Exercises :

2.2.0 : True or False .

a. If $x_n \to \infty$ and $y_n \to -\infty$ then $x_n + y_n \to 0$ as $n \to \infty$. False

$x_n = n^2$ and $y_n = -n \Rightarrow x_n + y_n = n^2 - n \to \infty$ as $n \to \infty$.

b. If $x_n \to -\infty$ then $\frac{1}{x_n} \to 0$ as $n \to \infty$. True

let $\varepsilon > 0$, If $x_n \to -\infty$ as $n \to \infty$ then let $N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow x_n < -\frac{1}{\varepsilon}$

Then $x_n < 0$ so $|x_n| = -x_n > 0$.

$\left(x_n < -\frac{1}{\varepsilon} \right)$ multiply $\frac{\varepsilon}{-x_n}$ which is positive.

we obtain $-\varepsilon < \frac{1}{x_n}$ i.e $\left| \frac{1}{x_n} \right| = -\frac{1}{x_n} < \varepsilon$. $\boxtimes$

c. If $x_n \to 0$ then $\frac{1}{x_n} \to \infty$ as $n \to \infty$ False

$x_n = \frac{(-1)^n}{n}$, Then $\frac{1}{x_n} = (-1)^n n$ has No limit as $n \to \infty$.

d. If $x_n \to \infty$, then $\left(\frac{1}{2}\right)^{x_n} \to 0$ as $n \to \infty$ True.

2.2.1 : prove that each of the following sequences converges to zero.

a. $x_n = \sin(\log n + n^2 + e^{n^2})/n$.

$|x_n| \leq \frac{1}{n} \to 0$ as $n \to \infty$

Then $-\frac{1}{n} < x_n < \frac{1}{n}$

since $\lim_{n\to\infty} \frac{1}{n} = \lim_{n\to\infty} -\frac{1}{n} = 0$ By def (proved on 2.1)

Then $\lim_{n\to\infty} x_n = 0$ By squeeze Theorem.

b. $x_n = \dfrac{2n}{n^2+\pi} = \dfrac{2n}{\frac{(1+\pi)}{n^2}}$

2.2.2 : use def to prove that each of the following seq. diverges to $+\infty$ or $-\infty$.

a. $x_n = n^2 - n \to \infty$

let $M \in \mathbb{R}$ we need to find $N \in \mathbb{N}$ s.t $n \geq N \Rightarrow n^2 - n > M$.

By Archimedian principle $\exists N \in \mathbb{N}$ s.t $N > \max\{2, M\}$

Then $n \geq N$ implise $n^2 - n$

$= n(n-1)$

$\geq N(N-1)$ want $> M$ so need $N > M$ and $N > 2$ so

$> M(2-1)$

$> M$ ⊠

b. on note.

c. $x_n = \frac{n^2+1}{n} = \frac{n^2}{n} + \frac{1}{n} = n + \frac{1}{n} \to \infty$

let $M \in \mathbb{R}$ we need to find $N \in \mathbb{N}$ s.t $n \geq N \Rightarrow n + \frac{1}{n} > M$.

By Archimedian principle $\exists N \in \mathbb{N}$ s.t $N > M$

Then $n \geq N$ implise $n + \frac{1}{n}$ $\to > 0$

$\geq N + 0$ want $N > M$ so

$> M$ ⊠

d. $x_n = n^2(2 + \sin(n^3 + n + 1)) = \infty$ $\to -1 < \sin x < 1 \to 2 + \sin x \geq 2 - 1 = 1$.

let $M \in \mathbb{R}$ we need to find $N \in \mathbb{N}$ s.t $n \geq N \Rightarrow n^2(2 + \sin\theta) > M$.

By Archimedian principle $\exists N \in \mathbb{N}$ s.t $N > \sqrt{M}$

Then $n \geq N$ impliese $n^2(2 + \sin\theta)$

$n^2 \geq N^2$ since $n, N \in \mathbb{N}$ $\quad = n^2(2-1)$

$= n^2$

$\geq N^2$ want $N^2 > M$ so

$> M$ ⊠

2.2.3 : Find the limit (if it exists) of each of the following sequences.

a. $x_n = \frac{(2+3n-4n^2)}{(1-2n+3n^2)} = -\frac{4}{3}$ درجة البسط = درجة المقام

b. $x_n = \frac{n^3+n-2}{2n^3+n-2} = \frac{1}{2}$ 〃 〃

الطاق مقام c. $x_n = \sqrt{3n+2} - \sqrt{n}$

$= \frac{(\sqrt{3n+2}-\sqrt{n})(\sqrt{3n+2}+\sqrt{n})}{(\sqrt{3n+2}+\sqrt{n})} = \frac{2n+2}{\sqrt{3n+2}+\sqrt{2}} \longrightarrow \infty$ (diverges)

d. $x_n = \frac{\sqrt{4n+1}-\sqrt{n-1}}{\sqrt{9n+1}-\sqrt{n+2}} = \frac{1}{2}$ يضرب بـ $\frac{1}{\sqrt{n}}$ ويكمل ويطلع الحل

2.2.4 :

a. ✓

b. prove corollary 2.16 :

By symmetry we may suppose that $x = y = \infty$

since $y_n \to \infty$ implise $y_n > 0$ for n large (we can apply Thm 2.15)

directly to obtain the conclusion when $\alpha > 0$

For the case $\alpha < 0$, $x_n > M$ implies $\alpha x_n < \alpha M$.

since any $M_0 \in \mathbb{R}$ can be written as αM for some $M \in \mathbb{R}$

we see by def that $x_n \to -\infty$ as $n \to \infty$

2.2.5: suppose that $x \in \mathbb{R}$, $x_n \geq 0$ and $x_n \to x$ as $n \to \infty$ prove that $\sqrt{x_n} \to \sqrt{x}$ as $n \to \infty$.

case 1: $x = 0$.

let $\varepsilon > 0$ and choose N so large that $n \geq N$ implies $|x_n| < \varepsilon^2$

$\sqrt{x_n} < \varepsilon$ for $n \geq N$ i.e $\sqrt{x_n} \to 0$ as $n \to \infty$.

case 2: $x > 0$ Then

$$\sqrt{x_n} - \sqrt{x} = (\sqrt{x_n} - \sqrt{x})\left(\frac{\sqrt{x_n} + \sqrt{x}}{\sqrt{x_n} + \sqrt{x}}\right) = \frac{x_n - x}{\sqrt{x_n} + \sqrt{x}}$$

Since $\sqrt{x_n} \geq 0$, it follows that

$$|\sqrt{x_n} - \sqrt{x}| \leq \frac{|x_n - x|}{\sqrt{x}}$$ converges to 0.

Hence, it follows from squeeze Thm. that $\sqrt{x_n} \to \sqrt{x}$ as $n \to \infty$

2.2.6: prove that given $x \in \mathbb{R}$ there is a sequence $r_n \in \mathbb{Q}$ s.t $r_n \to x$ as $n \to \infty$.

By the density Rationals, there is an r_n between $\frac{x+1}{n}$ and x $\forall n \in \mathbb{N}$.

since $|x - r_n| < \frac{1}{n}$ it follows from the squeeze Theorem $r_n \to \infty$ as $n \to \infty$.

$$-\frac{1}{n} < x - r_n < \frac{1}{n}$$

2.2.7: suppose that x and y are extended real numbers and that $\{x_n\}$, $\{y_n\}$ and $\{w_n\}$ real seq.

a. [squeeze Thm for $\bar{R}$] If $x_n \to x$ and $y_n \to x$ as $n \to \infty$ and

$x_n \leq w_n \leq y_n$ for $n \in \mathbb{N}$ prove that $w_n \to x$ as $n \to \infty$.

By Theorem 1 [squeeze Theorem] we may suppose that $|x| = \infty$

By symmetry we may suppose that $x = \infty$

By def given $M \in \mathbb{R}$ there is an $N \in \mathbb{N}$ s.t $n \geq N$ implies $x_n > M$.

since $w_n \geq x_n$ it follows that $w_n > M$ $\forall n > N$

By def, then $w_n \to \infty$ as $n \to \infty$.

b. [Comparison Theorem for $\bar{\mathbb{R}}$]. If $x_n \to x$ and $y_n \to y$ as $n \to \infty$ and $x_n \leq y_n$

for $n \in \mathbb{N}$ prove that $x \leq y$.

If x and y are finite, then the result follows from Thm 5

If $x = y = \pm\infty$ or $-x = y = \infty$ there is nothing to prove.

consider the case $x = \infty$ and $y = -\infty$.

But By def 1 with $M = 0 \to x_n > 0 > y_n$ for n sufficiently large.

Which contradiction with hypothesis $x_n \leq y_n$.

2.2.8: using the result in 2.2.5, prove the following results.

a. suppose that $0 \leq x_1 \leq 1$ and $x_{n+1} = 1 - \sqrt{1 - x_n}$ for $n \in \mathbb{N}$. If $x_n \to x$ as $n \to \infty$ then $x = 0$ or $x = 1$.

take the limit of $x_{n+1} = 1 - \sqrt{1 - x_n}$ as $n \to \infty$

we obtain $x = 1 - \sqrt{1-x} \to x+1 = -\sqrt{1-x} \to (1 - x = \sqrt{1-x})^2$

$$\cancel{1} - 2x + x^2 = \cancel{1} - \cancel{x}$$

$$x^2 - x = 0$$

$$x(x-1) = 0$$

$$x = 0 \quad \text{or} \quad x = 1$$

b. suppose that $x_1 > 3$ and $x_{n+1} = 2 + \sqrt{x_n - 2}$ for $n \in \mathbb{N}$. If $x_n \to x$ as $n \to \infty$ then $x = 3$.

take the limit of $x_{n+1} = 2 + \sqrt{x_n - 2}$ as $n \to \infty$

we obtain $x = 2 + \sqrt{x - 2}$

$-2 \quad -2$

$(x - 2 = \sqrt{x-2})^2$

$x^2 - 4x + 4 = x - 2 \longrightarrow x^2 - 5x + 6 = 0$

$-x + 2 \quad -x + 2$

$(x-2)(x-3) = 0$

$x = 2$ or $x = 3$

But $x_1 > 3$ so the limit must be $x = 3$.

c. suppose that $x_1 \geq 0$ and $x_{n+1} = \sqrt{2 + x_n}$ for $n \in \mathbb{N}$. If $x_n \to x$ as $n \to \infty$ then $x = 2$.

take the limit of $x_{n+1} = \sqrt{2 + x_n}$ as $n \to \infty$

we obtain $(x = \sqrt{2 + x})^2$

$x^2 = 2 + x$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

So $x = 2$ or $x = -1$

But $x_{n+1} = \sqrt{2 + x_n} \geq 0$ by def (all square roots are nonnegative)

so the limit must be $x = 2$.