permutations.

A permutation that can be expressed as a product of an even number of 2-cycles is called an even permutation. A permutation that can be expressed as a product of an odd number of 2-cycles is called an odd permutation.

S_n = All permutations on $\{1, 2, \dots, n\}$.

A_n = All even permutations on $\{1, 2, \dots n\}$, $\underline{\varepsilon = \rho_0} \in A_n$.

$A_n \subseteq S_n$, $|A_n| = \frac{n!}{2}$.

on book exp:

$|S_4| = 24$, $|A_n| = 12$.

Book.

exp: consider $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 8 & 4 & 3 & 7 & 1 & 6 & 2 & 9 \end{pmatrix}$, $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 5 & 7 & 9 & 1 & 2 & 4 & 6 & 8 \end{pmatrix}$

1. write α, β as product of disjoint cycles.

$$\alpha = (1576)(28)(34)(9) = (1576)(28)(34).$$

$$\beta = (137498625)$$

2. Find $\alpha\beta$, $\beta\alpha$.

$$\alpha\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 7 & 6 & 9 & 5 & 8 & 3 & 1 & 2 \end{pmatrix} = (14927368)(5) = (14927368)$$

$$\beta\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 6 & 9 & 7 & 4 & 3 & 2 & 5 & 8 \end{pmatrix} = (1)(26398547) = (26398547)$$

3. Find $|\alpha|$, $|\beta|$; $|\alpha\beta|$.

$$|\alpha| = \text{L.c.m}(4,2,2) = 4.$$

$$|\beta| = 9$$

$$|\alpha\beta| = 8$$

4. Is α, β even or odd?

الحل

$$\alpha = \overset{3}{(1576)}\overset{1}{(28)}\overset{1}{(34)} \overset{}{=5} \qquad \text{odd}$$

$$\beta = \overset{9-1=8}{(137498625)} \qquad \text{even}.$$