2.4] Frequency Response

"Sinusoidal Steady State Response"

$$x(t) = X_m \cos(\omega_0 t + \theta_x) \longrightarrow \boxed{LTI} \longrightarrow y(t) = Y_m \cos(\omega_0 t + \theta_y)$$

Let $x(t) = X_m e^{j(\omega_0 t + \theta_x)}$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau = \int_{-\infty}^{\infty} h(\tau)\, X_m e^{j(\omega_0 t - \omega_0 \tau + \theta_x)}\, d\tau$$

$$y(t) = X_m e^{j(\omega_0 t + \theta_x)} \int_{-\infty}^{\infty} h(\tau)\, e^{-j\omega_0 \tau}\, d\tau = X_m e^{j(\omega_0 t + \theta_x)} H(\omega)\Big|_{\omega = \omega_0}$$

where $H(\omega) = \int_{-\infty}^{\infty} h(\tau)\, e^{-j\omega \tau}\, d\tau$

$$H(\omega) = \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} \, d\tau$$

$H(\omega)$ is the frequency response of the system that characterizes the spectral response of the LTI system.

Note: $H(\omega)$ is the Fourier transform of $h(t)$

$$H(\omega) = \int_{-\infty}^{\infty} h(\tau) e^{j\omega\tau} \, d\tau = |H(\omega)| \, e^{j\theta_H(\omega)}$$

$$\Rightarrow \quad y(t) = \left[|H(\omega)| \Big|_{\omega=\omega_0} e^{j\theta_H(\omega_0)} \right] \left[X_m e^{j(\omega_0 t + \theta_x)} \right]$$

$$y(t) = Y_m e^{j(\omega_0 t + \theta_y)} = \left(|H(\omega_0)| X_m\right) e^{j(\omega_0 t + \theta_x + \theta_H(\omega))}$$

$$\Rightarrow Y_m = |H(\omega_0)| X_m$$ $\rightsquigarrow$ Magnitude

$$\Rightarrow \theta_y = \theta_x + \angle \theta_H(\omega_0)$$ $\rightsquigarrow$ phase

$$Re[y(t)] = Re\left[|H(\omega_0)| X_m e^{j(\omega_0 t + \theta_x + \theta_H(\omega_0))}\right]$$

$$\underbrace{Y_m}\cos(\omega_0 t + \underbrace{\theta_y}) = \underbrace{X_m |H(\omega_0)|}\cos(\omega_0 t + \underbrace{\theta_x + \theta(\omega_0)})$$

Ex :- Determine the frequency response of the system with impulse response of $h(t) = 10e^{-2t}u(t)$

$$H(\omega) = \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau = \int_{0}^{\infty} 10 e^{-(2+j\omega)\tau} d\tau$$

$$H(\omega) = \frac{10}{-(2+j\omega)} e^{-(2+j\omega)\tau} \Big|_{0}^{\infty} = \frac{10}{2+j\omega}$$

$$H(\omega) = \frac{10}{\sqrt{4+\omega^2}\ \angle \tan^{-1}(\omega/2)} = \frac{10}{\sqrt{4+\omega^2}} \angle -\tan^{-1}(\omega/2)$$

EX :- Determine the SS response of the system in the previous example to the following input signal:

$$x(t) = 2\cos(4t + \frac{\pi}{3}) + 5\sin(6t + \frac{\pi}{4})$$

superposition (LOA)

$$y(t) = 2\left(\frac{10}{\sqrt{4+(4)^2}}\right)\cos\left(4t + \frac{\pi}{3} - \tan^{-1}\left(\frac{4}{2}\right)\right)$$

$$+ 5\left(\frac{10}{\sqrt{4+(6)^2}}\right)\sin\left(6t + \frac{\pi}{4} - \tan^{-1}\left(\frac{6}{2}\right)\right)$$

$$y(t) = \frac{20}{\sqrt{20}}\cos\left(4t + \frac{\pi}{3} - \tan^{-1}(2)\right) + \frac{50}{\sqrt{40}}\sin\left(6t + \frac{\pi}{4} - \tan^{-1}(3)\right)$$

2.5] Stability of LTI system

$$x(t) \longrightarrow \boxed{LTI} \longrightarrow y(t)$$

$$\sum_{i=0}^{n} a_i \frac{d^{(i)} y(t)}{dt^{(i)}} = \sum_{j=0}^{m} b_j \frac{d^{(j)} x(t)}{dt^{(j)}}$$

$$y(t) = \underbrace{\text{zero input response}}_{y_{ZI}(t)} + \underbrace{\text{zero state response}}_{y_{ZS}(t)}$$

$y_{ZI}(t)$: due to initial conditions ($x(t) = 0$)

$y_{ZS}(t)$: due to $x(t)$ only (initial conditions $= 0$)

The LTI system is $\boxed{\text{BIBO stable}}$ if $y_{zs}(t)$ is bounded for a bounded $x(t)$

The LTI system is $\boxed{\text{asymptotically stable}}$ if $y_{zI}(t) = 0$ when $t \to \infty$

$$y(t) = y_{zI}(t) + y_{zs}(t)$$

$$Y(s) = Y_{zI}(s) + Y_{zs}(s) = Y_{zI}(s) + \underbrace{H(s)}_{\text{Transfer Function}} X(s)$$

$$H(s) = \frac{Y(s)}{X(s)}\Bigg|_{\text{zero initial conditions}}$$

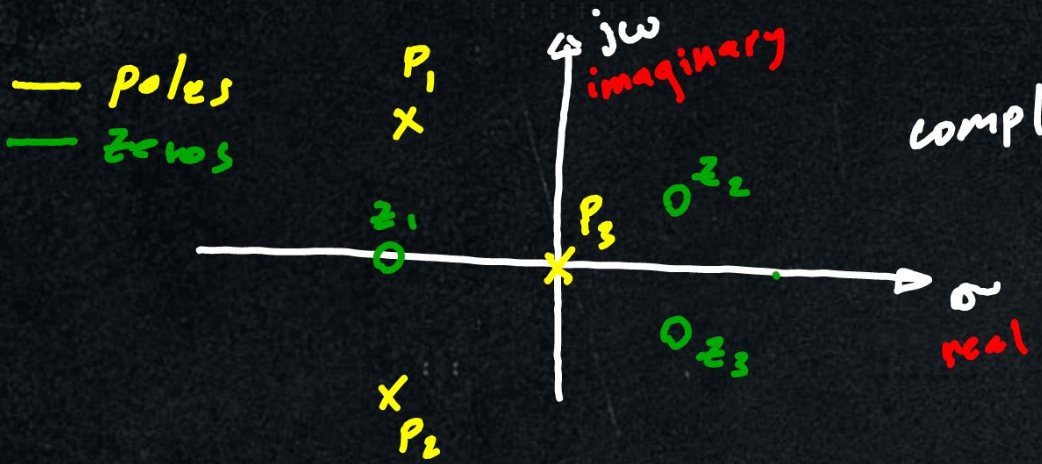
$$H(s) = \frac{b_0 + b_1 s + b_2 s^2 + \cdots + b_m s^m}{a_0 + a_1 s + a_2 s^2 + \cdots + a_n s^n}$$

$$H(s) = \frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-P_1)(s-P_2)\cdots(s-P_n)}$$

where $z_1, z_2, \ldots, z_m$ are zeros

$P_1, P_2, \ldots, P_n$ are poles

s is the Laplace variable $[s = \sigma + j\omega]$



$$y(t) = y_{zI}(t) + y_{zs}(t)$$

$$Y(s) = \underbrace{\frac{I(s)}{Q(s)}}_{Y_{zI}(s)} + \underbrace{\frac{P(s)}{Q(s)}}_{H(s)} X(s)$$

$\Rightarrow$ Assume that no pole/zero cancellation

$\Downarrow$

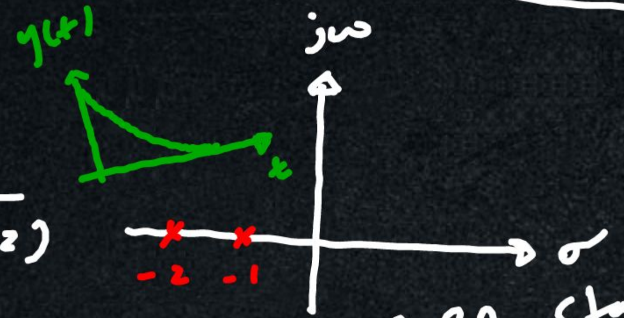
BIBO stable $\Longleftrightarrow$ Asymptotically stable (AS)

equivalent to

The system is BIBO stable or AS if all poles of $H(s)$ or roots of $Q(s)$ are real negative in the complex plane

ex:

$$H(s) = \frac{3}{(s+1)(s+2)}$$



$p_1 = -1, \ p_2 = -2$

BIBO stable

If $H(s)$ has at least one real positive pole or repeated poles with zero real parts, then the system is unstable.

ex:

$$H(s) = \frac{5}{(s+1)(s-2)}$$

$P_1 = -1$

$P_2 = 2$

$$H(s) = \frac{2}{(s+1)(s^2+1)^2}$$

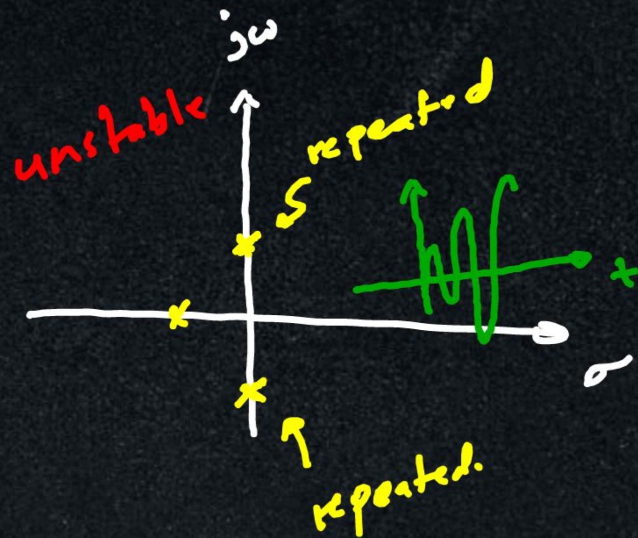
$P_1 = -1$

$P_2 = j$

$P_3 = -j$

$P_4 = j$

$P_5 = -j$



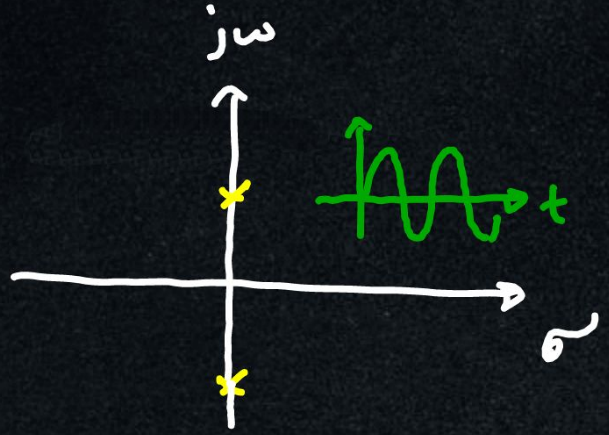
If $H(s)$ has poles (not repeated) and located on the imaginary axis, then the system is **marginally stable**.

ex:

$$H(s) = \frac{1}{s^2+1}$$

$P_1 = j$

$P_2 = -j$



EX :- Disuss the stability of the following systems

1) $H(s) = \dfrac{s+2}{(s+4)(s+1)}$ BIBO stable

2) $H(s) = \dfrac{s+4}{(s^2+4)(s+1)}$ marginally stable

3) $H(s) = \dfrac{s+4}{(s^2+4)^2(s+1)}$ unstable

4) $H(s) = \dfrac{s-2}{(s-2)(s+1)}$ BIBO stable

2.6] Modeling and simulation of LTI systems

* Differential equations
* Laplace transform
* State-space representation
* Observer representation model

* MatLab/simulink
* LabView
* Mathcad

↳ In this section, we will use the observer representation model (integrate and separate)

Note: modeling is usually used for simulation purposes to analyze system characteristics and its response

EX :- Determine the observer model

$$\frac{d^3y(t)}{dt^3}+2\frac{d^2y(t)}{dt^2}+3\frac{dy(t)}{dt}+y(t)=\frac{d^2x(t)}{dt^2}+7\frac{dx(t)}{dt}+2x(t)$$

$$y'''+2y''+3y'-x''-7x'=2x-y=q_0$$

$$y''+2y'-x'=\int_0^t q_0\,d\lambda+7x-3y=q_1$$

$$y'=\int_0^t q_1\,d\lambda+x-2y=q_2$$

$$y=\int_0^t q_2\,d\lambda$$

separate and integrate