3.3: Continuity

(ε-δ) defn

Df 1: let $\emptyset \neq E \subseteq \mathbb{R}$ and $f : E \to \mathbb{R}$

i. f is said to be continuous at a point $a \in E$ iff $\forall \varepsilon > 0$, $\exists$ a $\delta > 0$ (depends on ε, f and a) s.t $\boxed{|x-a| < \delta \text{ and } x \in E \Rightarrow |f(x)-f(a)| < \varepsilon.}$ *

ii. f is said to be continuous on E iff f is continuous at every $x \in E$.

RMK: let I be an open interval which contains a point a and $f : I \to \mathbb{R}$.
Then f is continuous at $a \in I$ iff $f(a) = \lim_{x \to a} f(x)$

Thm 1: sequential characterization of continuity:
suppose that E is a nonempty subset of $\mathbb{R}$, that $a \in E$ and that $f : E \to \mathbb{R}$
Then the following statements are equivalent:

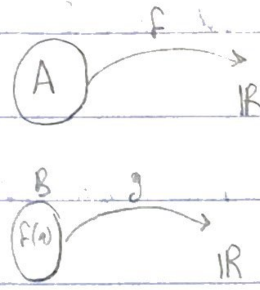
proof same

i. f is cont. at $a \in E$

ii. If $x_n \to a$ and $x_n \in E$, then $f(x_n) \to f(a)$ as $n \to \infty$.

Thm 2: let E be a nonempty subset of $\mathbb{R}$ and $f, g : E \to \mathbb{R}$. If f, g are continuous at a point $a \in E$ (resp. continuous on the set E), then so are $f+g$, fg, and αf (for any $\alpha \in \mathbb{R}$).
Moreover, $\frac{f}{g}$ is conti. at $a \in E$ when $g(a) \neq 0$ (resp. on E when $g(x) \neq 0$ $\forall x \in E$).

Df 2: suppose that A and B are subset of $\mathbb{R}$, that $f: A \to \mathbb{R}$ and $g: B \to \mathbb{R}$. If $f(a) \subseteq B$ for every $x \in A$ then the composition of g with f is the function $g \circ f: A \to \mathbb{R}$ defined by $(g \circ f)(x) := g(f(x))$, $x \in A$.



$(g \circ f)(x) = g(f(x))$, $y \in B$

$= g(y)$.

Thm 3: suppose that A and B are subsets of $\mathbb{R}$, that $f: A \to \mathbb{R}$ and $g: B \to \mathbb{R}$ and that $f(x) \in B$, $\forall x \in A$.

i. If $A := I \setminus \{a\}$, where I is a nondegenerate interval which either contains a or has a as one of its endpoints if $L := \lim_{\substack{x \to a \\ x \in I}} f(x)$ exists and belongs to B, and if g is cont. at $L \in B$ then ⟶ extended Real number

$$\boxed{\lim_{\substack{x \to a \\ x \in I}} (g \circ f)(x) = g\Big(\lim_{\substack{x \to a \\ x \in I}} f(x)\Big)} = g(L)$$

ii. If f is cont. at $a \in A$ and g is cont. at $f(a) \in B$, then $g \circ f$ is cont. at $a \in A$.

$$\lim_{x \to a} (g \circ f)(x) = (g \circ f)(a) \quad \text{OR} \quad \lim_{x \to a} g(f(x)) = g\Big(\lim_{x \to a} f(x)\Big) = g(f(x))$$

proof :

i) suppose that $x_n \in \overbrace{I \setminus \{a\}}^{A}$ and that $x_n \to a$ as $n \to \infty$. (we need $(g\circ f)(x_n) \to g(L)$ as $n \to \infty$)

since $f(A) \subseteq B \Rightarrow f(x_n) \in B$. Also by the sequential characterization of limit

$f(x_n) \longrightarrow L$ as $n \to \infty$ (since $\lim_{x\to a} f(x) = L$).

since g is cont. at $L \in B$, it follows by the seq. characterization of continuity, $(g\circ f)(x_n) = g(f(x_n)) \longrightarrow g(L)$ as $n \to \infty$

Hence, By the seq. charac. of limits, $(g\circ f)(x) \longrightarrow g(L)$ as $x \to a$ in I. ⊠

(ii) excersise.

Df 3: let $\phi \neq E \subseteq \mathbb{R}$. A function $f: E \to \mathbb{R}$ is said to be bounded on E iff $\exists$ an $M \in \mathbb{R}$ s.t $|f(x)| \leq M$, $\forall x \in E$. (f is dominated by M on E).

to prove unbdd → unbounded : $\exists x_0$ s.t $|f(x_0)| \geq M$ $\forall M \in \mathbb{R}$

RMK: Notice that whether a function f is bounded or not on a set E depends on E as well as on f. For exp. $f(x) = \frac{1}{x}$ is bounded on $[1, \infty)$ but unbounded on $(0, 2)$. $f(x) = x^2$ is bounded on $(-2, 2)$ (dominated by 4) But unbounded on $[0, \infty)$.

يتأثر بالفترة

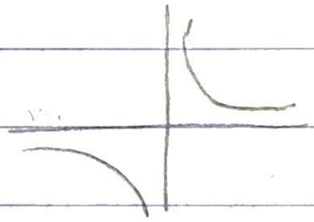
exp 1: $x \in [1, \infty) \to x \geq 1$

$f(x) = \frac{1}{x} \leq 1$

$|f(x)| \leq 1$.

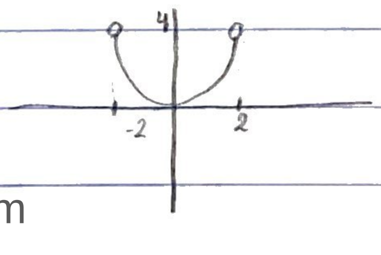
$(0, 2)$ unbdd :

$\lim_{x \to 0^+} \frac{1}{x} = +\infty$

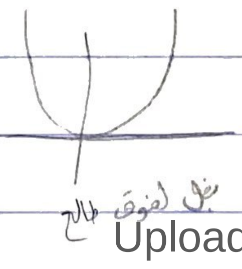


exp2 : $0 \leq x^2 < 4$ on $(-2, 2)$

bdd



$[0, \infty)$ unbdd

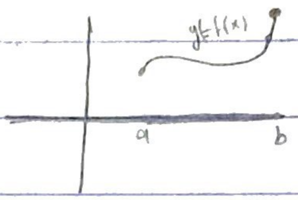


بضل لفوق طالع

Thm 4: Extreme Value Thm.

If I is closed, bounded interval and $f: I \to \mathbb{R}$ is continuous on I, then f is bounded on I. (Moreover, if $M = \sup_{x\in I} f(x)$ and $m = \inf_{x \in I} f(x)$, then $\exists$ points $x_m, x_M \in I$ s.t $f(x_M) = M$ and $f(x_m) = m$) ← 3 شروط

نفس absolute بالتالي max + min



proof: ← نفي DF3

suppose that f is unbounded on I, Then $\exists x_n \in I$ s.t $|f(x_n)| > n$, $\forall n \in \mathbb{N}$. *

since I is bdd, then x_n is bdd (since $x_n \in I$)

By the Bolzano-Weierstrass Thm, $\{x_n\}$ has a convergent subsequence

say $x_{n_k} \to a$ as $k \to \infty$

since I is closed, By the comparison thm $a \in I$.

continuity Inparticular $f(a) \in \mathbb{R}$, back to * substituting n_k for n

$|f(x_{n_k})| > n_k$ Taking the limit as $k \to \infty$ then $|f(a)| = \infty$

$|f(a)|$ ←

$|f(a)| = \infty$ a contradiction (f is conti. at a) $\times$.

Hence, f is bdd on I

We have proved that M and m are finite real numbers. To show that $\exists$ an $x_M \in I$ s.t $f(x_M) = M$, spse to the contrary, that ← النفي هون

بس $f>M$ or $f<M$ مش M هي sup منوجد النفي الاول

$f(x) < M \quad \forall x \in I$, then $g(x) = \dfrac{1}{M - f(x)}$ is continuous →

ثابت، معطى انه conti

So $g(x)$ is conti.

→ Hence, $g(x)$ is bdd on I, $\exists$ a $C > 0$ s.t $|g(x)| = g(x) \leq C$.

i.e. $\dfrac{1}{M - f(x)} \leq C$ it follows that $f(x) \leq M - \dfrac{1}{C}$ $\forall x \in I$.

شغل حلو ←

$$\Rightarrow \sup_{x \in I} f(x) \leq \sup_{x \in I} \left(M - \frac{1}{C}\right)$$

$$M \leq M - \frac{1}{C} < M \qquad \frac{1}{C} > 0$$

i.e. $M < M$!! (a contradiction)

Hence, $\exists\, x_M \in I$ s.t $f(x_M) = M$.

similarly, $\exists\, x_m \in I$ s.t $f(x_m) = m$. (do it ::).

اعطيها

RMK :

1. We also call the value M (resp. m) the maximum (resp. minimum) of f on I.

2. The Extreme value Thm is false if either closed or bounded is dropped from the hypothess.

counter example 2 :

$(0,1)$ is bounded interval but not closed and $f(x) = \frac{1}{x}$ is cont. and unbounded on $(0,1)$.

$[0,\infty)$ is closed but not bounded and the function $f(x) = x$ is cont. and unbounded on $[0,\infty)$.

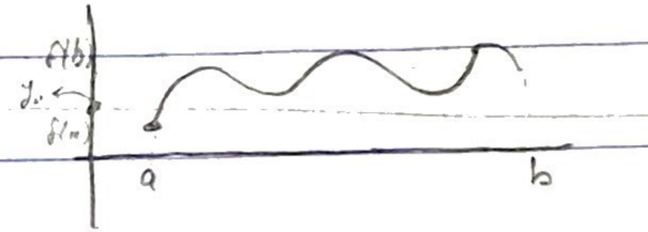
lemma : suppose that $a<b$ and that $f:(a,b) \to \mathbb{R}$ If f is continuous at $x_0 \in [a,b]$ and $f(x_0) > 0$ then $\exists$ an $\varepsilon > 0$ and a point $x_1 \in [a,b]$ such that $x_1 > x_0$ and $f(x) > \varepsilon$, $\forall x \in [x_0, x_1]$.

مش مطلوبة

Thm 5: Intermediate Value Theorem :

suppose that $a < b$ and that $f : [a,b] \to \mathbb{R}$ is continuous If y_0 lies between $f(a)$ and $f(b)$ then $\exists$ an $x_0 \in (a,b)$, such that $f(x_0) = y_0$.

يوصل proof:



$X = \cos X$

$f(x) = x - \cos x \qquad [0, \frac{\pi}{2}]$

$f(0) = -1 \quad < 0$

$f(\frac{\pi}{2}) = \frac{\pi}{2} \quad > 0$

expl: Prove that $f(x) = \begin{cases} \frac{|x|}{x} , & x \neq 0 \\ 1 , & x = 0 \end{cases}$ is continuous on $(-\infty, 0)$ and $[0, \infty)$, discontinuous at $x = 0$, and both $f(0^+)$ and $f(0^-)$ exist.

Sol:

$$f(x) = \begin{cases} 1 , & x \geq 0 \\ -1 , & x < 0 \end{cases}$$

By Thm

since $f(x) = 1$ for $x \geq 0$, it is clear that $f(0^+) = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 1 = 1$ exist

and $f(x) \to f(a)$ as $x \to a$ for any $a > 0$. In particular f is continuous on $[0, \infty)$. similarly, $f(0^-) = \lim_{x \to 0^-} f(x) = -1$ exist.

and f is continuous on $(-\infty, 0)$

Finally:

since $f(0^+) \neq f(0^-)$ then $\lim_{x \to 0} f(x)$ DNE.

$\therefore$ f is discontinuous at $x = 0$.

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or By

$x_n = \frac{1}{n} \to 0$ $\quad\quad$ $y_n = -\frac{1}{n} \to 0$

But $f(x_n) = 1 \to 1$ $\quad\quad$ $f(y_n) = -1 \to -1$

لو بيمنح 2-sequ. مشان أثبت انو discontinuous عند الصفر.

$(\sin \circ \frac{1}{x})(x)$

exp2: Assume that $\sin x$ is conti. on $(-\infty, \infty)$, prove that $f(x) = \begin{cases} \sin(\frac{1}{x}), & x \neq 0 \\ 1, & x = 0 \end{cases}$ is cont. on $(-\infty, 0)$ and $(0, \infty)$, discont at 0, and neither $f(0^+)$ nor $f(0^-)$ exsits.

ما بنربط بالنظرية لانه $f(0^+) \neq f(0^-)$

يعني زي exp1 ما بنربط

Sol: $g(x) = \frac{1}{x}$ is cont. for $x \neq 0$. يعني على $\mathbb{R} \setminus \{0\}$.

مش ضرب

Hence, By Thm 3, $f(x) = (\sin \circ g)(x)$

$= \sin(\frac{1}{x})$.

is conti. on $(-\infty, 0) \cup (0, \infty)$.

To prove $f(0^+)$ DNE. let $x_n = \dfrac{1}{\frac{\pi}{2} + n\pi} \to 0$ as $n \to \infty$

and observed that $\sin(\frac{1}{x_n}) = \sin(\frac{\pi}{2} + n\pi)$

$= (\sin \frac{\pi}{2})(\cos n\pi) + \cancel{\cos \frac{\pi}{2} \sin n\pi}$

$= (-1)^n, \; n \in \mathbb{N}$

convergе + dicreasing ← $x_n \downarrow 0$ But $(-1)^n$ doesn't converge.

By seq. charachterization of continuaty $f(0^+)$ DNE.

similarly: $f(0^-)$ DNE.

$y_n = -\dfrac{1}{\frac{\pi}{2} + n\pi} \longrightarrow 0$ as $n \to \infty$ But $f(y_n) = (-1)^{n+1}$ has No limit.

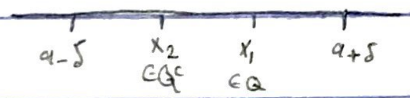
exp 3 : The Dirichlet function is defined by $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \text{ rational} \\ 0, & x \notin \mathbb{Q} \text{ irrational} \end{cases}$

prove that every point $x \in \mathbb{R}$ is a point of discontinuity of f (i.e.

(i.e. f is nowhere continuous).

proof: By density of rationals and Irrationals, given $a \in \mathbb{R}$ and $\delta > 0$

$\exists\, x_1 \in \mathbb{Q}$ and $x_2 \in \mathbb{Q}^c$ s.t $|x_1 - a| < \delta$ and $|x_2 - a| < \delta$



since $f(x_1) = 1$ and $f(x_2) = 0$, Then f cannot be cont. at $a \in \mathbb{R}$

$\rightarrow \; |f(x_1) - \underset{a \in \mathbb{Q}^c}{f(a)}| = |1 - 0| = 1 \not< \varepsilon, \forall \varepsilon > 0$ (take $\varepsilon_0 = \frac{1}{2}$)

$\rightarrow \; |f(x_2) - \overset{a \in \mathbb{Q}}{f(a)}| = |0 - 1| = 1 \not< \varepsilon, \forall \varepsilon > 0$ (take $\varepsilon_0 = \frac{1}{2}$).

OR suppose f is continuous at $a \in \mathbb{R}$.

$a \in \mathbb{R}, \exists\, x_n \in \mathbb{Q} : x_n \to a$

$f(x_n) \to f(a)$

$1 \qquad -1$ or 1

$\times$

If $a \in \mathbb{Q}^c$, we have a contradiction

$f(\frac{4}{9}) = \frac{1}{9}$

$f(\frac{2}{3}) = \frac{1}{3}$

$f(\frac{4}{11}) = \frac{1}{3}$

$f(\sqrt{2}) = 0$

اختصار تام

ex4: prove that $f(x) = \begin{cases} \frac{1}{q} , & x = \frac{p}{q} \in \mathbb{Q} \text{ (in reduced form)} \\ 0 , & x \notin \mathbb{Q} \end{cases}$

is continuous at every irrational in $(0,1)$ But discontinuous at every $\mathbb{Q}$ in $(0,1)$

First, we shall prove that f is discont. at every rational in $(0,1)$

By contradiction — نفرض أن f discont.

let $a \in (0,1)$ be a rational and spse that f is cont. at a

$\hookrightarrow x_n \to a$
$f(x_n) \to f(a)$

If x_n is a seq. of irrational s.t $x_n \to a$ as $n \to \infty$

Then $\underbrace{f(x_n)}_{=0, \forall n} \to f(a)$ as $n \to \infty$.

i.e $f(a) = 0$ But $f(a) \neq 0$ By def of f. a contradiction

By seq. char. of contin.

Next, we want to prove that f is conti. at every irrational in $(0,1)$.

Indeed, let $a \in (0,1)$ Be irrational

We must show that $f(x_n) \to f(a)$ as $n \to \infty$ for every seq.

$x_n \in (0,1)$ which satisfies $x_n \to a$ as $n \to \infty$

We may spse that $x_n \in \mathbb{Q}$, $\forall n \in \mathbb{N}$

We write $x_n = \frac{p_n}{q_n}$ in reduced form

since $f(a) = 0$, it sufficies to show that $q_n \to \infty$ as $n \to \infty$ $(f(x_n) = f(\frac{p_n}{q_n}) = \frac{1}{q_n} \to f(a) =$

By contradiction

spse to the contrary that there exist integers $n_1 < n_2 < \dots$ s.t

$|q_{n_k}| \leq M < \infty$ for $K \in \mathbb{N}$.

since $x_{n_k} \in (0,1)$, it follows that the set $E = \{ x_{n_k} : \frac{p_{n_k}}{q_{n_k}} ; K \in \mathbb{N} \}$

Contains only a finite number of pts.

Hence, the limit of any seq in E must belong to E, a contradiction. X.

since a is such a limit and a is irrational

عيل جداً ما دايك ؟!

RMK : The composition of two function $g \circ f$ can be nowhere continuous even though f is discont. only on $\mathbb{Q}$ and g is discont. at only one point.

proof. let $f(x) = \begin{cases} \frac{1}{q} , & x = \frac{p}{q} \in \mathbb{Q} \\ 0 , & x \notin \mathbb{Q} \end{cases}$ is discont. on $\mathbb{Q}$.

$g(x) = \begin{cases} 1 , & x \neq 0 \\ 0 , & x = 0 \end{cases}$ → discont at $x=0$

clearly $\underbrace{(g \circ f) = \begin{cases} 1 , & x \in \mathbb{Q} \\ 0 , & x \notin \mathbb{Q} \end{cases}}_{\text{is Nowhere conti.}}$ → $x \in \mathbb{Q}$: $g(f(x)) = g\left(\frac{1}{q}\right)$; $x \notin \mathbb{Q}$: $g(f(x)) = g(0)$

Hence, $g \circ f$ is Dirichlet function Nowhere continuous By exp 3.