Chapter 2. First Order Differential Equations.

2.2. Seperable Equations:

The general form of the first order D. E is

$$\frac{dy}{dx} = f(x,y) \quad \ldots (1)$$

We can write eq. (1) in the form:

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \quad \ldots (2)$$

by setting $M(x,y) = -f(x,y)$ and $N(x,y) = 1$.

If M is a function of x only and N is a function of y only, then eq. (2) can be written as:

$$M(x) + N(y)\frac{dy}{dx} = 0 \quad \ldots (3)$$

Eq. (3) is called seperable, because it can be written as: $M(x)\,dx + N(y)\,dy = 0$,

so we can solve it by integrating M and N

Thus, the general form of a first order seperable D.E is $\boxed{\dfrac{dy}{dx} = g(x)\,h(y)}$

Example (1): Solve $y' = \dfrac{x^2}{4-y^2}$

Sol: $\dfrac{dy}{dx} = \dfrac{x^2}{4-y^2} \Rightarrow (4-y^2)\,dy = x^2\,dx$.

$\Rightarrow \int (4-y^2)\,dy = \int x^2\,dx$

$\Rightarrow 4y - \dfrac{y^3}{3} = \dfrac{x^3}{3} + C$

$\Rightarrow 12y - y^3 - x^3 = C$ (Implicit form).

Note: If (x_0, y_0) is given, we can find C.

Example (2): Solve the IVP

$$\frac{dy}{dx} = \frac{3x^2+4x+2}{2(y-1)} \quad , \quad y(0) = -1 .$$

Sol: $\int 2(y-1)\,dy = \int 3x^2+4x+2\,dx$

$$\Rightarrow \quad y^2 - 2y = x^3 + 2x^2 + 2x + C_1 .$$

Using: $y(0) = -1$ for finding C_1:

$$(-1)^2 - 2(-1) = (0)^3 + 2(0)^2 + 2(0) + C_1 \Rightarrow \boxed{C_1 = 3}$$

Therefore: $y^2 - 2y = x^3 + 2x^2 + 2x + 3$. (Implicit)

To obtain the solution explicitly:

$$y^2 - 2y + 1 = x^3 + 2x^2 + 2x + 3 + 1$$ "إكمال المربع"

$$(y-1)^2 = x^3 + 2x^2 + 2x + 4$$

$$\Rightarrow \quad y = 1 \pm \sqrt{x^3+2x^2+2x+4}$$

(Explicit)

Now: $y(0) = 1 + \sqrt{4} = 3$

or $y(0) = 1 - \sqrt{2} = -1$

$$\Rightarrow \quad y = 1 - \sqrt{x^3+2x^2+2x+4} .$$

To determine the solution in which interval exists : "Domain"

$$x^3 + 2x^2 + 2x + 4 \geq 0$$

$$x^2(x+2) + 2(x+2) \geq 0$$

$$(x^2+2)(x+2) \geq 0 \iff x \geq -2$$

But at $x = 2$, we have $y = 1 - \sqrt{-8+8-4+4} = 1$

which makes the D.E undefined.

Finally, $y = 1 - \sqrt{x^3 + 2x^2 + 2x + 4}$, $x > -2$.

Example (3): Solve the following IVP:

$$\begin{cases} x e^{2x + \cos y} + (\sin y) y' = 0 \\ y(0) = \frac{\pi}{2} . \end{cases}$$

Sol:
$$x e^{2x} \cdot e^{\cos y} + (\sin y) \frac{dy}{dx} = 0$$

$$\Rightarrow \quad x e^{2x}\, dx = -(\sin y)\, e^{-\cos y}\, dy.$$

Now : $\int x e^{2x}\, dx$

$\left\{ \begin{array}{l} \text{by parts:} \\ u = x, \quad dv = e^{2x} dx \\ du = dx \quad v = \frac{e^{2x}}{2} \end{array} \right.$ $-\int$

$$= \frac{x}{2} e^{2x} - \int \frac{e^{2x}}{2}\, dx$$

$$= \frac{x}{2} e^{2x} - \frac{1}{4} e^{2x} + C_1. \rightsquigarrow (\mathrm{I})$$

$$\int \sin y\, e^{-\cos y}\, dy$$

$\left\{ \begin{array}{l} \text{by substitution:} \\ u = -\cos y \\ du = \sin y\, dy \end{array} \right.$

$$= \int e^{u}\, dy = e^{u} + C_2$$

$$= e^{-\cos y} + C_2. \rightsquigarrow (\mathrm{II})$$

(I) and (II) Imply:

$$\frac{x}{2} e^{2x} - \frac{1}{4} e^{2x} = -e^{-\cos y} + C$$

Using $y(0) = \frac{\pi}{2}$ we have : $0 - \frac{1}{4} = -e^{-\cos\frac{\pi}{2}} + C$

$$\Rightarrow -\frac{1}{4} = -1 + C \Rightarrow \boxed{C = \frac{3}{4}}$$

Finally, $\frac{1}{2} x e^{2x} - \frac{1}{4} e^{2x} = -e^{-\cos y} + \frac{3}{4}$. (Implicit)

Exemple (4): $\frac{dy}{dt} = y^{\frac{1}{3}}$, $y(0) = 0$.

$$\int y^{-\frac{1}{3}} dy = dt$$

$\Rightarrow \frac{3}{2} y^{\frac{2}{3}} = t + C$ (Implicit)

Using $y(0) = 0 \Rightarrow \boxed{C = 0}$

$\Rightarrow \frac{3}{2} y^{\frac{2}{3}} = t \Rightarrow y^2 = \left(\frac{2}{3}t\right)^3$

$\Rightarrow y = \pm \left(\frac{2}{3}t\right)^3$

Using $y(0) = 0 \Rightarrow y_1(t) = \left(\frac{2}{3}t\right)^3$

$y_2(t) = -\left(\frac{2}{3}t\right)^3$

and $y_3(t) = 0$ is another solution

So, we have $\boxed{3}$ solutions.

Exemple (5): $y y' = 1$, $y(0) = 0$

$\Rightarrow y\,dy = dx \Rightarrow \frac{y^2}{2} = x + C$, Using $y(0)=0$,

$\boxed{C=0}$, Therefore $y = \pm\sqrt{2x}$ (only 2 solutions)

$\cancel{y(t) = 0}$

Home work # 2 : Solve

1) $$\frac{dy}{dx} = \frac{xy - 3x - y + 3}{xy - 2x + 4y - 8}$$

2) $$\begin{cases} (x - xy^2) + (8y - x^2 y)y' = 0 \\ y(2) = 2. \end{cases}$$

Homogeneous Differential Equations:

The general form of a homogeneous D.E's is

$$\frac{dy}{dx} = f(x,y) = F\left(\frac{y}{x}\right) \quad \ldots (4)$$

Let $\frac{y}{x} = v$, or $\boxed{y = xv}$ ... (5)

then $$\frac{dy}{dx} = x\frac{dv}{dx} + v \quad \ldots (6)$$

Substitute (5) and (6) in (4) we get:

$$x\frac{dv}{dx} + v = F(v).$$ "Seperable D.E"

Example (6): Solve the following D.E Explicitly

(Q32) $y' = \dfrac{3y^2 + x^2}{2xy}$, $y(1) = 2$

Sol: $\dfrac{dy}{dx} = \dfrac{3}{2}\left(\dfrac{y}{x}\right) + \dfrac{1}{2}\left(\dfrac{x}{y}\right) = F\left(\dfrac{y}{x}\right)$ is Homog.

Let $\dfrac{y}{x} = v$ or $y = vx \Rightarrow \dfrac{dy}{dx} = x\dfrac{dv}{dx} + v$

$\Rightarrow x\dfrac{dv}{dx} + v = \dfrac{3}{2}v + \dfrac{1}{2v}$.

$\Rightarrow x\dfrac{dv}{dx} = \dfrac{1}{2}\left(v + \dfrac{1}{v}\right) = \dfrac{1}{2}\left(\dfrac{v^2+1}{v}\right)$

$\Rightarrow \dfrac{2v}{v^2+1}\,dv = \dfrac{1}{\cancel{2}x}\,dx$

$\Rightarrow \ln(v^2+1) = \ln|x| + C$

$\Rightarrow \ln\left(\dfrac{y^2}{x^2} + 1\right) = \ln|x| + C$

Using $y(1) = 2$: $\boxed{\ln 5 = C}$

$$\Rightarrow \ln\left(\frac{y^2}{x^2}+1\right) = \ln|x| + \ln 5 = \ln|5x|$$

$$\Rightarrow \frac{y^2}{x^2}+1 = |5x|$$

So either $\frac{y^2}{x^2}+1 = 5x$ or $\underbrace{\frac{y^2}{x^2}+1 = -5x}_{\frac{4}{1}+1 \neq -5 \text{ (reject)}}$

Using IC. $y(1) = 2$, then

$$\frac{y^2}{x^2}+1 = 5x$$

$$\Rightarrow y^2 = x^2(5x-1)$$

$$\Rightarrow y = \pm\sqrt{x^2(5x-1)}$$ $\left(y = -\sqrt{x^2(5x-1)}, \times \text{ since } 2 \neq -\sqrt{\cdots} \right)$

Again Using IC. $y(1) = 2 \Rightarrow y = +\sqrt{x^2(5x-1)}$

$$\Rightarrow y = x\sqrt{5x-1}$$ $\sqrt{x^2} = |x| = \pm x$

To find the Interval on which the Solution exists

$$5x-1 \geq 0 \iff x \geq \frac{1}{5}.$$

But at $x = \frac{1}{5} \Rightarrow y = 0$, so the D.E is Undefined

therefore, $x > \frac{1}{5}$

Example (7): Solve $x\,dy = \left(x e^{\frac{y}{x}} + y + x\right) dx$

Sol: $\dfrac{dy}{dx} = e^{\frac{y}{x}} + \dfrac{y}{x} + 1 = F\left(\dfrac{y}{x}\right)$.

Let $\dfrac{y}{x} = v \Rightarrow y = vx \Rightarrow \dfrac{dy}{dx} = x\dfrac{dv}{dx} + v$.

$$\Rightarrow x\frac{dv}{dx} + \cancel{v} = e^{v} + \cancel{v} + 1$$

$$\Rightarrow \int \frac{dv}{e^{v}+1} = \int \frac{dx}{x}$$

$$\Rightarrow -\int \frac{-e^{-v}}{1+e^{-v}}\,dv = \int \frac{dx}{x}$$

$$\Rightarrow -\ln|1+e^{-v}| = \ln|x| + C.$$

$$\Rightarrow -\ln\left(1+e^{-\frac{y}{x}}\right) = \ln|x| + C \text{ . (Implicit)}$$

Home work # 3: Solve:

$$\left[x^2 \sin\left(\frac{y^2}{x^2}\right) - 2y^2 \cos\left(\frac{y^2}{x^2}\right)\right] dx + 2xy \cos\left(\frac{y^2}{x^2}\right) dy = 0$$

2.1 Linear Equations; Method of Integrating Factors

Recall: The general form of first order D.E is

$$\frac{dy}{dt} = f(t, y) \quad \cdots \quad (1)$$

where f is a given function of two variables t, y.

If f in Eq. (1) depends linearly on y, then eq (1) is Linear D.E. If f is not linear in y, then eq (1) is nonlinear.

Thus, the general form of first order linear D.E is

$$\frac{dy}{dt} + p(t)\,y = g(t) \quad \cdots (2)$$

where $p(t)$ and $g(t)$ are given functions of t.

Example ① $y' + xy = \sin x$ 1st order Linear

② $\dfrac{d\beta}{dx} + \alpha\beta^2 = \tan x$ 1st order Nonlinear

Note : If $p(t)$ and $g(t)$ are constants, we learned how to solve the equation in Section (1.2)

Note : If $p(t)$ and $g(t)$ are functions of t, then we will use the method of Integrating factor.

$$\frac{dy}{dt} + p(t)\, y = g(t)$$

positive func.

Multiply both sides of this equation by $\mu(t) > 0$.

$$\Rightarrow \mu(t) \frac{dy}{dt} + \mu(t) p(t) y = \mu(t) g(t) \quad \cdots (3)$$

Need to find $\mu(t)$ so that the L.H.S of eq (3) is the derivative of $\mu(t)\, y$. That is, Comparing the L.H.S of eq (3) with

$$\frac{d}{dt}\big(\mu(t)\, y(t)\big) = \mu(t) \frac{dy}{dt} + \frac{d\mu}{dt} y(t) \quad \cdots (4)$$

we observe that:

$$\frac{d\mu(t)}{dt} = p(t)\,\mu(t) \qquad \text{(seperable)}$$

$$\Rightarrow \int \frac{d\mu}{\mu(t)} = \int p(t)\,dt$$

$$\Rightarrow \ln|\mu(t)| = \int p(t)\,dt + C \quad , \quad \mu(t) > 0$$

$$\Rightarrow \mu(t) = A\,e^{\int p(t)\,dt} \quad , \quad [A = e^{c} \,,\ \text{take } A = 1]$$

$$\therefore \mu(t) = e^{\int p(t)\,dt} \quad \text{"Integrating factor of eq (2)"}$$

Now, back to eq (2) and Multiply it by $\mu(t) = e^{\int p(t)\,dt}$

$$e^{\int p(t)\,dt} \cdot \frac{dy}{dt} + p(t)\,e^{\int p(t)\,dt}\,y = g(t)\,e^{\int p(t)\,dt}$$

$$\Leftrightarrow \frac{d}{dt}\left[\mu(t)\,y(t)\right] = g(t)\,\mu(t)$$

$$\Leftrightarrow \mu(t)\,y(t) = \int g(t)\,\mu(t)\,dt + C$$

Finally, $y(t) = \dfrac{1}{\mu(t)}\left[\int g(t)\mu(t)\,dt + C\right].$

Theorem: If $y' + p(x)y = g(t)$, where

$p(t)$ and $g(t)$ are continuous functions , then

the integrating factor is $\mu(x) = e^{\int p(x)\,dx}$,

and $y(x) = \frac{1}{\mu(x)}\left[\int \mu(x)\,g(x)\,dx + C_1\right]$.

Example (3): Solve $y' + 2y = 4$, $y(0) = 1$

Sol(1): $\frac{dy}{dx} = 4 - 2y \iff \frac{1}{2(2-y)}\,dy = dx$

$\iff -\frac{1}{2}\ln|2-y| = x + C$

Using $y(0) = 1$: $-\frac{1}{2}\ln(1) = 0 + C \Rightarrow \boxed{C = 0}$

Therefore : $-\frac{1}{2}\ln|2-y| = x$

$\Rightarrow \ln|2-y| = -2x$

$\Rightarrow 2 - y = \pm e^{-2x}$

$\Rightarrow y = 2 \mp e^{-2x}$

Using $y(0) = 1 \Rightarrow y = 2 - e^{-2x}$

Or, Sol(2) : $y' + 2y = 4$, $y(0) = 1$, with

$P(x) = 2$ and $g(x) = 4$. Therefore,

$$\mu(x) = e^{\int p(x)\,dx} = e^{\int 2\,dx} = e^{2x}$$

$$y(x) = \frac{1}{e^{2x}}\left[\int 4e^{2x}\,dx + C\right]$$

$$= e^{-2x}\left[2e^{2x} + C\right]$$

Using I.C. $y(0) = 1$, we get $1 = 2 + C \Rightarrow \boxed{C = -1}$

$$\Rightarrow y(x) = e^{-2x}\left[2e^{2x} - 1\right]$$

$$y(x) = 2 - e^{-2x}$$

Example (4) : Solve $t\frac{dy}{dt} + 2y = 4t^2$, $y(1) = 2$.

$$\frac{dy}{dt} + \underbrace{\frac{2}{t}}_{p(t)}\, y = \underbrace{4t}_{g(t)} \quad , t \neq 0 .$$

$$\Rightarrow \mu(t) = e^{\int p(t)\,dt} = e^{\int \frac{2}{t}\,dt} = e^{2\ln t} = t^2 \;, t \neq 0$$

$$\Rightarrow y(t) = \frac{1}{\mu(t)}\left[\int \mu(t)\, g(t)\,dt + C\right]$$

$$y(t) = \frac{1}{t^2}\left[\int t^2 \cdot (4t)\,dt + C\right]$$

$$= \frac{1}{t^2}\left[\int 4t^3\,dt + C\right]$$

$$y(t) = t^2 + \frac{C}{t^2} \quad , t \neq 0$$

Using IC. $y(1) = 2$, then $2 = 1 + C \Rightarrow \boxed{C = 1}$

$$\therefore y(t) = t^2 + \frac{1}{t^2} \quad , t \neq 0 .$$

Note : the largest Interval where the Solution exists in $(0, \infty)$

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Example (5): Solve $\frac{dy}{dx} = \frac{y}{ye^y - 2x}$.

The equation is Not linear in y, but its Linear in x, therefore we can write it as:

$$\frac{dx}{dy} = e^y - \frac{2x}{y}$$

$$\Rightarrow \boxed{\frac{dx}{dy} + \frac{2}{y} x = e^y} \text{, Linear in } x$$

$$p(y) = \frac{2}{y} \quad , \; g(y) = e^y ,$$

$$\mu(y) = e^{\int \frac{2}{y} dy} = e^{2\ln|y|} = e^{\ln y^2} = y^2 , \quad y > 0$$

$$x(y) = \frac{1}{y^2}\left[\int y^2 . e^y \, dy + C\right]$$

$$u = y^2 \qquad dv = e^y dy$$

$$2y \xrightarrow{+} e^y$$

$$2 \xrightarrow{-} e^y$$

$$0 \xrightarrow{+} e^y$$

$$= \frac{1}{y^2}\left[y^2 e^y - 2ye^y + 2e^y + C\right]$$

$$x(y) = e^y - \frac{2}{y} e^y + \frac{2}{y^2} e^y + \frac{C}{y^2} .$$

(Q27) page 40. Consider the IVP:

$$y' + \frac{1}{2}y = 2\cos t \quad , y(0) = -1$$

Find the Coordinates of the first local Maximum point of the solution for $t > 0$.

sol: $\mu(t) = e^{\int \frac{1}{2} dt} = e^{\frac{1}{2}t} \quad , t > 0$

$$y(t) = \frac{1}{e^{\frac{1}{2}t}} \left[\int \underbrace{e^{\frac{1}{2}t} \cdot (2\cos t)}_{\text{by parts twice}} dt + C \right]$$

$$y(t) = 2e^{-\frac{1}{2}t} \left[\frac{4}{5} e^{\frac{1}{2}t} \left(\sin t + \frac{1}{2}\cos t \right) \right] + C e^{-\frac{1}{2}t}$$

Using IC. $y(0) = -1 \implies -1 = \frac{8}{5}\left(\frac{1}{2}\right) + C \implies \boxed{C = -\frac{9}{5}}$

$$\implies y(t) = \frac{8}{5}\sin t + \frac{4}{5}\cos t - \frac{9}{5} e^{-\frac{1}{2}t}.$$

Now, $y'(t) = \frac{8}{5}\cos t - \frac{4}{5}\sin t + \frac{9}{10} e^{-\frac{1}{2}t}$

$$y''(t) = -\frac{8}{5}\sin t - \frac{4}{5}\cos t - \frac{9}{20} e^{-\frac{1}{2}t}.$$

Solving $y'(t) = 0 \implies t^* = 1.3643$

Since $y''(t^*) < 0$, then the point $(t^*, y(t^*))$ is a local Max. The Coordinates $(1.3643, 0.82008)$.

(Q31) page 41. Consider the IVP:

$$y' - \frac{3}{2} y = 3t + 2e^{t} \quad , y(0) = y_0$$

Find the value of y_0 that seperates solutions that grow positively as $t \to \infty$ from those that grow negatively. How does the solution that corresponds to this critical value of y_0 behave as $t \to \infty$?

Sol: $\mu(t) = e^{-\int p(t)\,dt} = e^{-\int \frac{3}{2} dt} = e^{-\frac{3}{2} t}$

$$y(t) = \frac{1}{e^{-\frac{3}{2}t}} \left[\int e^{-\frac{3}{2}t} \cdot (3t + 2e^{t})\,dt + C \right]$$

$$= e^{\frac{3}{2}t} \left[3\int t e^{-\frac{3}{2}t} dt + 2\int e^{-\frac{1}{2}t} dt + C \right]$$

$$= e^{\frac{3}{2}t} \left[3\left(-\frac{2te^{-\frac{3}{2}t}}{3}\right) - 3\int \frac{e^{-\frac{3}{2}t}(-2)}{3} dt + 2\int e^{-\frac{1}{2}t} dt + C \right]$$

$$y(t) = -2t - \frac{4}{3} - 4e^{t} + C e^{\frac{3}{2}t}.$$

Now: Using IC. $y(0) = y_0$, we have

$$y_0 = -\frac{4}{3} - 4 + C \implies C = y_0 + \frac{16}{3}$$

$$\Rightarrow y(t) = -2t - \frac{4}{3} - 4e^{t} + \left(y_0 + \frac{16}{3}\right) e^{\frac{3}{2}t}.$$

As $t \longrightarrow \infty$, the term $e^{\frac{3}{2}t}$ will dominate the Solution

So the Coefficient sign of $e^{\frac{3}{2}t}$ will determine

the divergence property; that is;

the critical value of IC is $y_0 = \frac{-16}{3}$.

- If $y_0 > \frac{-16}{3}$, then $\lim\limits_{t \to \infty} y(t) = \infty$
- If $y_0 < -\frac{16}{3}$, then $\lim\limits_{t \to \infty} y(t) = -\infty$
- If $y_0 = \frac{-16}{3} \Rightarrow y(t) = -2t - \frac{4}{3} - 4e^{t}$.

which grows negatively as $t \longrightarrow \infty$.

Home Work #4. Solve the IVP:

(1) $(\sin y)\, dx + 2(x - 3\sin y) \cos y \, dy = 0$

$x\left(\frac{\pi}{2}\right) = \frac{1}{2}$

(2) $x y' + x y = 1 - y, \quad y(1) = 1.$

2.3 Modeling with First order Equations:

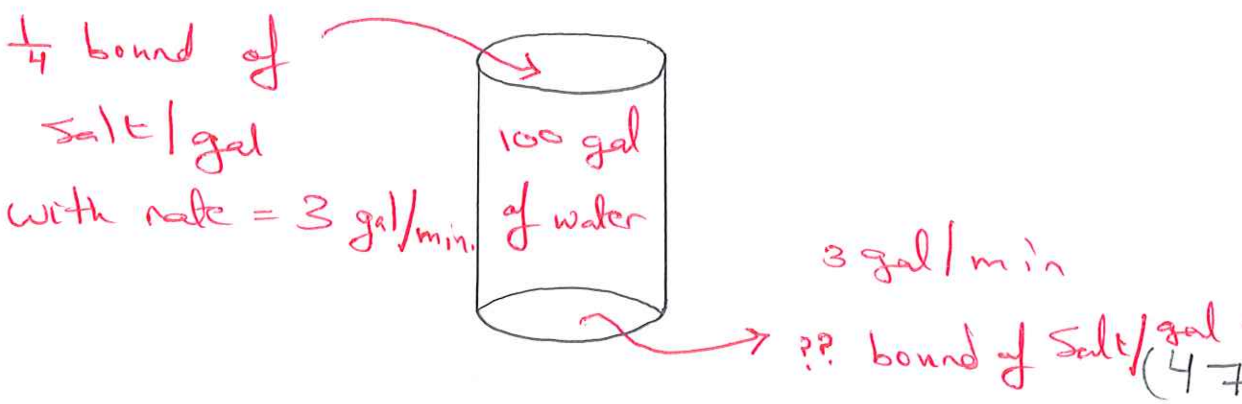
Example(1): At time $t=0$ a tank Contains 50 bound of salt dissolved in 100 gal of water. Assume that water Containing $\frac{1}{4}$ bound of salt/gal is entering the tank at a rate of 3 gal/min. and leave it at the same rate.

(1) Set up the IVP that describes this process.

(2) Find the amount of salt $Q(t)$ in the tank at any time t.

(3) Find the limiting amount of salt Q_L in the tank after a very long time.

(4) Find the time T when $Q(t) = 25.5$.



Sol: Let $Q(t)$ be the amount of salt in the tank at any time t, $Q(0) = 50$ bound.

(1) $\frac{dQ}{dt}$ = rate in − rate out

= (concentration × flow in) − (concentration × flow out).

$$= \left(\frac{1}{4}\right)(3) - \left(\frac{Q}{100}\right)(3)$$

So, the IVP is

$$\frac{dQ}{dt} = \frac{3}{4} - \frac{3Q}{100}, \quad Q(0) = 50.$$

(2) $\frac{dQ}{dt} + \frac{3}{100} Q = \frac{3}{4}$, $p(t) = \frac{3}{100}$, $g(t) = \frac{3}{4}$

$$\Rightarrow \mu(t) = e^{\int p(t)\,dt} = e^{\int \frac{3}{100}\,dt} = e^{\frac{3}{100}t}.$$

$$y(t) = \frac{1}{e^{\frac{3}{100}t}} \left[\int e^{\frac{3}{100}t} \cdot \frac{3}{4}\,dt + C \right]$$

$$= 25 + e^{-\frac{3}{100}t} C.$$

Using IC: $Q(0) = 50 \Rightarrow 50 = 25 + C \Rightarrow \boxed{C = 25}$

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$\therefore \quad Q(t) = 25 + 25 e^{-\frac{3}{100} t}$

(3) $\quad Q_L = \lim_{t \to \infty} Q(t) = 25$

(4) $\quad Q(t) = 25.5$

$\Rightarrow \quad 25.5 = 25 + 25 e^{-\frac{3}{100} t}$

$\Rightarrow \quad \dfrac{0.5}{25} = e^{-\frac{3}{100} t}$

$\Rightarrow \quad \dfrac{5}{250} = e^{-\frac{3}{100} t}$

$\Rightarrow \quad \dfrac{1}{50} = e^{-\frac{3}{100} t}$

$\Rightarrow \quad \ln \frac{1}{50} = -\frac{3}{100} t$

$\Rightarrow \quad \ln 1 - \ln 50 = -\frac{3}{100} t$

$\Rightarrow \quad \frac{100}{3} \ln 50 = t.$

Example(2) : A tank of Capacity 200 gal has initially 0.1 gm of toxic Wastes dissolved in 80 gal of water. Water with toxic wastes starts flow into the tank at a rate 4 gal / min. and flow out of the tank at a rate 2 gal / min.

The incoming water contains $\frac{1}{4}$ gm / gal of toxic wastes.

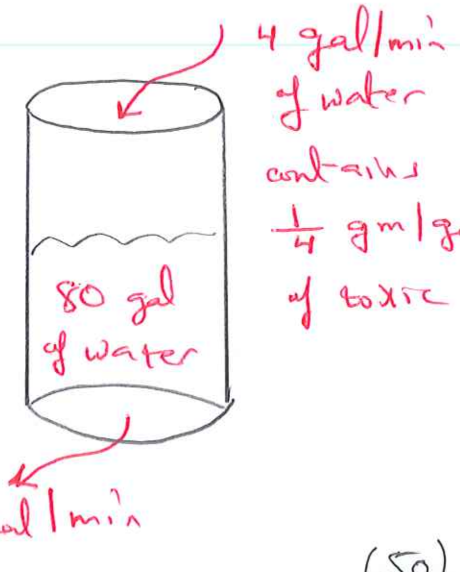
(1) Write the IVP that describes this process.

(2) Find the amount of toxic wastes in the tank at time t.

(3) Find the amount of toxic wastes in the tank when it becomes to over flow.

Sol: Let $Q(t)$ be the amount of toxic wastes in the tank at any time t, with

$$Q(0) = 0.1$$



The Capacity of the tank = 200 gal.

At $t=1 \rightsquigarrow 82$ gal of water

At $t=2 \rightsquigarrow 80+2(2) = 84$ gal of water.

At $t=3 \rightsquigarrow 80+2(3) = 86$ gal of water.

$\vdots$

At any time t, Volume $= 80+2t$.

Then $\dfrac{dQ}{dt}$ = rate in − rate out.

= (concentration x flow in) − (Concentration x flow out).

$$= \left(\frac{1}{4} \times 4\right) - \left(\frac{Q(t)}{80+2t} \times 2\right)$$

So, the IVP becomes:

$$\frac{dQ}{dt} = 1 - \frac{Q(t)}{40+t}, \quad Q(0) = 0.1$$

$$\Longleftrightarrow \begin{cases} \dfrac{dQ}{dt} + \dfrac{1}{40+t} Q(t) = 1. \\ Q(0) = 0.1. \end{cases}$$

(2) $p(t) = \dfrac{1}{40+t}$, $g(t) = 1$.

$$\Rightarrow \quad \mu(t) = e^{\int \frac{1}{40+t}\,dt} = e^{\ln|40+t|} = 40+t$$

$$Q(t) = \frac{1}{\mu(t)}\left[\int \mu(t)\, g(t)\, dt + C\right]$$

$$= \frac{1}{40+t}\left[\int (40+t)1\, dt + C\right]$$

$$= \frac{1}{40+t}\left(40t + \frac{t^2}{2}\right) + \frac{C}{40+t}$$

Using I.C : $\dfrac{1}{10} = Q(0) = \dfrac{C}{40} \Rightarrow \boxed{C = 4}$.

$$\therefore \quad Q(t) = \frac{40t + \frac{t^2}{2} + 4}{40+t}$$

(3) The tank will overflow at $t = 60$

$$\Rightarrow \quad Q(60) = \frac{40(60) + \frac{(60)^2}{2} + 4}{40+60} = 42.04.$$

Compound Interest:

Suppose that a sum of money is deposited in a bank that pays interest at an annual rate r.

Let $S(t)$ be the value of investment at any time t, (s.t) $S(t)$ depends on the frequency with which interest is compounded as well as on the interest.

If we assume that compounding takes place continuously, then the IVP the describes the growth of the investment is given by:

$$\begin{cases} \dfrac{dS}{dt} = rS, \\ S(0) = S_0 \end{cases}$$

"Interest · Current value of Invest."

Then solving the IVP: $S(t) = S_0 e^{rt}$.

Now, suppose that there maybe deposits or withdrawals in addition of the accrual of interest, then IVP

$$\frac{dS}{dt} = rS + k \quad , S(0) = S_0 \quad ---$$

$k > 0$ deposits.

$k < 0$ withdrawals.

Recall: Newton's law of Heating & Cooling:

States that the temprature of an object changes at a rate proportional to the difference between the temprature of the object itself and the temprature of its surroundings. (Ambient air temprature in most cases).

Let $T(t)$ be the temprature of the object at any time t.

T_R : temprature of the surrounding.

Then $$\frac{dT}{dt} = \overset{\text{Cooling}}{-}C\left(T - T_R\right), \quad C > 0$$

Example: Suppose that the temprature of a cup of coffee obeys Newton's Law of Cooling. If the coffee has a temprature of 90°C when freshly poured, and 1 min Later has Cooled to 85°C in a room at 20°C, determine when the coffee reaches a tempreture of 65 °c.

Sol: Let $T(t)$ be the tempreture of the Cup of coffee. Then the IVP is given by:

$$\frac{dT}{dt} = -c(T - T_R) \quad , \quad T(0) = 90\,°c$$

Given that $T_R = 20\,°c$ and $T(1) = 85$

Need to find t such that $T(t) = 65\,°c$?

$$\int \frac{dT}{(T-20)} = \int -c \, dt$$ (seperable)

$$\Rightarrow \ln|T-20| = -ct + k$$

$$\Rightarrow T - 20 = \pm e^{k} \cdot e^{-ct}$$

$$\Rightarrow T(t) = 20 + A e^{-ct}$$, where $A = \pm e^{k}$

Now Using I.C: $T(0) = 90 \Rightarrow 90 = 20 + A \Rightarrow \boxed{A = 70}$

$T(1) = 85 \Rightarrow 85 = 20 + 70 e^{-c} \Rightarrow 65 = 70 e^{-c}$

$$\Rightarrow e^{-c} = \frac{65}{70} \Rightarrow C = -\ln\left(\frac{65}{70}\right).$$

$$\Rightarrow \quad T(t) = 20 + 70\, e^{+\left(\ln\left(\frac{65}{70}\right)\right)t}$$

$t \ln\left(\frac{65}{70}\right)$

$$= 20 + 70\left(\frac{65}{70}\right)^t.$$

Now, Need t such that $T(t) = 65$.

$$\Rightarrow \quad 65 = 20 + 70\left(\frac{65}{70}\right)^t$$

$$\frac{45}{70} = \left(\frac{65}{70}\right)^t$$

$$\ln\left(\frac{45}{70}\right) = t \ln\left(\frac{65}{70}\right)$$

$$\Rightarrow \quad t = \frac{\ln\left(\frac{45}{70}\right)}{\ln\left(\frac{65}{70}\right)} \approx 5.96 \text{ min.}$$

Home work # 5.

1) (Q 4) page 60

2) (Q 9) page 61

(Q1) page (60). Consider a tank used in certain hydrodynamic experiments. After one experiment the tank contains 200 L of a dye solution with a concentration of 1 g/L. To prepare for the next experiment, the tank is to be rinsed with fresh water flowing in at a rate of 2 L/min, the well-stirred solution flowing out at the same rate. Find the time that will elapse before the concentration of dye in the tank reaches 1 % of its original value.

Sol: Let $Q(t)$ be the amount of dye solution at time t.

The Initial amount : $Q(0) = \text{Volume} \times \text{Concentration}$

$$= 200 \times 1$$

$$= 200.$$

$$\Rightarrow \text{IVP}: \begin{cases} \dfrac{dQ}{dt} = \text{rate in} - \text{rate out} \\ Q(0) = 200. \end{cases}$$

$$\Rightarrow \frac{dQ}{dt} = \underset{\text{fresh water}}{0} \times 2 - \frac{Q(t)}{200} \times 2$$

$$\Rightarrow \frac{dQ}{dt} = -\frac{Q(t)}{100} \quad \text{with} \quad Q(0) = 200.$$

$$\Rightarrow \frac{dQ}{Q} = -\frac{1}{100}\,dt.$$

$$\Rightarrow \ln|Q(t)| = -\frac{1}{100}t + C$$

$$\Rightarrow Q(t) = \underbrace{\pm e^{C}}_{A} \cdot e^{-\frac{1}{100}t} = A e^{-\frac{1}{100}t}$$

Using I.C : $Q(0) = 200 \Rightarrow \boxed{200 = A}$

$$\Rightarrow Q(t) = 200\, e^{-\frac{1}{100}t}$$

Now: 1 % of 200 = 2 of dye solution.

$$\Rightarrow 2 = 200\, e^{-\frac{1}{100}t} \Rightarrow \frac{2}{200} = e^{-\frac{1}{100}t}$$

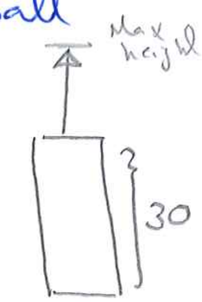
$$\Rightarrow \ln\left(\frac{1}{100}\right) = -\frac{1}{100}t \Rightarrow t = \frac{\ln(1/100)}{-1/100} \approx 460.5 \text{ min}$$

(Q20) page 64. A ball with mass 0.15 Kg is thrown upward with initial velocity 20 m/s from the roof of a building 30 m high. Neglect air resistance.

(a) Find the maximum height above the ground that the ball reaches.

(b) Assuming that the ball misses the building on the way down, find the time that it hits the ground.

Sol: (a). Let $y(t)$ be the height of the ball above the ground at time t.



Then Net force $= m\,a = -mg + 0$

Air resistence

$$\Rightarrow \quad m \cdot \frac{d^2 y}{dt^2} = -mg$$

$$\Rightarrow \quad \int d^2 y = \int -g \, dt$$

$$\Rightarrow \quad y'(t) = -gt + C$$

$$\Rightarrow \quad y(t) = -g\frac{t^2}{2} + C\,t + D.$$

We know that $y'(0) = 20$ (Initial velocity)

then $\boxed{20 = C}$

Also, we know that $y(0) = 30$ (starting point).

then $30 = D.$ Therefore,

$$y(t) = -g\frac{t^2}{2} + 20t + 30.$$

The Maximum height occured when $v = y'(t) = 0$

$$\Rightarrow \quad -gt + 20 = 0 \Rightarrow t = \frac{20}{g} = \frac{20}{9.8} \approx 2.04\text{ s}$$

Therefore, $y(2.04) \approx 50.4$ m.

(b) When the ball hits the ground, we have $y(t) = 0$.

$$\Rightarrow \quad -\frac{9.8}{2}t^2 + 20t + 30 = 0 \quad \text{(quadratic eq.)}$$

$$t_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-20 \pm \sqrt{(20)^2 - 4(-\frac{9.8}{2})(30)}}{2(-\frac{9.8}{2})}$$

$\Rightarrow$ $t_1 = 5.25$ second , $t_2 = \cancel{\ }-1.12$ (Ignore)

2.4. Differences Between Linear and Nonlinear Equations

Recall that the first order Ordinary D.E has the general form:

$$\frac{dy}{dt} = f(t, y) \quad \dots (1)$$

If f is linear in y, then eq.(1) is linear D.E.

If f is Nonlinear in y, then eq.(1) is Nonlinear D.E.

Question: Does every IVP have exactly one solution? (Existence and Uniqueness).

For Linear equations, the answer is given by the following theorem.

Thm (2.4.1) Existence and Uniqueness of Solutions.

Consider the Linear D.E with Initial Condition

$$y' + p(t)\, y = g(t), \quad y(t_0) = y_0 \quad \dots (2), \text{ then}$$

If $p(t)$ and $g(t)$ are Continuous on an open Interval $I := (\alpha, \beta)$ containing t_0, then there exists a Unique function $y = \phi(t)$ that satisfies the IVP eq.(2).

Example: Without Solving, Find the largest interval in which the Solution exists? (If there exists).

1) $\begin{cases} t y' + 2y = 4t^2 \\ y(1) = 2. \end{cases}$

Rewriting the equation in the standard form:

$$y' + \frac{2}{t} y = 4t \quad , \text{ so}$$

$p(t) = \dfrac{2}{t}$ is Contineous on $(-\infty, 0) \cup (0, \infty)$

$g(t) = 4t$ is Continuous on $(-\infty, \infty)$.

$\Rightarrow$ $p(t)$ and $g(t)$ are continuous on $(-\infty, 0) \cup (0, \infty)$

The largest interval Containing $t_0 = 1$ is $(0, \infty)$.

Therefore the problem has a Unique Solution on $(0, \infty)$.

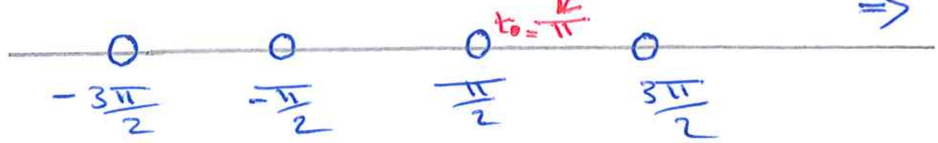
---

2) $y' + (\tan t) y = \sin t$, with $y(\pi) = 0$.

$p(t) = \tan t$ is Cont. on $(-\infty, \infty) \setminus \{\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \dots\}$

$g(t) = \sin t$ is Cont. on $(-\infty, \infty)$.

$\Rightarrow$ $p(t)$ and $g(t)$ are Cont. on $(-\infty, \infty) \setminus \{\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \dots\}$



$\Rightarrow$ Largest Interval is $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

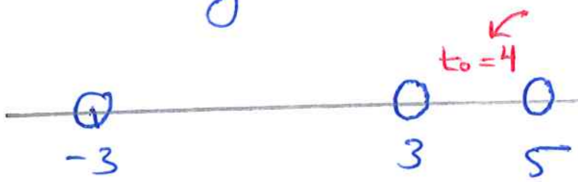
3) $(t^2-9)y' + 2y = \ln|20-4t|$, $y(4) = -3$.

$$\Rightarrow \quad y' + \frac{2}{t^2-9}y = \frac{\ln|20-4t|}{t^2-9}.$$

$p(t) = \dfrac{2}{t^2-9}$ is Contin. on $(-\infty,\infty)\setminus\{\pm 3\}$.

$g(t) = \dfrac{\ln|20-4t|}{t^2-9}$ is Cont. on $(-\infty,\infty)\setminus\{\pm 3, 5\}$.

There fore $p(t)$ and $g(t)$ are Cont. on $(-\infty,\infty)\setminus\{\pm 3, 5\}$.



$\Rightarrow$ Largest Interval is $(3,5)$.

Home work # 6:

Determine the Largest interval where the Solution exists? Does the Solution Unique?

1) $(\ln t)y' + y = \cot(t)$, $y(2) = 3$.

Ans. $(1, \pi)$

2) $y' + (\ln t)y = \cot(t)$, $y(2) = 3$

Ans. $(0, \pi)$.

Thm (2.4.2) Nonlinear Equations.

Consider the IVP $\left\{ \frac{dy}{dt} = f(t,y) \right.$, $y(t_0) = y_0$ ---(3)

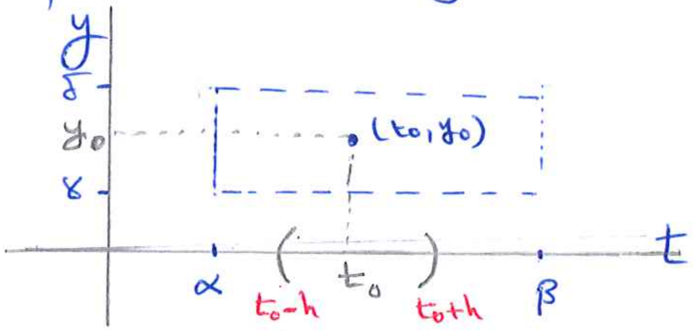
If $f(t,y)$ and $\frac{\partial f}{\partial y}$ are Continuous in some

rectangle $\alpha < t < \beta$, $\gamma < y < \delta$ containing

the point (t_0, y_0) , then in some interval

$t_0 - h < t < t_0 + h$ Contained in $\alpha < t < \beta$

there is a Unique Solution $y = \phi(t)$ of IVP (3).



Example : Does the IVP : $\frac{dy}{dt} = \sqrt{y - t^2}$, $y(0) = 1$

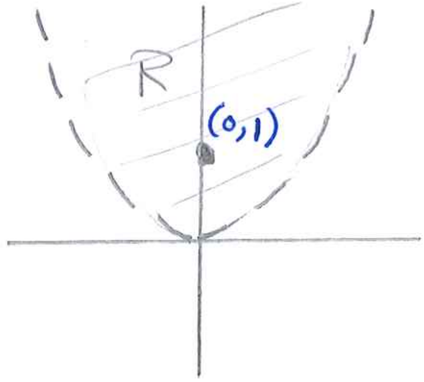
have a Unique Solution ?

Sol: $f(t,y) = \sqrt{y - t^2}$

$$\frac{\partial f}{\partial y} = \frac{1}{2\sqrt{y - t^2}}$$

f and $\frac{\partial f}{\partial y}$ are Contin. on the Region :

$$R = \{(t,y) : y - t^2 > 0\}$$
$$= \{(t,y) : y > t^2\}$$



Now, $(0,1) \in R$, Consequently, a rectangle can be drawn around $(0,1)$ in which f and $\frac{\partial f}{\partial y}$ are Cont.

Example: Determine whether (thm 2.4.2) gurantees that the IVP : $y' = y^{\frac{2}{3}}$, $y(0) = 0$ posses a Unique Solution?

Sol: $f(t,y) = y^{\frac{2}{3}}$

$$\frac{\partial f}{\partial y} = \frac{2}{3} y^{-\frac{1}{3}} = \frac{2}{3\sqrt[3]{y}}$$

$\Rightarrow$ f and $\frac{\partial f}{\partial y}$ are Cont. on $R = \{(t,y) : y \neq 0\}$

The Initial point $(0,0) \notin R$, Hence,

Thm 2.4.2 Failed.

In this case, we must Solve the IVP:

$$\frac{dy}{dt} = y^{\frac{2}{3}}$$

$$\Rightarrow \int y^{-\frac{2}{3}}\, dy = \int dt \quad , \; y \neq 0$$

$$\Rightarrow 3\, y^{\frac{1}{3}} = t + c$$

Using I.C : $y(0) = 0 \Rightarrow \boxed{c = 0}$

$\therefore \quad 3\, y^{\frac{1}{3}} = t \Rightarrow y = \left(\frac{t}{3}\right)^3 = \frac{t^3}{27}$.

And by inspection we can see that $y(t) = 0$ is another Solution.

$\Rightarrow$ The IVP has more than one Solution (Not Unique)

Note : We can ensure the Existence of the Solution by Continuity of f alone (Not Uniqueness)

Note : The interval of Validity for Nonlinear D.E can't be determine without solving the IVP, and y_0 has an Influence.

Example: Solve the given IVP, then determine how the interval in which the solution exists depends on the initial value y_0.

$$\begin{cases} \dfrac{dy}{dt} = y^2 \\ y(0) = y_0 \end{cases}$$

Sol: $\dfrac{dy}{dt} = y^2 \Rightarrow \displaystyle\int \frac{dy}{y^2} = \int dt \quad , y \neq 0.$

$\Rightarrow \quad -y^{-1} = t + c \Rightarrow y = \dfrac{-1}{t+c}$

Using I.C: $y(0) = y_0 \Rightarrow y_0 = \dfrac{-1}{c} \Rightarrow \boxed{c = \dfrac{-1}{y_0}}$

$\therefore \quad y(t) = \dfrac{-1}{t - \frac{1}{y_0}} = \dfrac{y_0}{1 - y_0 t}$

Note that the solution becomes unbounded as $t \to \frac{1}{y_0}$, Vertical Asym.
so the Interval of existence of the solution is

$$-\infty < t < \frac{1}{y_0} \quad , \text{if } y_0 > 0$$

and $\quad \dfrac{1}{y_0} < t < \infty \quad , \text{if } y_0 < 0$

Note: If $y(0) = y_0 = 0$, then $y(t) = 0$ is another Sol. with Interval of Validity $(-\infty, \infty)$.

(Q14) page 76. Solve the following IVP, then find the Interval of Validity:

$$y' = 2ty^2 \quad , \quad y(0) = y_0$$

Sol: $\displaystyle\int \frac{dy}{y^2} = \int 2t\, dt$

$$\Rightarrow \quad \frac{-1}{y} = t^2 + C \quad \Rightarrow \quad y(t) = \frac{-1}{t^2 + C}$$

Using I.C: $y(0) = y_0 \Rightarrow y_0 = \frac{-1}{C} \Rightarrow \boxed{C = \frac{-1}{y_0}}$

$$\Rightarrow \quad y(t) = \frac{y_0}{1 - y_0 t^2} \quad , \quad 1 - y_0 t^2 \neq 0$$

$$\frac{1}{y_0} \neq t^2 \qquad \frac{-1}{t^2 - \frac{1}{y_0}}$$

- If $y_0 > 0$, then $\frac{1}{y_0} > 0$, solutions exist as long as long as $t^2 < \frac{1}{y_0} \Leftrightarrow |t| < \frac{1}{\sqrt{y_0}}$

$$\Leftrightarrow \quad \frac{-1}{\sqrt{y_0}} < t < \frac{1}{\sqrt{y_0}}$$

- If $y_0 < 0$, then $1 - y_0 t^2$ is Never Zero, so the solutions are defined $\forall t \in \mathbb{R}$

- If $y_0 = 0$, then $y(t) = 0$ is a solution defined $\forall t \in \mathbb{R}$

(Q16) page 76. Solve the IVP, then find the Interval where the Solution exists:

$$y' = \frac{t^2}{y(1+t^3)}, \quad y(0) = y_0 \quad , \begin{matrix} t \neq -1 \\ y \neq 0 \end{matrix}$$

sol: $$\int y\,dy = \int \frac{t^2}{1+t^3}\,dt$$

$$\frac{y^2}{2} = \frac{1}{3}\ln|1+t^3| + C$$

Using I.C: $y(0) = y_0 \Rightarrow \boxed{\frac{y_0^2}{2} = C}$

$$\Rightarrow y(t) = \sqrt{\frac{2}{3}\ln|1+t^3| + y_0^2}$$

Now, the Solution exists for $\frac{2}{3}\ln|1+t^3| + y_0^2 \geq 0$

$$\Leftrightarrow \quad y_0^2 \geq -\frac{2}{3}\ln|1+t^3|$$

$$\Leftrightarrow \quad -\frac{3}{2}y_0^2 \leq \ln|1+t^3|$$

$$\Leftrightarrow \quad e^{-\frac{3}{2}y_0^2} - 1 \leq t^3$$

$$\Leftrightarrow \quad \sqrt[3]{e^{-\frac{3}{2}y_0^2} - 1} \cancel{\leq} t < \infty.$$ (open Interval)

(Q13) page 76. Solve the given IVP, then determine how the interval in which the solution exists depends on the initial value y_0.

$$y' = \frac{-4t}{y}, \quad y(0) = y_0, \quad \boxed{y \neq 0}$$

$$\Rightarrow \quad y\,dy = -4t\,dt \qquad \text{(seperable)}$$

$$\Rightarrow \quad \frac{y^2}{2} = -2t^2 + C.$$

$$\Rightarrow \quad y^2 = -4t^2 + A, \quad \text{where } A = 2C.$$

$$\Rightarrow \quad y = \pm\sqrt{-4t^2 + A}$$

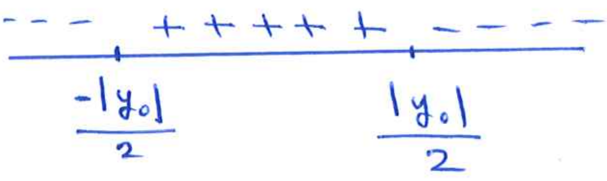
Using I.C : $y(0) = y_0 \Rightarrow \quad y_0 = \pm\sqrt{A} \quad \Rightarrow \boxed{A = y_0^2}$

$$\Rightarrow \quad y = \pm\sqrt{-4t^2 + y_0^2}.$$

Now, $y_0^2 - 4t^2 = (y_0 - 2t)(y_0 + 2t) \geq 0$

$y_0^2 \geq 4t^2 \Leftrightarrow \frac{y_0}{2} \geq |t|$

$\Rightarrow$ Interval of validity:



$$\left(\frac{-|y_0|}{2}, \frac{|y_0|}{2} \right).$$

Note: at $t = \frac{-|y_0|}{2}, \frac{|y_0|}{2}$, $y(t) = 0$, so we ignore them.

Bernoulli Equations

Sometimes it's possible to solve a Nonlinear equation by making a change of the dependent variable that converts it into a Linear equation.

Such as : $y' + p(t)\, y = q(t)\, y^n$. ...(I)

This equation is called Bernoulli equation.

Note that if $\boxed{n = 0} \Rightarrow$ eq. (I) : $y' + p(t) y = q(t)$

which is Linear in y.

(Q27) page 77. (a) Solve Bernoulli equation for $n = 1$.

(b) Show that if $n \neq 0, 1$, then the substitution $v = y^{1-n}$ reduces Bernoulli's equation to a Linear equation.

Sol: (a) If $n = 1 \Rightarrow$ eq. (I) becomes :

$$y' + p(t)\, y = q(t)\, y^1.$$

$$\Rightarrow \quad y' = \big(q(t) - p(t) \big)\, y^1$$ which is Seperable

$$\Rightarrow \int \frac{dy}{y} = \int (q(t) - p(t))\, dt$$

$$\Rightarrow \ln|y| = \int (q(t) - p(t))\, dt + C$$

$$\Rightarrow y = D\, e^{\int (q(t) - p(t))\, dt}, \quad D = \pm e^{C}.$$

(b) Let $v = y^{1-n}$, $n \neq 0, 1$

then $\frac{dv}{dt} = (1-n)\, y^{-n} \cdot \frac{dy}{dt}$

Nonlinear.

$\Rightarrow y' + p(t)\, y = q(t)\, y^{n}$, divide both sides

by $y^{n} \Rightarrow y^{-n} \frac{dy}{dt} + p(t) \frac{y}{y^{n}} = q(t)$

$$\Rightarrow \frac{v'}{(1-n)} + p(t)\, v = q(t)$$

$\Rightarrow v' + (1-n)\, p(t)\, v = q(t)(1-n)$, which is Linear.

So, we solve it as a 1st order Linear D.E for v, then substitute $v = y^{1-n}$.

(Q28) page 78. Solve the following D.E.

$$t^2 y' + 2ty - y^3 = 0, \quad t > 0.$$

Sol: $(t^2)y' + (2t)y = 1 \cdot y^3$

$\Rightarrow \quad y' + \frac{2}{t} y = \frac{1}{t^2} y^3$ ... (II) (Bernoulli Equation).

$\Rightarrow$ Let $V = y^{1-3} = y^{-2}$, then

$$V' = -2 y^{-3} y'.$$

Divide eq.(II) by y^3, then

$$\Rightarrow \quad \frac{y'}{y^3} + \frac{2}{t} \frac{y}{y^3} = \frac{1}{t^2}$$

$\Rightarrow \quad \frac{v'}{-2} + \frac{2}{t} v = \frac{1}{t^2}$, which is Linear.

Then: $\mu(t) = e^{\int \frac{-4}{t} dt} = e^{-4 \ln|t|} = e^{\ln|t|^{-4}} = \frac{1}{t^4}, \ t > 0$

$$v(t) = \frac{1}{\mu(t)} \left[\int \mu(t) g(t) \, dt + C \right]$$

$$= t^4 \left[\int \frac{1}{t^4} \cdot \left(\frac{-2}{t^2} \right) dt + C \right]$$

$$\Longrightarrow \quad v(t) = t^4 \left[\int -2t^{-6}\, dt + C \right]$$

$$= t^4 \left[\frac{2}{5} t^{-5} + C \right]$$

$$v(t) = \frac{2}{5t} + C\, t^4 .$$

Now, $v(t) = y^{-2} = \dfrac{2}{5t} + \dfrac{C\, t^4}{1}$

$$\Longrightarrow \quad y = \frac{1}{\sqrt{\dfrac{2}{5t} + C t^4}} \; .$$

Home Work #7 : Solve the following D.E's

(1) $(t^2+1)\dfrac{dy}{dt} = 4ty + 4t\sqrt{y}$. (Bernulli , $n = \frac{1}{2}$)

(2) $\dfrac{dy}{dx} + \dfrac{2y}{6x+1} = \dfrac{-3x^2}{(6x+1)\,y^2}$, (Bernulli , $n = -2$)

(3) $\dfrac{dy}{dx} = \dfrac{3y^2 - x^2}{2xy}$, $y(1) = 2$.

(Bernulli , $n = -1$)
and also homogeneous.

2.6 Exact Equations and Integrating Factors.

Consider the D.E $2x + y^2 + 2xy \frac{dy}{dx} = 0$.

This equation is neither linear nor seperable.

Thm 2.6.1. Consider a D.E. of the form

$$M(x,y)\,dx + N(x,y)\,dy = 0 \quad \ldots (1)$$

where M, N, M_y, N_x are all continuous on the Region $R: \alpha < x < \beta,\ \gamma < y < \delta$. Then Eq. (1) is an Exact equation in R iff $M_y = N_x$.

That is, there exists a function f satisfies

$$f_x = M \text{ and } f_y = N \quad \text{iff} \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Note: When $M(x,y)\,dx + N(x,y)\,dy = 0$ is Exact then:

$$f_x\,dx + f_y\,dy = 0$$

$$\Rightarrow f_x + f_y \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{d}{dx} f(x,y) = 0 \quad \text{(chain Rule)}$$

$$\Rightarrow f(x,y) = C \quad \text{is the Solution.}$$

Example (1). Solve $2x+y^2+2xy\frac{dy}{dx}=0$

Sol: $(2x+y^2)\,dx + 2xy\,dy = 0$. ...(*)

Here $M(x,y)=(2x+y^2)$ and $N(x,y)=(2xy)$.

Note that $\frac{\partial M}{\partial y} = 2y = \frac{\partial N}{\partial x}$, So the equation (*) is Exact. By thm 2.6.1, $\exists$ a function $f(x,y)$ such that

$$f_x = M(x,y) = 2x+y^2 \quad \text{...(1)}$$

$$f_y = N(x,y) = 2xy \quad \text{...(2)}$$

Integrating (1) with respect to x, we have

$$f(x,y) = x^2 + y^2x + h(y) \quad \text{...(3)}$$

Now from eq.(3) : $f_y(x,y) \overset{\text{Eq.(3)}}{=} 2xy + h'(y) \overset{\text{Eq.(2)}}{=} 2xy$

$\Rightarrow h'(y) = 0 \Rightarrow \boxed{h(y) = C_1}$ ...(4)

substitute (4) into (3) we get:

$$f(x,y) = x^2 + y^2x + C_1 = C_2 \quad \text{(Solution)}$$

Hence the Solution is given Implicitly:

$$x^2 + xy^2 = C$$

Example (2) : Verify that the D.E is Exact, then Solve it.

$$\left(\frac{y}{1+x^2} - \frac{e^y}{x}\right)dx = \left(e^y \ln x - \tan^{-1} x + 2\right)dy$$

Sol:
$$\Big(\underbrace{\frac{e^y}{x} - \frac{y}{1+x^2}}_{M(x,y)}\Big)dx + \Big(\underbrace{e^y \ln x - \tan^{-1} x + 2}_{N(x,y)}\Big)dy = 0.$$

$$M_y = \frac{\partial M}{\partial y} = \frac{e^y}{x} - \frac{1}{1+x^2} = \frac{\partial N}{\partial x} = N_x.$$

$\Rightarrow$ The equation is Exact. Thus, $\exists$ function $f(x,y)$

such that : $f_x = M(x,y) = \dfrac{e^y}{x} - \dfrac{y}{1+x^2}$ $\cdots$ ①

$f_y = N(x,y) = e^y \ln x - \tan^{-1} x + 2$ $\cdots$ ②

Integrating ② with respect to y :

$$f(x,y) = e^y \ln x - y \tan^{-1} x + 2y + h(x) \quad \cdots ③$$

$$f_x(x,y) \overset{Eq.(3)}{=} \frac{e^y}{\ln x} - \frac{y}{1+x^2} + h'(x) \overset{Eq.(1)}{=} \frac{e^y}{x} - \frac{y}{1+x^2}$$

$\Rightarrow h'(x) = 0 \Rightarrow h(x) = C_1$

$\Rightarrow f(x,y) = e^y \ln x - y \tan^{-1} x + 2y + C_1 = C_2$

$\Rightarrow$ $$e^y \ln x - y \tan^{-1} x + 2y = C$$

Example (3): Solve the following D.E.

$$(y\cos x + 2x e^{y}) + (\sin x + x^2 e^{y} - 1)y' = 0.$$

Sol: $\underbrace{(y\cos x + 2x e^{y})}_{M(x,y)}dx + \underbrace{(\sin x + x^2 e^{y} - 1)}_{N(x,y)}dy = 0$

$$M_y = \frac{\partial M}{\partial y} = \cos x + 2x e^{y} = \frac{\partial N}{\partial x} = N_x$$

$\therefore$ The equation is Exact, then $\exists$ a function $f(x,y)$ such that

$$f_x = y\cos x + 2x e^{y} = M(x,y) \quad \ldots (1)$$

$$f_y = \sin x + x^2 e^{y} - 1 = N(x,y) \quad \ldots (2)$$

Integrate (2) with respect to y:

$$f(x,y) = (\sin x)y + x^2 e^{y} - y + h(x) \quad \ldots (3)$$

Now: $f_x(x,y) \overset{Eq.(3)}{=} (\cos x)y + 2x e^{y} + h'(x) \overset{Eq(1)}{=} M(x,y)$

$\Rightarrow \quad h'(x) = 0 \quad \Rightarrow \quad \boxed{h(x) = C_1}$

$\Rightarrow \quad f(x,y) = y(\sin x) + x^2 e^{y} - y + C_1 = C_2$

$\Rightarrow \quad y(\sin x) + x^2 e^{y} - y = C$

Note: If the equation is Not Exact, we will find an Integrating factor which makes the given differential equation Exact.

Integrating Factor theorem:

Consider the D.E: $M(x,y)\,dx + N(x,y)\,dy = 0$.
which is Not exact, then one of the following could be an Integrating factor that makes the D.E Exact.

1) If $\dfrac{M_y - N_x}{N} = h(x)$, then $e^{\int h(x)\,dx}$ is the Integrating factor.

2) If $\dfrac{N_x - M_y}{M} = k(y)$, then $e^{\int k(y)\,dy}$ is the Integrating factor.

3) If $\dfrac{N_x - M_y}{x \cdot M - y \cdot N} = R(v)$, where $v = xy$

then $e^{\int R(v)\,dv}$ is the Integrating factor.

Example (4): Show that the following D.E

$$\left(3x^2y - 8x\right)y' = 4y - 2xy^2 \quad , \text{ is Not exact}$$

then find an appropriate integrating factor which can be Used to make it exact, then Solve it.

Sol: $$\underbrace{\left(3x^2y - 8x\right)}_{N(x,y)}dy + \underbrace{\left(2xy^2 - 4y\right)}_{M(x,y)}dx = 0 \quad \ldots(*)$$

$$M_y = 4xy - 4 \neq , \quad N_x = 6xy - 8$$

1) $$\frac{M_y - N_x}{N} = \frac{4xy - 4 - (6xy - 8)}{(3x^2y - 8x)} = \frac{4 - 2xy}{3x^2y - 8x}$$

which is $\neq h(x)$ & $\neq k(y)$.

2) $$\frac{N_x - M_y}{M} = \frac{(6xy - 8) - (4xy - 4)}{(2xy^2 - 4y)} = \frac{-4 + 2xy}{2xy^2 - 4y}$$

$$= \frac{+2(xy - 2)}{2y(xy - 2)} = \frac{1}{y} = k(y).$$

$\Rightarrow$ Integrating factor is $e^{\int \frac{1}{y}\, dy} = e^{\ln|y|} = \boxed{y}\ , y>0$

Multiply both sides of eq. (*) by y:

$$\underbrace{\left(3x^2y^2 - 8xy\right)}_{N(x,y)} dy + \underbrace{\left(2xy^3 - 4y^2\right)}_{M(x,y)} dx = 0 \quad \cdots (**)$$

Note that: $M_y = 6xy^2 - 8y = N_x$ (Exact).

$\therefore\ \exists$ a function $f(x,y)$ such that

$$f_x = 2xy^3 - 4y^2 \quad \cdots (1)$$

$$f_y = 3x^2y^2 - 8xy \quad \cdots (2)$$

Integrating eq. (2) with respect to y:

$$f(x,y) = x^2y^3 - 4xy^2 + h(x) \quad \cdots (3)$$

Diff. Eq. (3)

$$f_x = 2xy^3 - 4y^2 + h'(x) = 2xy^3 - 4y^2 = M(x,y)$$

$$\Rightarrow h'(x) = 0 \Rightarrow h(x) = C_1$$

Thus: $f(x,y) = x^2y^3 - 4xy^2 + C_1 = C_2$

$$\Rightarrow x^2y^3 - 4xy^2 = C$$

Example (5): Solve $\underbrace{(x+2)\sin y}_{M(x,y)}\,dx + \underbrace{(x\cos y)}_{N(x,y)}\,dy = 0$

Sol: $M_y = (x+2)\cos y \neq,\ N_x = \cos y$ (Not Exact)

1) $$\frac{M_y - N_x}{N} = \frac{(x+2)\cos y - \cos y}{x\cos y} = \frac{\cancel{\cos y}\,(x+2-1)}{x\,\cancel{\cos y}}$$

$$= \frac{x+1}{x} = 1 + \frac{1}{x} = h(x)$$

$\Rightarrow$ Integrating factor : $e^{\int 1+\frac{1}{x}\,dx} = e^{x+\ln|x|}$

$$= x e^{x}, \quad x>0$$

Multiply both Sides of the D.E by xe^x, we get.

$$x(x+2)\sin y\, e^{x}\,dx + \underbrace{x^2(\cos y)\,e^{x}}_{\text{ضرب اقترانات}}\,dy = 0$$

Now $M_y = x(x+2)e^{x}\cos y = N_x$

$\therefore\ \exists$ a function $f(x,y)$ such that

$$f_x(x,y) = x(x+2)e^{x}\sin y \quad \text{--- (1)}$$

$$f_y(x,y) = x^2 e^{x}\cos y \quad \text{... (2)}$$

Integrating eq.(2) with respect to y:

$$f(x,y) = x^2 e^x \sin y + h(x) \qquad \cdots (3)$$

$$\Rightarrow f_x \overset{\text{Diff. Eq (3)}}{=} x^2 e^x \sin y + 2x e^x \sin y + h'(x)$$

$$\overset{\text{Eq (1)}}{=} x^2 e^x \sin y + 2x e^x \sin y .$$

$$\Rightarrow h'(x) = 0 \Rightarrow h(x) = C_1$$

$$\Rightarrow x^2 e^x \sin y + C_1 = C_2$$

$$\Rightarrow x^2 e^x \sin y = C.$$

Home Work # 8:

1) Solve the following IVP:

$$x y^3 + (x^2 y^2 + 1) y' = 0 \quad , y(2) = 1 , x > 0, y > 0$$

(Non exact $\Rightarrow$ Exact , then Bernulli with $n = -1$).

2) $$y' = \frac{3y^2 - x^2}{2xy} \quad , y(1) = 2$$

(Non exact $\Rightarrow$ Exact , then Homogeneous & Bernulli with $n=-1$)

3) $$2x^2 + y + (x^2 y - x) y' = 0 \; , \; y(1) = 1.$$

(Q10) page 101. Solve the following:

$$\left(\frac{y}{x} + 6x\right) + (\ln x - 2)y' = 0 \quad , \quad x > 0.$$

sol: $$\underbrace{\left(\frac{y}{x} + 6x\right)}_{M(x,y)} dx + \underbrace{(\ln x - 2)}_{N(x,y)} dy = 0$$

$$M_y = \frac{1}{x} = N_x \Rightarrow \text{Exact} , \exists f(x,y) \ (s.t)$$

$$f_x(x,y) = \frac{y}{x} + 6x \qquad \text{--- (1)}$$

$$f_y = \ln x - 2 \qquad \text{--- (2)}$$

Integrate (1) w.r.t x :

$$f(x,y) = y \ln x + 3x^2 + h(y) \quad \text{---(3)}$$

Now $$f_y \overset{\text{Diff. Eq (3)}}{=} \ln x + h'(y) \overset{\text{Eq.(2)}}{=} \ln x - 2$$

$$\Rightarrow h'(y) = -2 \Rightarrow h(y) = -2y.$$

$$\Rightarrow f(x,y) = y \ln x + 3x^2 - 2y = C.$$

(Q15) page 101. Find b that makes the eq. Exact

$$(xy^2 + bx^2y) + (x+y)x^2y' = 0$$

$$(xy^2 + bx^2y)\,dx + \overbrace{(x+y)x^2}^{x^3 + x^2y}\,dy = 0$$

$$M_y = 2xy + bx^2 = 3x^2 + 2xy \Rightarrow \boxed{b = 3}$$

(Q23) page 101 : show that if $\frac{(N_x - M_y)}{M} = Q$

where Q is a function of y only, then the D.E

$$M + N y' = 0$$

has an Integrating factor of the form:

$$\mu(y) = \exp \int Q(y)\, dy.$$

Proof: Suppose that $M + N y' = 0$ is Not Exact.

and Consider $\underbrace{\mu(y) M}_{\tilde{M}(x,y)} dx + \underbrace{\mu(y) N}_{\tilde{N}(x,y)} dy = 0$ is Exact

then $\left.\begin{aligned} \tilde{M}_y &= \mu(y) M_y + \mu'(y) M \\ \tilde{N}_x &= \mu(y) N_x + N \cdot 0 \end{aligned}\right\}$ $\tilde{M}_y = \tilde{N}_x$ (Exact)

$$\Rightarrow 0 = \tilde{N}_x - \tilde{M}_y = \mu(y) [N_x - M_y] - \mu'(y) \cdot M$$

$$\Rightarrow 0 = \mu(y) [N_x - M_y] - \mu'(y) M$$

$$\Rightarrow \frac{\mu'(y)}{\mu(y)} = \frac{[N_x - M_y]}{M} := Q(y)$$

$$\Rightarrow \ln|\mu(y)| = \int Q(y)\, dy$$

$$\Rightarrow \mu(y) = \exp \int Q(y) dy.$$

2.8 The Existence and Uniqueness Theorem.

Theorem 2.8.1. Consider the IVP

$$\frac{dy}{dt} = f(t,y) \quad , y(0) = 0 \qquad \ldots (1)$$

If f and $\frac{\partial f}{\partial y}$ are Continuous in a rectangle

$R: |t| \leq a \; , \; |y| \leq b$, then there is some

Interval $|t| \leq h \leq a$ in which there exists

a Unique solution $y = \phi(t)$ of the IVP (1).

Note: Theorem 2.8.1 differs from theorem 2.4.2

only in the Initial Condition.

In Thm 2.4.2. we have $y(t_0) = y_0$

Note: We can transform any IVP with $y(t_0) = y_0$

to an equivalent starting from $y(0) = 0$.

Example (1): Consider the IVP :

$$y' - y^3 = t^3 \quad , \quad y(2) = 5 .$$

To transform the I.C to $y(0) = 0$:

Let $s = t - 2 \Rightarrow t = s + 2$

$z = y - 5 \Rightarrow y = z + 5$

& $z' = y'$

So the Equivalent IVP is :

$$z' - (z+5)^3 = (s+2)^3 \quad , \quad z(0) = 0 .$$

Method of Successive approximation (Picard's Iteration Method)

To use this method, we generate a sequence of functions $\{\phi_n(t)\}$ where $\phi_n(t)$ satisfy the following integral equation

$$y = \phi(t) = \int_0^t f(s, \phi(s))\, ds . \quad \dots (2)$$

This equation sometimes is hard to be solved because of its Implicit Form.

We know from thm 2.4.2 that such $\phi(t)$ is exist.

The Question is How to find such $\phi(t)$?

Step 1 : Let $\phi(0) = 0 = \phi_0(t)$ (I.C. $y(0) = 0$)

Step 2 : Set $\phi_{n+1}(t) = \int_0^t f(s, \phi_n(s))\, ds$.

(i.e)

$$\phi_1(t) = \int_0^t f(s, \phi_0(s))\, ds$$

$$\phi_2(t) = \int_0^t f(s, \phi_1(s))\, ds$$

$$\vdots$$

$$\phi_{n+1}(t) = \int_0^t f(s, \phi_n(s))\, ds.$$

Taking the limit for both sides as $n \longrightarrow \infty$.

$$\phi(t) = \lim_{n\to\infty} \phi_{n+1}(t) = \lim_{n\to\infty} \int_0^t f(s, \phi_n(s))\, ds.$$

So $y(t) = \phi(t)$ will be the Solution of the IVP (1). This Iteration is called Picard Iteration.

Example (2) : Solve the following IVP Using Picard Itera

$$y' = 2t(1+y) \quad , \quad y(0) = 0 .$$

Sol: $f(t,y) = 2t(1+y)$. with $\phi_0(t) = 0$.

$$\Rightarrow \phi_1(t) = \int_0^t f(s,0)\,ds = \int_0^t 2s(1+0)\,ds = t^2 .$$

$$\phi_2(t) = \int_0^t f(s, s^2)\,ds = \int_0^t 2s(1+s^2)\,ds = t^2 + \frac{t^4}{2} .$$

$$\phi_3(t) = \int_0^t f\left(s, s^2 + \frac{s^4}{2}\right)ds = \int_0^t 2s\left(1+s^2+\frac{s^4}{2}\right)ds =$$

$$= t^2 + \frac{1}{2}t^4 + \frac{1}{6}t^6 .$$

$\vdots$

$$\phi_n(t) = t^2 + \frac{1}{2}t^4 + \frac{1}{6}t^6 + \cdots + \frac{t^{2n}}{n!}$$

$$= \frac{(t^2)^1}{1!} + \frac{(t^2)^2}{2!} + \frac{(t^2)^3}{3!} + \cdots + \frac{(t^2)^n}{n!}$$

$$\Rightarrow \phi_n(t) = \sum_{k=1}^{n} \frac{(t^2)^k}{k!} \quad \ldots (I)$$

Note that $\lim_{n\to\infty} \phi_n(t)$ converges iff $\sum_{k=1}^{\infty} \frac{(t^2)^k}{k!}$ converges. So we use the Ratio test to check convergence.

$$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \left| \frac{(t^2)^{k+1}}{(k+1)!} \cdot \frac{(k!)}{(t^2)^k} \right| = \lim_{k\to\infty} \left| \frac{t^2}{k+1} \right|$$

$$= t^2 \lim_{k\to\infty} \frac{1}{k+1} = 0 < 1, \quad \forall t.$$

Thus the series $\sum_{k=1}^{\infty} \frac{(t^2)^k}{k!}$ converges, $\forall t$.

and $y(t) = \lim_{n\to\infty} \phi_n(t) = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{(t^2)^k}{k!} = \sum_{k=1}^{\infty} \frac{(t^2)^k}{k!}$

$$y(t) = e^{t^2} - 1$$

Recall: $e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots$

$$\Rightarrow e^{t^2} = 1 + (t^2) + \frac{(t^2)^2}{2!} + \frac{(t^2)^3}{3!} + \cdots$$

$$\Rightarrow e^{t^2} - 1 = t^2 + \frac{(t^2)^2}{2!} + \frac{(t^2)^3}{3!} + \cdots = \sum_{k=1}^{\infty} \frac{(t^2)^k}{k!}$$

(Q1) page 120. Transform the IVP into Equivalent with $y(0)=0$:

$$y' = t^2 + y^2 \quad , \quad y(1) = 2$$

Let $s = t-1 \Rightarrow t = s+1$

$z = y-2 \Rightarrow y = z+2$ & $z' = y'$

$\Rightarrow \quad z' = (s+1)^2 + (z+2)^2 \quad , \quad z(0) = 0 .$

(Q6) page 120. Solve the IVP Using Picard Iteration

$$y' = y+1-t \quad , \quad y(0) = 0$$

Let $\phi_0(t) = 0$, then:

$$\phi_1(t) = \int_0^t f(s,0)\,ds = \int_0^t (0+1-s)\,ds = t - \frac{t^2}{2} .$$

$$\phi_2(t) = \int_0^t f\left(s, s-\frac{s^2}{2}\right) ds = \int_0^t \left(s - \frac{s^2}{2} + 1 - s\right) ds = t - \frac{t^3}{6}$$

$$\phi_3(t) = \int_0^t f\left(s, s-\frac{s^3}{6}\right) ds = \int_0^t \left(s - \frac{s^3}{6} + 1 - s\right) ds = t - \frac{t^4}{24}$$

$\vdots$

$$\phi_n(t) = t - \frac{t^{(n+1)}}{(n+1)!} = \left[t - \frac{t^2}{2!}\right] + \left[\frac{t^2}{2!} - \frac{t^3}{3!}\right] + \dots + \left[\frac{t^{n+1}}{(n+1)!} - \frac{t^n}{n!}\right]$$

$$\Rightarrow y(t) = \lim_{n\to\infty} \phi_n(t) = t + 0 + 0 \dots = \boxed{t} .$$

Miscellaneous Problems. Page 133.

Second order Equations (some special Cases):

Consider $y'' = f(t, y, y')$ ... (1)

be a second order D.E. Such an Equation Usually Can't be solved by Methods designed for 1st order Equations.

But there are two types of eq. (1) that can be transformed into 1st order equations by suitable change of variables.

Type 1: Equations with dependent Variable (y) Missing.

Assume $y'' = f(t, y')$, then:

Let $v = y' \Rightarrow v' = y''$

so $v' = f(t, v)$ which is first order differential equation, so we solve it for v then for y. (Q 36-41) page 135

(Q36) page 135. Solve:

$$t^2 y'' + 2t y' - 1 = 0 \quad , t > 0.$$

Sol: Let $v = y' \Rightarrow v' = y''$

$\Rightarrow \quad t^2 v' + 2t v = 1$

$\Rightarrow \quad v' + \frac{2}{t} v = \frac{1}{t^2} \quad , t > 0 \quad$, Linear in v

$\Rightarrow \quad \mu(t) = e^{\int \frac{2}{t} dt} = e^{2\ln(t)} = t^2. \quad , t > 0$

$\Rightarrow \quad v(t) = \frac{1}{t^2}\left[\int t^2 \cdot \frac{1}{t^2} dt + C\right]$

$y' = v(t) = \frac{1}{t^2}\left[\int 1 dt + C\right] = \frac{1}{t} + \frac{C}{t^2}$

Therefore, $\int dy = \int \left(\frac{1}{t} + C t^{-2}\right) dt$

$\Rightarrow \quad y(t) = \ln t + \frac{C t^{-1}}{-1} + D$

$y(t) = \ln t + \frac{k}{t} + D \quad$, where $k = -C$

(Q51) page 136. Solve the following IVP:

$$y' y'' - t = 0 \quad , \; y(1) = 2 \quad , \; y'(1) = 1$$

Sol: Let $v = y' \Rightarrow v' = y''$

$\Rightarrow \quad v v' - t = 0 \quad \Rightarrow \int v \, dv = \int t \, dt$

$$\frac{v^2}{2} = \frac{t^2}{2} + C$$

Using I.C : $y'(1) = 1 \Rightarrow v(1) = 1 \Rightarrow \frac{1}{2} = \frac{1}{2} + C \Rightarrow \boxed{C = 0}$

$\Rightarrow \quad \frac{v^2}{2} = \frac{t^2}{2} \quad \Rightarrow \quad v = t \quad \Rightarrow \quad y' = t$

$\Rightarrow \quad \int dy = \int t \, dt$

$$y = \frac{t^2}{2} + C$$

Using I.C: $y(1) = 2 \Rightarrow 2 = \frac{1}{2} + C \Rightarrow \boxed{C = 1.5}$

$\therefore \quad y(t) = \frac{t^2}{2} + 1.5.$

Type (II): Equations with the Independent Variable (t) Missing:

Assume $y'' = f(y, y')$. ... (*)

Let $v = y' \Rightarrow v' = y''$, then Eq. (*) becomes

$$\boxed{v' = f(y, v)} \Leftrightarrow \frac{dv}{dt} = f(y(t), v(t))$$

Note that we have two dependent functions $y(t), v(t)$.

We will use chain Rule:

$$y'' = \frac{dv}{dt} = \frac{dv}{dy} \cdot \frac{dy}{dt} = \frac{dv}{dy} \cdot v .$$

Therefore, Eq. (*) becomes:

$$v \cdot \frac{dv}{dy} = f(y, v)$$ which is

1 order Non Linear Differential Equation.

(Q42) page 136. $yy'' + (y')^2 = 0$.

Let $v = y' \Rightarrow v' = y''$

& $y'' = \frac{dv}{dt} = v.\frac{dv}{dy}$

$\Rightarrow y\left(v.\frac{dv}{dy}\right) + v^2 = 0$ (seperable).

$\Rightarrow \int \frac{1}{v}\,dv = \int -\frac{1}{y}\,dy$, $v \neq 0$

$\Rightarrow \ln|v| = -\ln|y| + C$

$\Rightarrow v = \underbrace{\pm e^{C}}_{A} \cdot e^{-\ln|y|} = A \cdot y^{-1}$

Now $y' = v = A \cdot y^{-1}$

$\Rightarrow y\,dy = A\,dt$

$\Rightarrow \frac{y^2}{2} = At + C$

Note: If $v = 0$, then $y' = 0 \Rightarrow y = \text{Constant}$

$\Rightarrow y = C$ satisfies the equation Also (i.e Solution).

"End of chapter 2"