Chapter (4): Fourier Transform

Fourier Transform (FT): mathematical transformation employed to ~~transfer~~ transfer signal between time domain and frequency domain.

FS $\longrightarrow$ for periodic signals

FT $\longrightarrow$ for periodic and non periodic signals

$$X(f) = F(x(t)) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$$

$$x(t) = F^{-1}(X(f)) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi f t}\, df$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

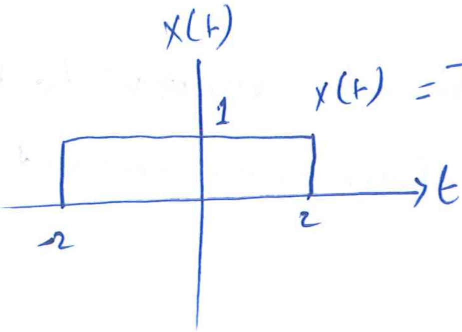
in general $X(f)$ is a complex, so it can be written

as $X(f) = |X(f)| \angle \theta_{X(f)} = |X(f)|\, e^{j\theta_{X(f)}}$

where $|X(f)| = |X(-f)|$

$$\theta_{X(f)} = -\theta_{X(-f)}$$

example 8 - for the following signal, find $X(f)$



$x(t) = \Pi(t/4)$

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$$

$$= \int_{-2}^{2} (1)\, e^{-j2\pi f t}\, dt = \frac{-1}{j2\pi f}\, e^{-j2\pi f t}\Big|_{-2}^{2}$$

$$= \frac{-1}{j2\pi f}\left[e^{-j2\pi f(2)} - e^{-j2\pi f(-2)}\right]$$

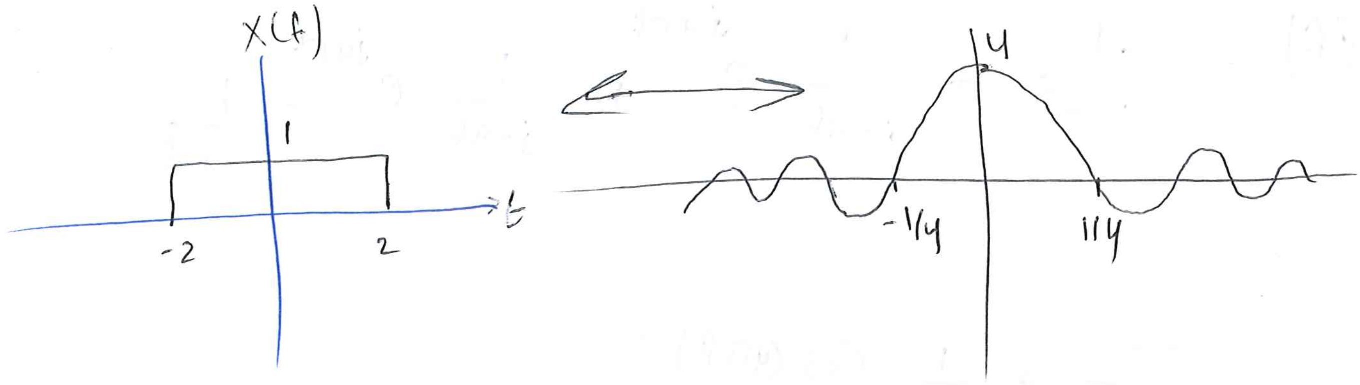
$$= \frac{-1}{j2\pi f}\left[e^{-j4\pi f} - e^{+j4\pi f}\right]$$

$$= \frac{1}{j2\pi f}\left[e^{j4\pi f} - e^{-j4\pi f}\right] = \frac{1}{\pi f}\left[\frac{e^{j4\pi f} - e^{-j4\pi f}}{j2}\right]$$

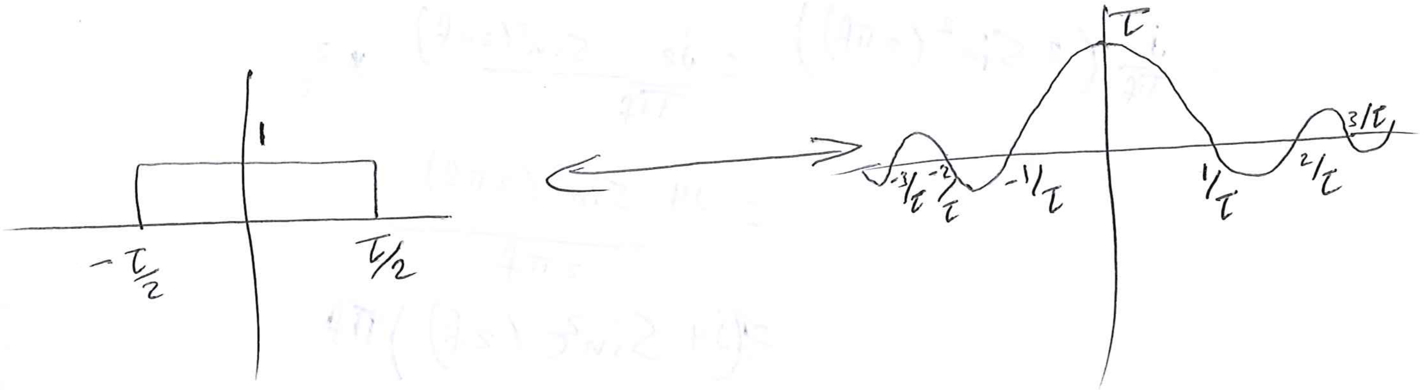
$$= \frac{1}{\pi f}\sin(4\pi f) = \frac{4}{4\pi f}\sin(4\pi f)$$

$$\text{sinc}(\theta) = \frac{\sin(\pi\theta)}{\pi\theta}$$

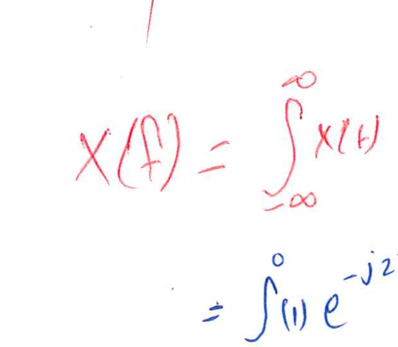
$$X(f) = 4\,\text{sinc}(4f)$$



$$\Pi\left(\frac{t}{\tau}\right) \overset{F}{\longleftrightarrow} \tau \, \text{Sinc}(\tau f)$$ Fourier pair



Example :- for the following signal, find $X(f)$



$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$$

$$= \int_{-2}^{0} (1)\, e^{-j2\pi f t}\, dt + \int_{0}^{2} (-1)\, e^{-j2\pi f t}\, dt$$

$$= \frac{-1}{j2\pi f}\, e^{-j2\pi f t}\Big|_{-2}^{0} + \frac{1}{j2\pi f}\, e^{-j2\pi f}\Big|_{0}^{2}$$

$$X(f) = \frac{-1}{j2\pi f} + \frac{1}{j2\pi f} e^{j4\pi f} + \frac{1}{j2\pi f} e^{-j4\pi f} - \frac{1}{j2\pi f}$$

$$= \frac{-2}{j2\pi f} + \frac{1}{j\pi f} \cos(4\pi f)$$

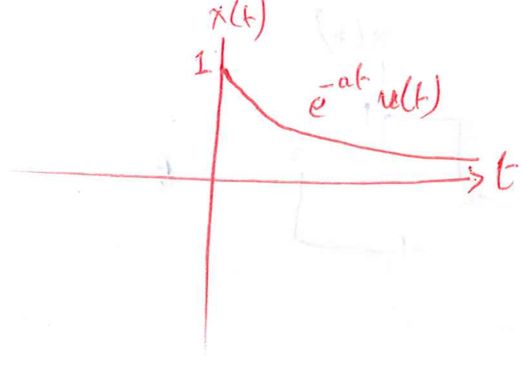
$$= \frac{j}{\pi f} - \frac{j}{\pi f} \cos(4\pi f) = \frac{j}{\pi f}\left(1 - \cos(4\pi f)\right)$$

$$= \frac{j}{\pi f}\left(2 \sin^2(2\pi f)\right) = \frac{j2 \sin^2(2\pi f)}{\pi f} \times \frac{2}{2}$$

$$= \frac{j4 \sin^2(2\pi f)}{2\pi f}$$

$$= \left(j4 \sin^2 c\,(2f)\right) \pi f$$

Example :- Find the FT of $x(t) = e^{-at} u(t)$



$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$$

$$= \int_{-\infty}^{\infty} e^{-at} e^{-j2\pi f t}\, dt$$

$$= \int_{0}^{\infty} e^{-(a+j2\pi f) t}\, dt$$

$$= \frac{-1}{a+j2\pi f} e^{-(a+j2\pi f) t} \Big|_{0}^{\infty}$$

$$= \frac{1}{a+j2\pi f} \quad a > 0 \quad = \frac{1}{\sqrt{a^2+(2\pi f)^2}} e^{-j\tan^{-1}\left(\frac{2\pi f}{a}\right)}$$

Properties of Fourier Transform

[1] Linearity :- (super position theorim)

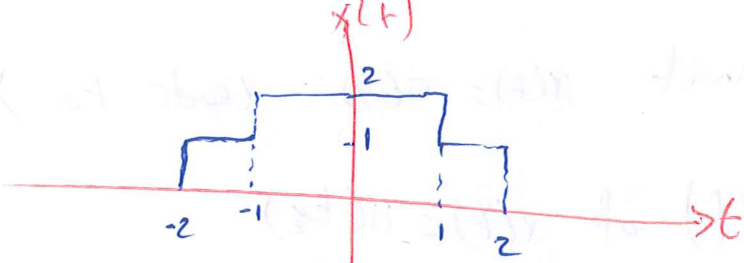
If $x(t) \longrightarrow X(f)$

$y(t) \longrightarrow Y(f)$

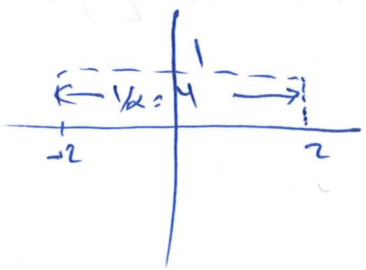
then

$$\alpha x(t) + \beta y(t) \longrightarrow \alpha X(f) + \beta Y(f)$$

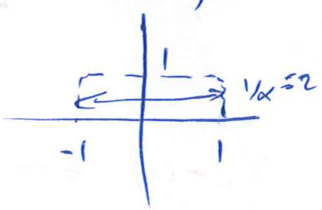
Example : for the following signal $x(t)$, find $X(f)$



$x_1(t) = \Pi(t/4)$ $\qquad$ $x_2(t) = \Pi(t/2)$



+



$$x(t) = x_1(t) + x_2(t)$$

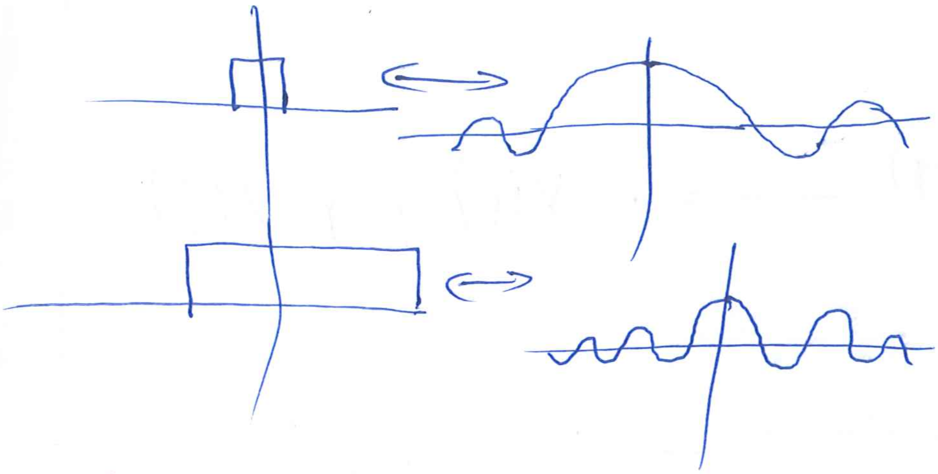
$$= \Pi(t/4) + \Pi(t/2) \qquad = 4\,\text{Sinc}(4f) + 2\,\text{Sinc}(2f)$$

2 Time scaling property

$$x(t) \longrightarrow X(f)$$

$$x(at) \longrightarrow \frac{1}{|a|} X\left(\frac{f}{a}\right)$$

Time compression of a signal results in its spectral expansion



Examples

given that $x(t) = \Pi(t/4)$ leads to $X(f) = 4\,\mathrm{Sinc}(4f)$

find $X(f)$ of $x(t) = \Pi(t/2)$

$$x_2(t) = \Pi(t/2) = x_1(2 * t/4) = \frac{1}{2}\left(4\,\mathrm{Sinc}\left(\frac{4f}{2}\right)\right) = 2\,\mathrm{sinc}(2f)$$

Example: Show that $x(-t) \longleftrightarrow X(-f)$

$$x(at) \longleftrightarrow \frac{1}{|a|} X\left(\frac{f}{a}\right)$$

$$a = -1 \Rightarrow \mathcal{F}(x(-t)) = \frac{1}{|-1|} X\left(\frac{f}{-1}\right) = (1) X(-f)$$

$$= X(-f)$$

Example : Given that $e^{-at} u(t) \xleftrightarrow{F} \frac{1}{a + j2\pi f}$

① Find the FT of $e^{at} u(-t)$

$$e^{at} u(-t) \xleftrightarrow{F} \frac{1}{a - j2\pi f}$$

② Find the FT of $e^{-a|t|}$

$$e^{-a|t|} = e^{-at} u(t) + e^{at} u(-t)$$

$$\Rightarrow F\left(e^{-a|t|}\right) = F\left(e^{-at} u(t)\right) + F\left(e^{at} u(-t)\right)$$

$$= \frac{1}{a + j2\pi f} + \frac{1}{a - j2\pi f} = \frac{2a}{a^2 + (2\pi f)^2}$$

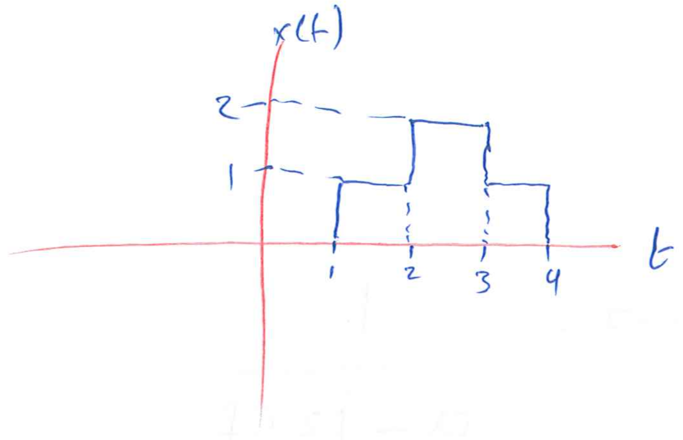
$\boxed{3}$ Time shifting property

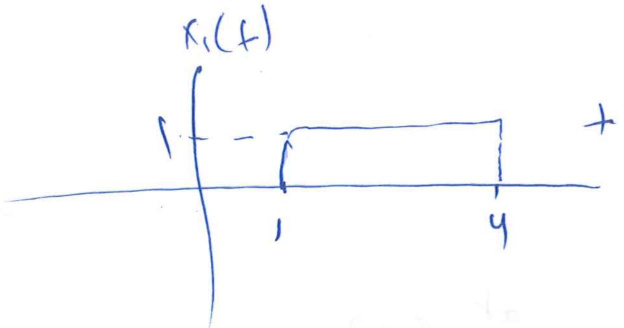
$$x(t) \longrightarrow X(f)$$

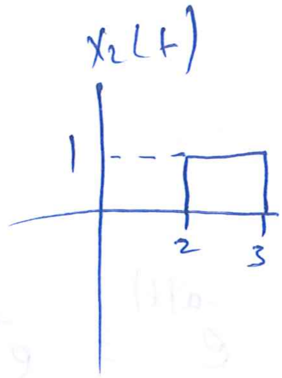
$$x(t - t_0) \longrightarrow X(f)\, e^{-j2\pi f t_0}$$

delaying a signal by t_0 seconds does not change the amplitude spectrum, while the phase spectrum is changed by a linear phase $(-2\pi f t_0)$

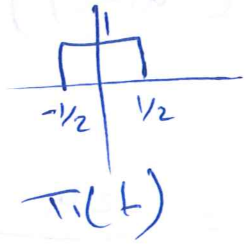
Examples:- for the following signal $x(t)$, find $X(f)$







$$x(t) = x_1(t) + x_2(t)$$



$$x_2(t) = \Pi(t - 2.5)$$

$$x_1(t) = \Pi\left(\frac{1}{3}(t - 2.5)\right)$$

$$X_2(f) = \text{sinc}(f)\, e^{-j2\pi f(2.5)}$$

$$X_1(f) = 3\,\text{sinc}(3f)\, e^{-j2\pi f(2.5)}$$

$$X(f) = \left[3\,\text{Sinc}(3f) + \text{Sinc}(f)\right] e^{-j2\pi f(2.5)}$$

$\boxed{4}$ Time Transformation (Time scaling + time shifting)

$$x(t) \longleftrightarrow X(f)$$

$$x(at - t_0) = x\left(a\left(t - \frac{t_0}{a}\right)\right) \longleftrightarrow \frac{1}{|a|} X\left(\frac{f}{a}\right) e^{-j2\pi \frac{f}{a} t_0}$$

$\boxed{5}$ Duality

If $x(t) \longrightarrow X(f)$

then $X(t) \longrightarrow x(-f)$

If a function $x(t)$ has a fourier transform $X(f)$, then if we have a function of time $X(t)$ such that

$$X(t) = X(f)\Big|_{f=t}$$

then $$F(X(t)) = x(-f) = x(t)\Big|_{t=-f}$$

$$x(t) \longrightarrow X(\omega)$$

$$X(t) \longrightarrow 2\pi x(-\omega)$$

Examples:- $x(t) = 10\ \text{Sinc}(30t)$, find $X(f)$

$$\Pi\left(\frac{t}{\tau}\right) \longrightarrow \tau\ \text{Sinc}(\tau f)$$

then using duality,

$$\tau\ \text{Sinc}(\tau t) \longrightarrow \Pi\left(-\frac{f}{\tau}\right)$$

but $\Pi(t) = \Pi(-t)$ even function

So: $\tau \, \text{Sinc}(\tau t) \longrightarrow \Pi\left(\frac{f}{\tau}\right)$

$$x(t) = 10 \, \text{Sinc}(30t)$$

$$= \frac{1}{3} \, 30 \, \text{Sinc}(30t)$$

$$= \frac{1}{3} \, \Pi\left(\frac{-f}{30}\right)$$

$$= \frac{1}{3} \, \Pi\left(\frac{f}{30}\right) \quad \text{even function}$$

$\boxed{6}$ Convolution:-

$$x(t) \longrightarrow \underset{\text{LTI}}{\boxed{h(t)}} \longrightarrow y(t) = x(t) * h(t)$$

Convolution Results into multiplication in frequency domain

$$x(t) \longrightarrow X(f)$$
$$h(t) \longrightarrow H(f)$$

$$Y(f) = X(f) \cdot H(f)$$

using the duality property then multiplication in time domain leads to Convolution in frequency domain

$$x(t) \cdot h(t) \longleftrightarrow X(f) * H(f)$$

$$x(t) \cdot x(t) \longleftrightarrow X(f) * X(f)$$

Example :- Find the output y(t) for the following system

$$10e^{-2t}u(t) \longrightarrow \boxed{50\, e^{-3t} u(t)} \longrightarrow y(t) = ??$$

$$y(t) = F^{-1}(Y(f))$$

$$Y(f) = X(f) \cdot H(f)$$

$$e^{-at}u(t) \longleftrightarrow \frac{1}{a + j2\pi f}$$

$$\therefore X(f) = \frac{10}{2 + j2\pi f} \qquad H(f) = \frac{50}{3 + j2\pi f}$$

$$Y(f) = \frac{500}{(2 + j2\pi f)(3 + j2\pi f)}$$ Using partial fraction expansion

$$Y(f) = \frac{A}{2 + j2\pi f} + \frac{B}{3 + j2\pi f}$$

$$500 = A(3 + j2\pi) + B(2 + j2\pi f)$$

$$A\Big|_{j2\pi f = -2} = \frac{500}{3-2} = 500$$

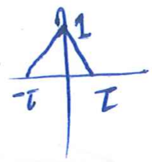
$$B\Big|_{j2\pi f = -3} = \frac{500}{2-3} = -500$$

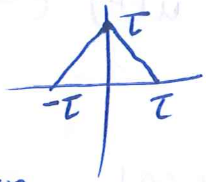
$$\therefore Y(f) = 500\left(\frac{1}{2+j2\pi f} - \frac{1}{3+j2\pi f}\right)$$

$$\therefore y(t) = F^{-1}(Y(f)) = 500\left(e^{-2t} - e^{-3t}\right)u(t)$$

Example :- consider the triangular signal, find the F.T

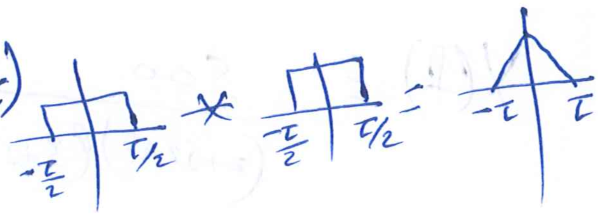
$\tau \Lambda\left(\frac{t}{\tau}\right)$

$\Lambda(t/\tau) \rightarrow$ 

$\tau \Delta(t/\tau) \rightarrow$ 

we can find the FT using the definition

or using convolution properity

$\Pi(t/\tau) * \Pi(t/\tau) = \tau\Lambda(t/\tau)$ 

$$\therefore F(\tau\Lambda(t/\tau)) = F(\Pi(t/\tau)) \cdot F(\Pi(t/\tau))$$

$$= F\{\Pi(t/\tau)\}^2$$

$$= \tau^2 \operatorname{sinc}^2(f\tau)$$

7 frequency shifting (Modulation Property)

It is a dual representation

$x(t) \longleftrightarrow X(f)$

$$F(x(t)\, e^{j2\pi f_0 t}) = X(f - f_0)$$

$$F(x(t)\cos(2\pi f_0 t)) = F\left(x(t) \cdot \left(\frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2}\right)\right)$$

$$= F\left(x(t)\tfrac{1}{2} e^{j2\pi f_0 t}\right) + F\left(x(t)\tfrac{1}{2} e^{-j2\pi f_0 t}\right)$$

$$= \frac{1}{2} X(f - f_0) + \frac{1}{2} X(f + f_0)$$

Examples for the following signals

1) $x_1(t) = \Pi\left(\frac{t}{3}\right)\cos(8\pi t)$

2) $x_2(t) = \Lambda\left(\frac{t}{2}\right)\cos(10\pi t)$

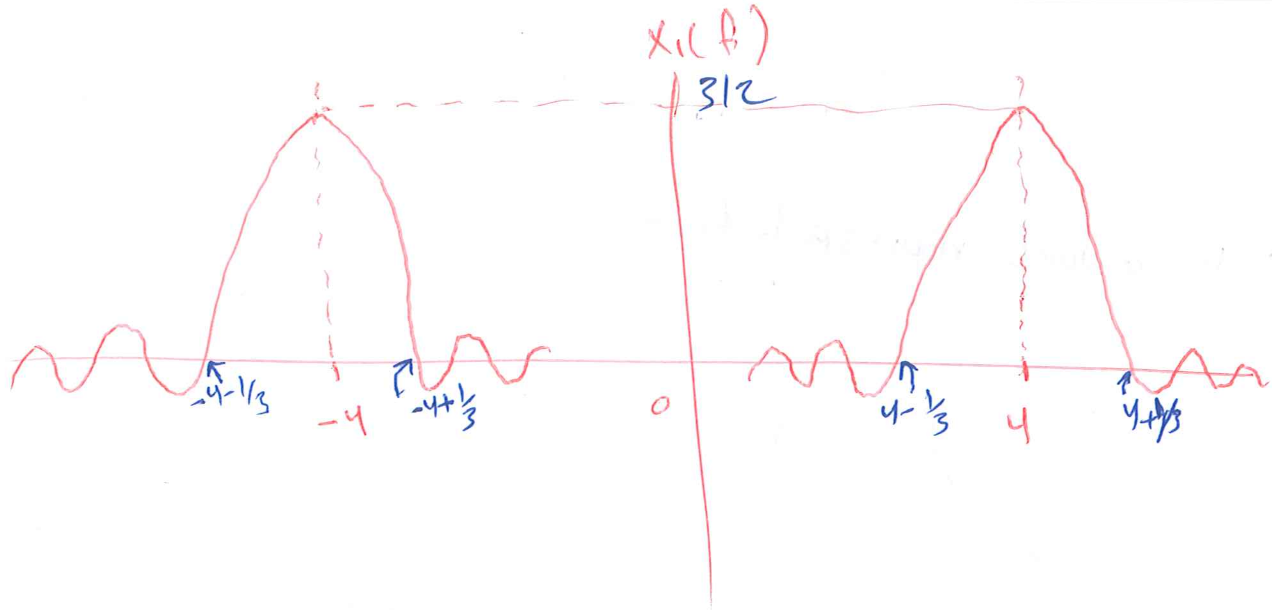
a) Find the FT of each signal

b) skethe the signal obtained in part (a)

a) $x_1(t) = \Pi(t/3)\cos(8\pi t)$

$$= \frac{1}{2}\Pi(t/3)\left[e^{j2\pi(4)} + e^{-j2\pi(4)}\right]$$

$$= \frac{1}{2}\, 3\, \mathrm{Sin}(3(f-4)) + \frac{1}{2}\, 3\, \mathrm{sinc}(3(f+4))$$

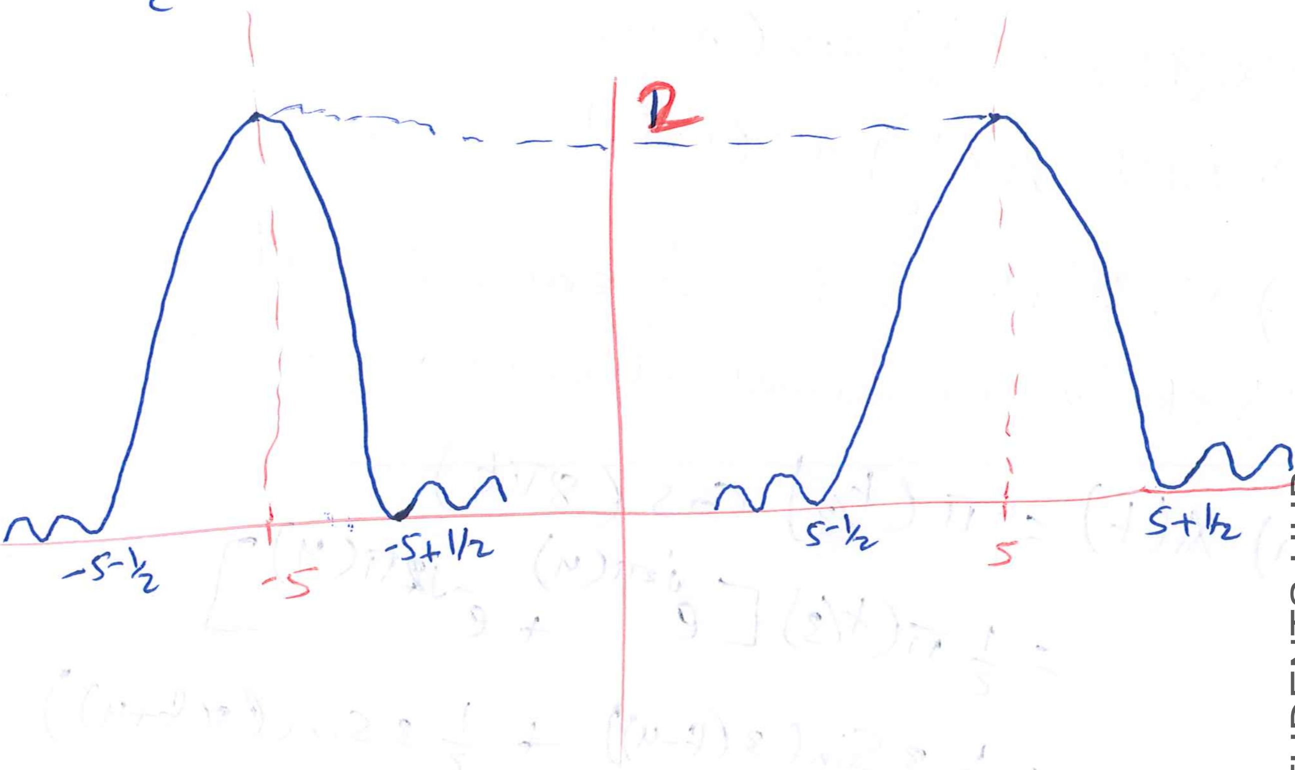


2) $x_2(t) = \Lambda\left(\frac{t}{2}\right)\cos(10\pi t)$

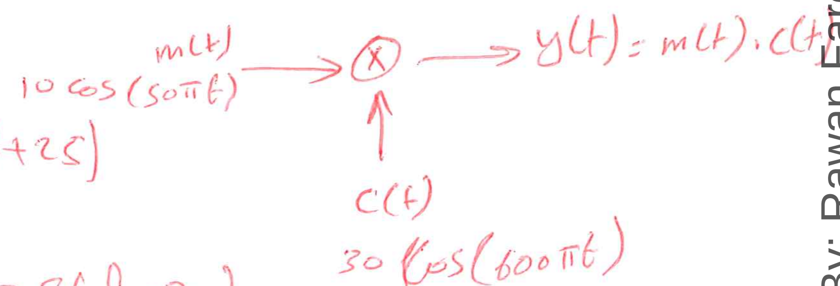
$= \Lambda\left(\frac{t}{2}\right)\left[\frac{1}{2}e^{j2\pi(5)t} + \frac{1}{2}e^{-j2\pi(5)t}\right]$

$= \frac{1}{2}\Lambda\left(\frac{t}{2}\right)e^{j2\pi(5)t} + \frac{1}{2}\Lambda(t/2)e^{-j2\pi(5)t}$

$= \frac{1}{2}\,4\,\mathrm{sinc}^2(2(f-5)) + \frac{1}{2}\,4\,\mathrm{sinc}^2(2(f+5))$

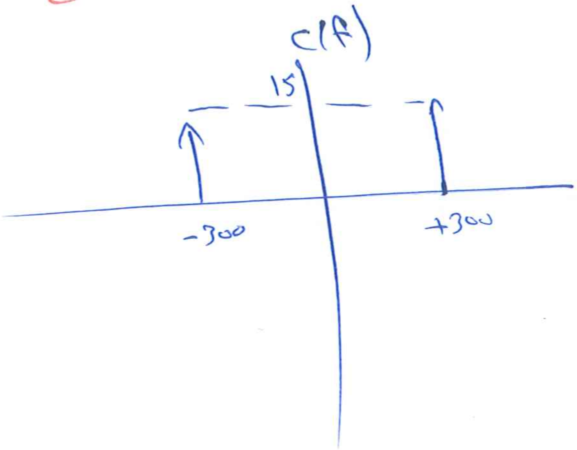


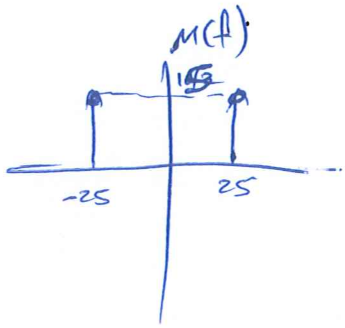
Example



$$M(f) = \frac{10}{2}\delta(f-25) + \frac{10}{2}\delta(f+25)$$

$$C(f) = \frac{30}{2}\delta(f-300) + \frac{30}{2}\delta(f+300)$$

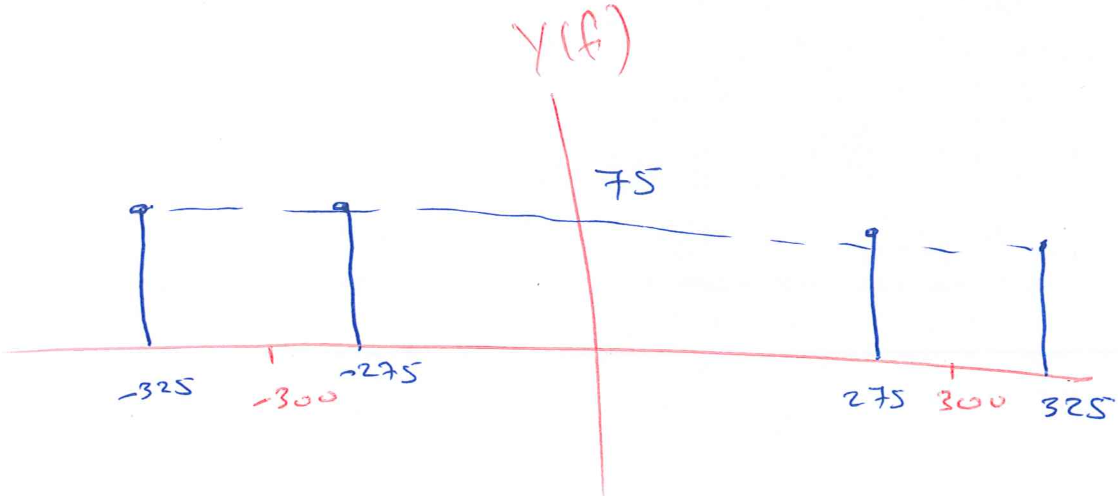




$$y(t) = 300\cos(50\pi t)\cos(600\pi t)$$

$$= 150\cos(550\pi t) + 150\cos(650\pi t)$$

$$Y(f) = \frac{150}{2}\delta(f-275) + \frac{150}{2}\delta(f+275) + \frac{150}{2}\delta(f-325) + \frac{150}{2}\delta(f+325)$$

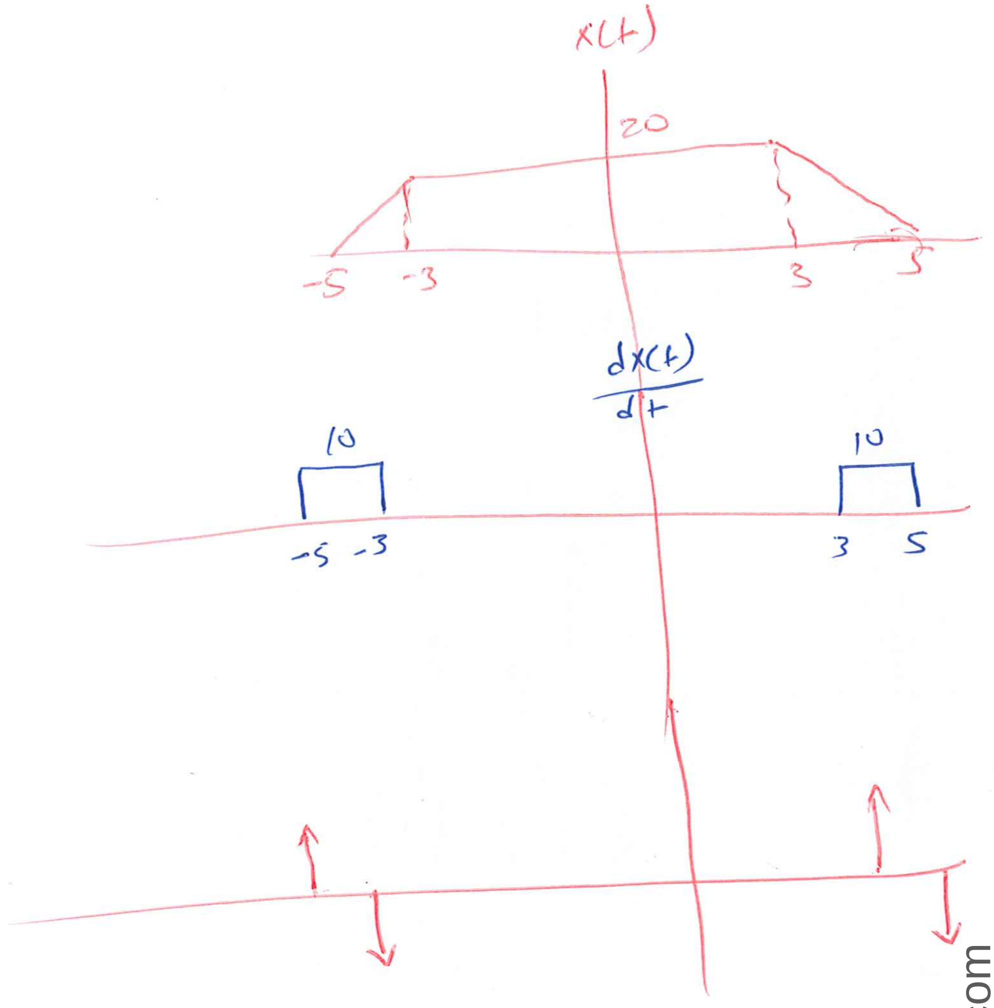


$\boxed{8}$ Differentiation Property

$$x(t) \longleftrightarrow X(f)$$

$$F\left(\frac{d}{dt}(x(t))\right) \longleftrightarrow j2\pi f\, X(f)$$

$$F\left(\frac{d^n x(t)}{dt^n}\right) = (j2\pi f)^n\, X(f)$$



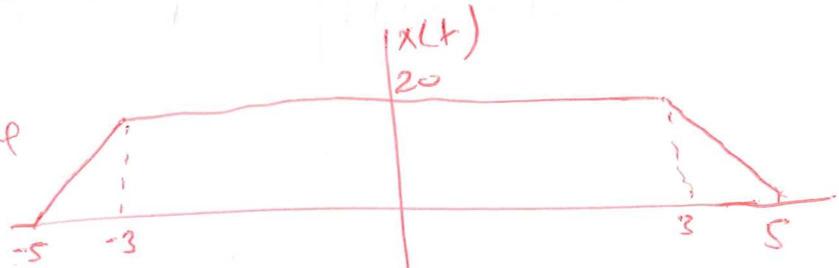
$\boxed{8}$ Differentiation Property

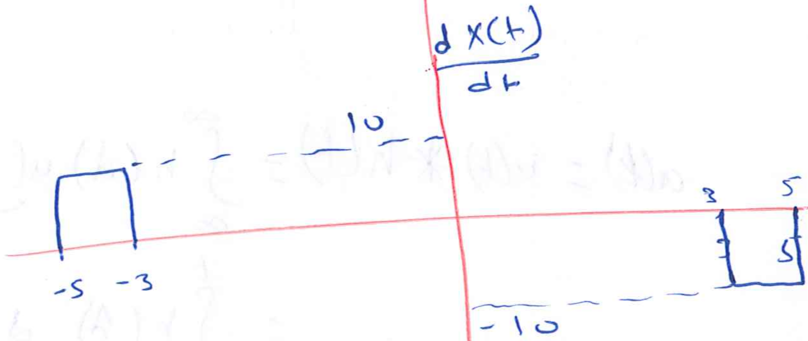
$$x(t) \longleftrightarrow X(f)$$

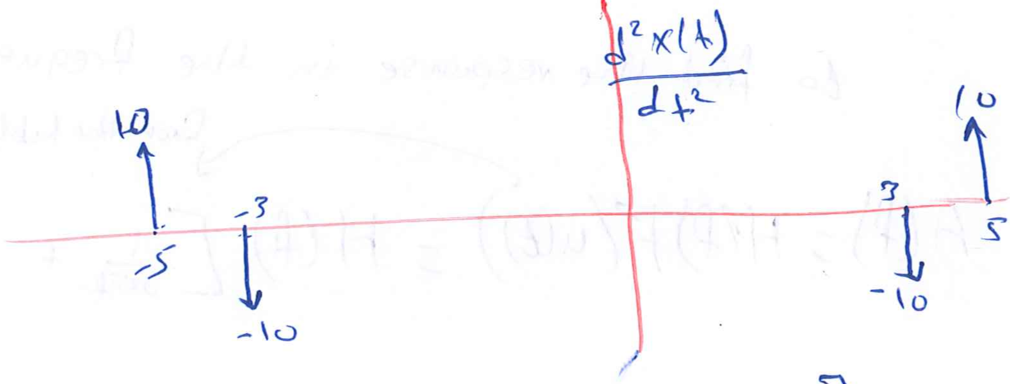
$$F\left(\frac{d}{dt}x(t)\right) \longleftrightarrow j2\pi f\, X(f)$$

$$F\left(\frac{d^n}{dt^n}x(t)\right) \longleftrightarrow (j2\pi f)^n X(f)$$

Example:- find $X(f)$ of the following signal:-







$$y(t) = \frac{d^2}{dx^2}x(t) = 10\left[\delta(t+5) - \delta(t+3) - \delta(t-3) + \delta(t-5)\right]$$

$$Y(f) = (j2\pi f)^2\, X(f) = 10\left[e^{j2\pi f(5)} - e^{j2\pi f(3)} - e^{j2\pi f(-3)} + e^{j2\pi f(-5)}\right]$$

$$= 10 \cdot 2\left[\cos(2\pi f(5)) - \cos(2\pi f(3))\right]$$

$$\therefore\ X(f) = -\frac{20}{4\pi f^2}\left[\cos(10\pi f) - \cos(6\pi f)\right]$$

9 Time Integration

$$x(t) \longleftrightarrow X(f)$$

$$F\left(\int_{-\infty}^{t} x(\tau)\,d\tau\right) \Longleftrightarrow \frac{X(f)}{j2\pi f} + \frac{1}{2}X(0)\,\delta(f)$$

$$X(0) = X(f)\Big|_{f=0} = \text{average value} = \int_{-\infty}^{\infty} x(t)\,dt$$

Example:- Consider finding the unit step response $a(t)$ of LTI system as a function of $h(t)$

$$a(t) = u(t) * h(t) = \int_{-\infty}^{\infty} h(\lambda)\,\underbrace{u(t-\lambda)}_{=1 \text{ for } t \geq \lambda}\,d\lambda$$

$$= \int_{-\infty}^{t} h(\lambda)\,d\lambda$$

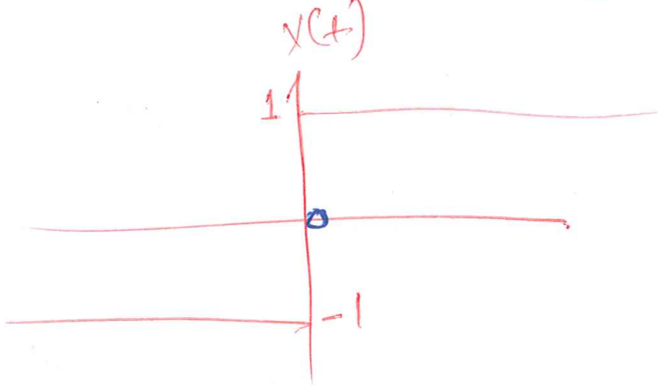
to find the response in the frequency domain

from the table

$$A(f) = H(f)\,F(u(t)) = H(f)\left[\frac{1}{j2\pi f} + \frac{1}{2}\delta(f)\right]$$

$$= \frac{H(f)}{j2\pi f} + \frac{1}{2}H(0)\,\delta(f)$$

Example 8 - Find the Fourier transform of the Signum function



$$sgn(t) = \begin{cases} 1, & t>0 \\ 0, & t=0 \\ -1, & t<0 \end{cases}$$

$$sgn(t) = 2u(t) - 1$$

$$\frac{d}{dt}(sgn(t)) = 2\delta(t)$$

$$F\left(\frac{d}{dt}(sgn(t))\right) = 2F(\delta(t))$$

$$(j2\pi f)F(sgn(t)) = 2$$

$$\therefore F(sgn(t)) = \frac{2}{j2\pi f} = \frac{1}{j\pi f}$$

find the F.T of u(t)

$$u(t) = \frac{1}{2} + \frac{1}{2}sgn(t)$$

$$F(u(t)) = F\left(\frac{1}{2} + \frac{1}{2}sgn(t)\right)$$

$$= \frac{1}{2}\delta(f) + \frac{1}{2}\cdot\frac{1}{j\pi f}$$

$$= \frac{1}{2}\delta(f) + \frac{1}{j2\pi f}$$

Example:- Hilber Transform

Hilber Transform $\hat{x}(t)$ is obtained by convolving $x(t)$ with $\frac{1}{\pi t}$.

$$x(t) \longrightarrow \boxed{\text{LTI}}^{\,1/\pi t} \longrightarrow y(t) = \hat{x}(t)$$

$$\hat{x}(t) = x(t) * \frac{1}{\pi t}$$

in the frequency domain

$$\text{Sgn}(t) \xleftrightarrow{FT} \frac{1}{j\pi f}$$

using duality property

$$\frac{1}{j\pi t} \longleftrightarrow \text{Sgn}(-f) = -\text{sgn}(f)$$

$$\Rightarrow \frac{1}{\pi t} \longleftrightarrow -j\,\text{Sgn}(f)$$

$$F(\hat{x}(t)) = F\left(\frac{1}{\pi t}\right) \cdot F(x(t))$$

$$\hat{X}(f) = -j\,\text{sgn}(f) \cdot X(f)$$

$$|\hat{X}(f)| = |X(f)| \qquad \Theta_{\hat{X}(f)} = \begin{cases} \Theta_{X(f)} - 90, & f > 0 \\ \Theta_{X(f)} + 90, & f < 0 \end{cases}$$

* FT of periodic signals:-

we can find the FT for single period and then samples the FT as follows

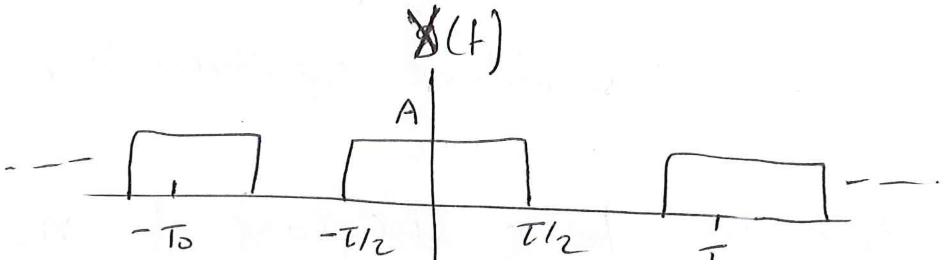
$$X_n = f_0\, X(kf_0) = f_0\, X(f)\Big|_{f=kf_0}$$

(fourier series coefficients)

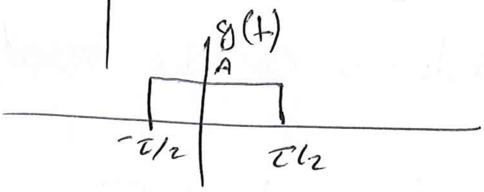
$$X(f) = \sum_{n=-\infty}^{\infty} \underbrace{X_n}_{\text{FS coefficient}}\, \delta(f-nf_0)$$

$$X(f) = f_0 \sum_{n=-\infty}^{\infty} X(nf_0)\, \delta(f-nf_0)$$

Example:-



$$X(f) = f_0 \sum_{n=-\infty}^{\infty} G(nf_0)\, \delta(f-nf_0)$$

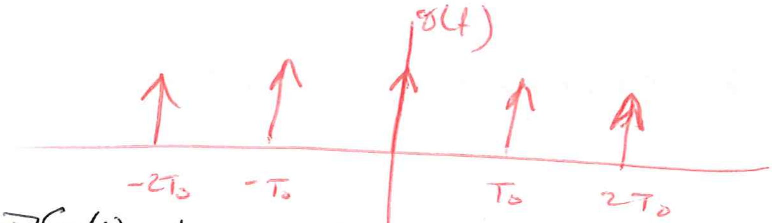


$$g(t) = A\, \text{rect}(t/\tau) = A\, \Pi(t/\tau) \xleftrightarrow{FT} G(f) = A\tau\, \text{Sinc}(\tau f)$$

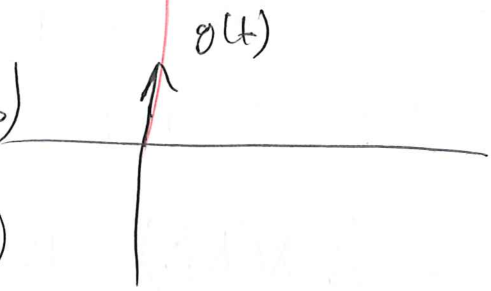
$$X(f) = f_0 \sum G(nf_0)\, \delta(f-nf_0)$$

$$= f_0 \sum A\tau\, \text{Sinc}(\tau n f_0)\, \delta(f-nf_0)$$

Example:-



$g(t) = \delta(t) \xleftrightarrow{FT} G(f) = 1$



$\therefore X(f) = f_0 \sum G(nf_0)\, \delta(f - nf_0)$

$= f_0 \sum 1\, \delta(f - nf_0)$

$= f_0 \sum \delta(f - nf_0)$

Example:- obtain the fourier transform of the periodic raised-cosine pulse train

$$x(t) = \frac{1}{2} A \sum_{n=-\infty}^{\infty} [1 + \cos(20\pi(t - nT_0)]\, \Pi\left(\frac{t - nT_0}{\tau}\right)$$

wher $T_0 \geq \tau$, sketch the wave form and the amplitude spectrum for the case $\tau = T_0$

let us take the case of $n = 0$, the same procedures can be used for different values of n

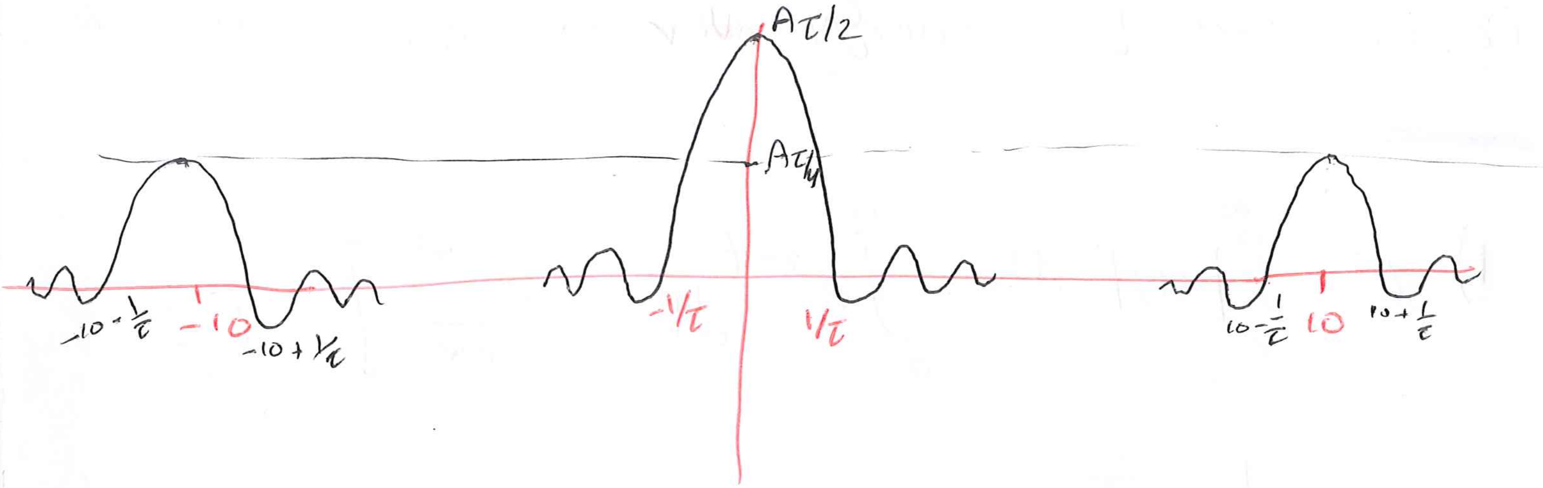
$$g(t) = \frac{1}{2} A [1 + \cos(20\pi t)]\, \Pi(t/\tau)$$

$$G(f) = \frac{1}{2} A \tau \,\text{Sinc}(\tau f) + \frac{A}{4} \tau \,\text{Sinc}(\tau(f - 10))$$

$$+ \frac{A}{4} \tau \,\text{Sinc}(\tau(f + 10))$$

$$X(f) = f_0 \sum G(nf_0)\,\delta(f - nf_0)$$

$$X(f) = f_0 \sum_{n=-\infty}^{\infty} \Big[\tfrac{1}{2} A \tau \, \mathrm{Sinc}(n\tau f_0) + \frac{A}{4}\tau \, \mathrm{Sinc}(n\tau(f_0 - 10)) + \frac{A}{4}\tau \, \mathrm{Sinc}(n\tau(f_0+10)) \Big] \, \delta(f - nf_0)$$



* Energy spectral density

- Energy spectral density (ESD) is used to determine energy distribution of an energy signal in the frequency spectrum

- Knowledge of these distributions is valuable in the analysis and design of communication systems and other systems.

- Parseval's theorem shows that Energy and power in time domain remains unchanged in frequency domain.

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |X(f)|^2 \, df$$

$G(f) \triangleq |X(f)|^2 \longrightarrow$ Parseval's theorem

$\therefore E = \int_{-\infty}^{\infty} G(f)\, df$

Example :- Consider $x(t) = e^{-\alpha t} u(t)$, $\alpha > 0$

① Determine E

② Determine E_B , energy ~~over~~ frequency range $-B < f < B$

1) $E = \int_{-\infty}^{\infty} |x(t)|^2\, dt = \int_0^{\infty} e^{-2\alpha t}\, dt = \left.\frac{e^{-2\alpha t}}{-2\alpha}\right|_0^{\infty} = \frac{0-1}{-2\alpha}$

$= \frac{1}{2\alpha}$ J

2) $E_B = \int_{-B}^{B} G(f)\, df$

$X(f) = \frac{1}{\alpha + j2\pi f}$

$|X(f)| = \frac{1}{\sqrt{\alpha^2 + (2\pi f)^2}} \Rightarrow G(f) = |X(f)|^2 = \frac{1}{\alpha^2 + (2\pi f)^2}$

$= \left(\frac{1}{\alpha + j2\pi f}\right)\left(\frac{1}{\alpha + j2\pi f}\right)^*$

$= \frac{1}{\alpha^2 + (2\pi f)^2}$

$E_B = \int_{-B}^{B} \frac{1}{\alpha^2 + (2\pi f)^2}\, df = \int_0^{B} \frac{2}{\alpha^2 + (2\pi f)^2}\, df$

$= \int_0^{B} \frac{2/\alpha^2}{1 + \frac{(2\pi f)^2}{\alpha^2}} = \frac{2}{\alpha^2} \int_0^{B} \frac{1}{1 + \left(\frac{2\pi f}{\alpha}\right)^2}$

$$E_B = \frac{2}{\alpha^2} \int_0^B \frac{1}{1+\left(\frac{2\pi f}{\alpha}\right)^2} \, df$$

If $\frac{2\pi f}{\alpha} = z$

$dz = \frac{2\pi}{\alpha} df$

$df = \frac{\alpha}{2\pi} dz$

$$= \frac{2}{\alpha^2} \int_0^{\frac{2\pi}{\alpha}B} \frac{1}{1+(z)^2} \frac{\alpha}{2\pi} \, dz$$

$$= \frac{1}{\alpha\pi} \int_0^{\frac{2\pi}{\alpha}B} \frac{1}{1+z^2} \, dz$$

$$= \frac{1}{\alpha\pi} \tan^{-1}\left(\frac{2\pi B}{\alpha}\right) \; J$$

note that as $B \to \infty$ $\lim_{B\to\infty} E_B = \frac{1}{\pi\alpha} \cdot \frac{\pi}{2} = \frac{1}{2\alpha} J$

* In terms of percentage of Energy Contained Bandwidth B to total energy is

$$= \frac{\frac{1}{\alpha\pi} \tan^{-1}\left(\frac{2\pi B}{\alpha}\right)}{\frac{1}{2\alpha}} * 100\%$$

$$= \frac{2}{\pi} \tan^{-1}\left(\frac{2\pi B}{\alpha}\right) * 100\%$$

If you ~~are~~ asked to find the bandwidth such that 85% or 90% of the energy is preserved then find B such that 90% of energy is preserved

$$\frac{200}{\pi} \tan^{-1}\left(\frac{2\pi B}{\alpha}\right)\% = 90\%$$

$$\tan^{-1}\left(\frac{2\pi B}{\alpha}\right) = \frac{90\pi}{200}$$

$$\frac{2\pi B}{\alpha} = \tan\left(\frac{90\pi}{200}\right)^{\text{radian}}$$

$$= \tan 1.413$$

$$= 6.28$$

$$\Rightarrow B = \frac{6.28\,\alpha}{2\pi} \qquad \text{if } \alpha = 4$$

$$\therefore B = 4 \text{ HZ}$$

* System Analysis with the FT

$$x(t) \longrightarrow \underset{}{\overset{\text{LTI}}{\boxed{h(t)}}} \longrightarrow y(t) = x(t) * h(t)$$

$$= \int_{-\infty}^{\infty} x(\lambda)\, h(t-\lambda)\, d\lambda$$

$$X(f) \longrightarrow \boxed{H(f)} \longrightarrow Y(f) = X(f)\, H(f)$$

$$y(t) = F^{-1}(Y(f))$$

$h(t)$ is the impulse response

$H(f)$ is the frequency response (transfer function)

$h(t)$ and $H(f)$ are good characterization of the system.

Since $H(f)$ in general a complex quantity

$$H(f) = |H(f)| \, e^{j\angle H(f)}$$

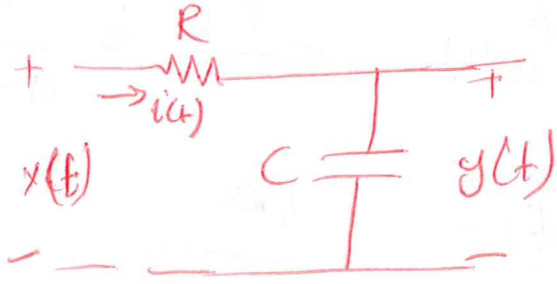
$|H(f)|$: amplitude response function

$\angle H(f)$: phase response function

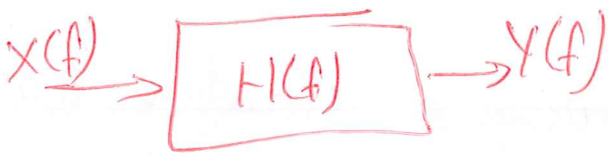
$$|Y(f)| = |H(f)| \, |X(f)|$$

$$\angle Y(f) = \angle H(f) + \angle X(f)$$

Example :- obtain $H(f)$ for the following System.



Sol :- we can solve the problem with 3 different ways



1) Fourier Transform of the differential equation.

$x(t) = R\,i(t) + y(t)$ $\quad$ $i(t) = C\dfrac{dy(t)}{dt}$

$x(t) = RC\dfrac{dy(t)}{dt} + y(t)$ $\quad$ initial conditions are zero

the differential equation can be presented in frequency domain if initially at rest

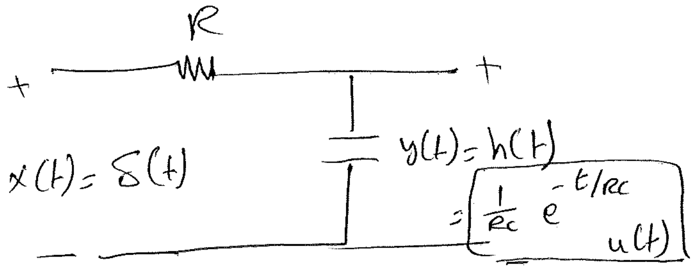
$\therefore X(f) = RC(j2\pi f)\,Y(f) + Y(f)$

$\Rightarrow X(f) = Y(f)\,[1 + j2\pi f RC]$

$\therefore H(f) = \dfrac{Y(f)}{X(f)} = \dfrac{1}{1 + j2\pi f RC}$ ✓

$= \dfrac{1}{\sqrt{1 + (j2\pi RCf)^2}} \tan^{-1}\left(\dfrac{2\pi f RC}{1}\right)$

(2) find $h(t)$ then $H(f)$



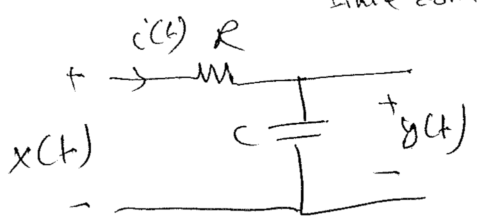
$h(t) = \dfrac{1}{RC} e^{-t/RC}$

$e^{-at} u(t) \xleftrightarrow{FT} \dfrac{1}{a + j2\pi f}$

$H(f) = \dfrac{1/RC}{\frac{1}{RC} + j2\pi f} = \dfrac{1}{1 + j(2\pi f RC)}$

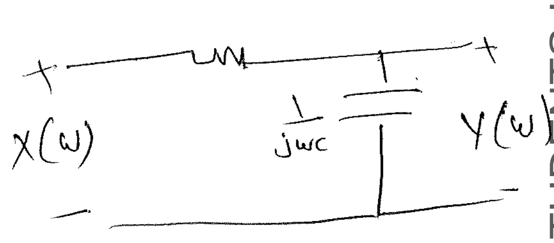
(3) using AC steady-state analysis

time domain



$\Rightarrow$

frequency domain



$Y(w) = X(w)\left(\dfrac{\frac{1}{jwC}}{R + \frac{1}{jwC}}\right)$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1/j\omega c}{R + \frac{1}{j\omega c}} = \frac{1}{1 + j\omega cR}$$

$$H(f) = \frac{1}{1 + j2\pi f Rc}$$

Example :- find the tranfer function of the system described by the following differential equation :-

$$\frac{d^2 y(t)}{dt^2} + 7\frac{dy(t)}{dt} + 12\, y(t) = x(t)$$

$$(j2\pi f)^2 Y(f) + 7(j2\pi f)Y(f) + 12\, Y(f) = X(f)$$

$$X(f) = Y(f)\left[12 + 7(j2\pi f) + (j2\pi f)^2\right]$$

$$H(f) = \frac{Y(f)}{X(f)} = \frac{1}{(j2\pi f)^2 + 7(j2\pi f) + 12}$$

$$= \frac{1}{(j2\pi f + 3)(j2\pi f + 4)}$$

we Can find h(t)

we Can find Y(f)

we Can find y(t)

$$Y(f) = H(f) \cdot X(f)$$

$$= \frac{X(f)}{(j2\pi f+3)(j2\pi f+4)}$$

if $X(f) = 1$

$$\therefore \; Y(f) = \frac{1}{(j2\pi f+3)(j2\pi f+4)}$$

and using partial fraction expansion we can find $y(t)$

if $X(f) = \delta(f)$

$$\therefore \; Y(f) = \frac{\delta(f)}{(j2\pi f+3)(j2\pi f+4)}$$

$$= \frac{\delta(f)}{12}$$

from the table we can find $y(t)$