Laplace Transform

In this Chapter we will learn how to solve some DE's using new technique called Laplace Transform.

Question: Why Laplace Transform?

Answer:
- → To solve nonhomogeneous DE's for y_h and y_p in different way
- → To solve some DE's with discontinuity factors or external factors or forces
- → To solve some Integral Equations

Recall that the Improper Integral of $f(t)$ defined on an unbounded interval $[a,\infty)$ is defined by

$$\int_a^{\infty} f(t)\,dt = \lim_{b\to\infty} \int_a^{b} f(t)\,dt$$

This Improper Integral converges if ① $\int_a^b f(t)\,dt$ exists

and ② limit exists

To guarantee ①, we assume $f(t)$ Piecewise Continuous (PC)

Def The function $f(t)$ is PC on interval $I=(\alpha,\beta)$ if I can be partioned into small K sub-intervals

$$\alpha = t_0 < t_1 < \cdots < t_K = \beta$$

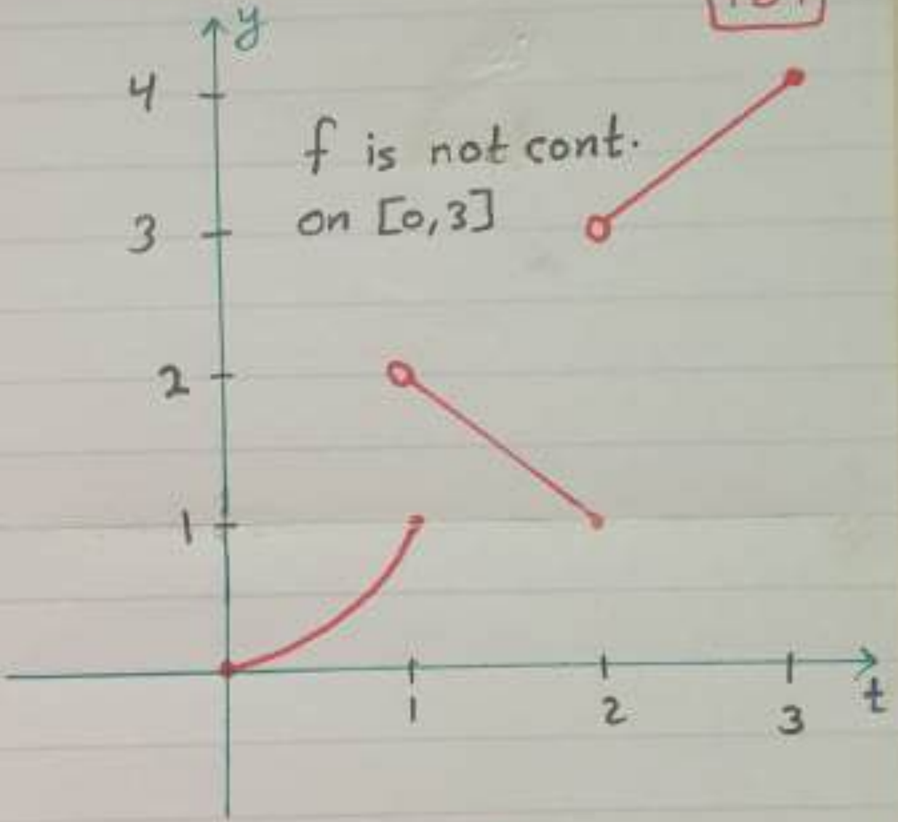
s.t:

① $f(t)$ is cont. on each sub-interval and
② $f(t)$ has finite limit at the boundary of each sub-interval

<u>Exp</u> show that

$$f(t)=\begin{cases} t^2 , & 0\le t\le 1 \\ 3-t , & 1<t\le 2 \\ t+1 , & 2<t\le 3 \end{cases}$$

is PC on [0,3]



f is not cont. on [0,3]

- f is cont. on each sub-interval : (0,1) , (1,2) , (2,3)

- $\lim_{t\to 0^+} f(t)=0$, $\lim_{t\to 1^-} f(t)=1$, $\lim_{t\to 1^+} f(t)=2$,
  $\lim_{t\to 2^-} f(t)=1$, $\lim_{t\to 2^+} f(t)=3$, $\lim_{t\to 3^-} f(t)=4$ } All finite

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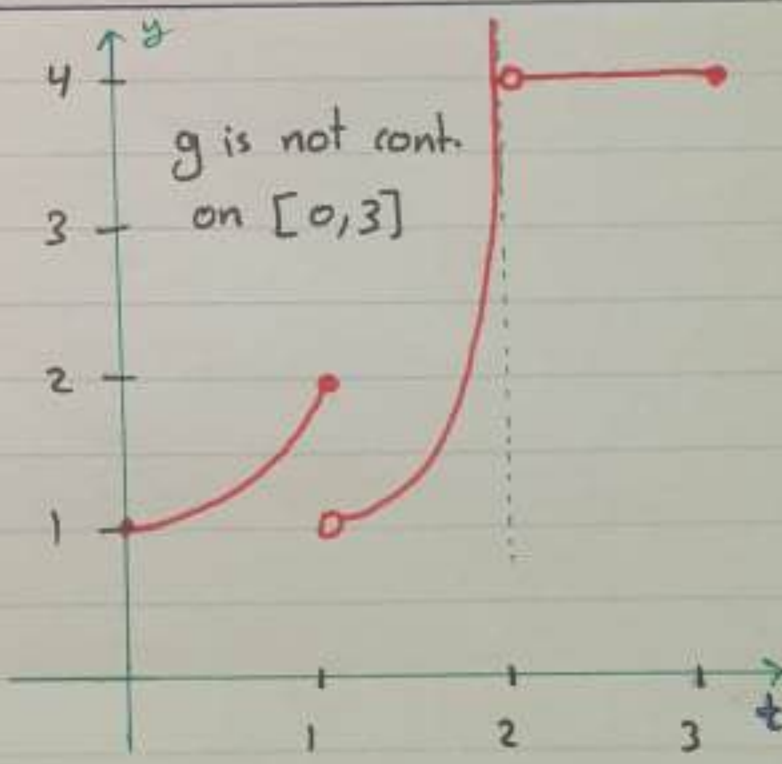
<u>Exp</u> Show that

$$g(t)=\begin{cases} t^2+1 & , 0\le t\le 1 \\ \dfrac{1}{2-t} & , 1<t< 2 \\ 4 & , 2<t\le 3 \end{cases}$$

is not PC on [0,3]



g is not cont. on [0,3]

- g is cont. on each subinterval (0,1), (1,2), (2,3)

- But $\lim_{t\to 2^-} g(t)=\infty$

Def The Laplace Transform of the function $f(t)$ is defined by

$$L\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\, dt = F(s) \quad , \quad s \in \mathbb{R}^+, \; f \text{ is PC}$$

$$\xrightarrow[f(t)]{\text{input}} \boxed{L\{f(t)\}} \xrightarrow[F(s)]{\text{output}}$$

Laplace Transform (LT)

$$\xleftarrow[f(t)]{\text{output}} \boxed{L^{-1}(F(s))} \xleftarrow[F(s)]{\text{input}}$$

Inverse Transform

Exp Find Laplace Transform of the following functions:

① $f(t) = c \quad , \quad c$ is constant

$$L\{f(t)\} = L\{c\} = \int_0^{\infty} e^{-st} c \, dt$$

$$= c \lim_{b \to \infty} \int_0^b e^{-st} dt = c \lim_{b \to \infty} \frac{-1}{s} e^{-st} \Big|_0^b$$

$$= \frac{-c}{s} \lim_{b \to \infty} \left[e^{-sb} - e^{0} \right] = \frac{-c}{s} \left[0 - 1 \right]$$

$$= \frac{c}{s}$$

$$= F(s)$$

Hence, $L^{-1}(F(s)) = L^{-1}\left(\frac{c}{s}\right) = c = f(t)$

Exp $L\{2\} = \frac{2}{s}$

$L\{\pi\} = \frac{\pi}{s}$

$L\{\sqrt{5}\} = \frac{\sqrt{5}}{s}$

$L^{-1}\left(\frac{e}{s}\right) = e$

$L^{-1}\left(\frac{\sqrt{3}}{2s}\right) = L^{-1}\left(\frac{\frac{\sqrt{3}}{2}}{s}\right) = \frac{\sqrt{3}}{2}$

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② $f(t) = t$

$$F(s) = L\{f(t)\} = L\{t\} = \int_0^\infty e^{-st}\, t\, dt$$

$$= \lim_{b\to\infty} \int_0^b t\, e^{-st}\, dt$$

t	$(+)$	e^{-st}
1	$(-)$	$-\frac{1}{s} e^{-st}$
0		$\frac{1}{s^2} e^{-st}$

$$= \lim_{b\to\infty} \left[\frac{-t}{s} e^{-st} - \frac{1}{s^2} e^{-st}\right]\Bigg|_0^b$$

$$= \lim_{b\to\infty} \left[\frac{-1}{s} \frac{b}{e^{sb}} - \frac{1}{s^2} e^{-sb} - \left(0 - \frac{1}{s^2}\right)\right]$$

$$= 0 - 0 + \frac{1}{s^2}$$

$$\lim_{b\to\infty} \frac{b}{e^{sb}} = \lim_{b\to\infty} \frac{1}{s e^{sb}} = 0$$

$$= \frac{1}{s^2}$$

Hence, $L^{-1}(F(s)) = L^{-1}\left(\frac{1}{s^2}\right) = t = f(t)$

③ $f(t) = t^2$

$$F(s) = L\{f(t)\} = L\{t^2\} = \int_0^\infty e^{-st}\, t^2\, dt$$

$$= \lim_{b\to\infty} \int_0^b t^2 e^{-st}\, dt$$

$$\begin{array}{lll} t^2 & \xrightarrow{(+)} & e^{-st} \\ 2t & \xrightarrow{(-)} & \frac{-1}{s} e^{-st} \\ 2 & \xrightarrow{(+)} & \frac{1}{s^2} e^{-st} \\ 0 & & \frac{-1}{s^3} e^{-st} \end{array}$$

$$= \lim_{b\to\infty} \left[\frac{-t^2}{s} e^{-st} - \frac{2t}{s^2} e^{-st} - \frac{2}{s^3} e^{-st} \right]\Big|_0^b$$

$$= \lim_{b\to\infty} \left[\frac{-1}{s} \frac{b^2}{e^{sb}} - \frac{2}{s^2} \frac{b}{e^{sb}} - \frac{2}{s^3} e^{-sb} - \left(0 - 0 - \frac{2}{s^3}\right) \right]$$

$$= 0 - 0 - 0 + \frac{2}{s^3}$$

$$= \frac{2}{s^3}$$

Hence, $L^{-1}(F(s)) = L^{-1}\left(\frac{2}{s^3}\right) = t^2 = f(t)$

One can show that if $f(t) = t^n$ then $F(s) = \frac{n!}{s^{n+1}}$

$$L\{t^n\} = \frac{n!}{s^{n+1}}$$

Hence, $L^{-1}\left(\frac{n!}{s^{n+1}}\right) = t^n$

Exp $L\{t\} = L\{t^1\} = \frac{1!}{s^{1+1}} = \frac{1}{s^2}$

$$L\{t^2\} = \frac{2!}{s^{2+1}} = \frac{2}{s^3}$$

$$L\{t^3\} = \frac{3!}{s^{3+1}} = \frac{6}{s^4}$$

Exp $L^{-1}\left(\frac{4}{s^3}\right) = \frac{4}{2!} L^{-1}\left(\frac{2!}{s^3}\right) = 2t^2$

$$L^{-1}\left(\frac{7}{s^6}\right) = \frac{7}{5!} L^{-1}\left(\frac{5!}{s^6}\right) = \frac{7}{5!} t^5$$

$$L^{-1}\left(\frac{2s^2-4s}{s^3}\right) = L^{-1}\left(\frac{2}{s} - \frac{4}{s^2}\right) = 2L^{-1}\left(\frac{1}{s}\right) - 4L^{-1}\left(\frac{1}{s^2}\right)$$

$$= 2(1) - (4)t$$

$$= 2 - 4t$$

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(4) $f(t) = c_1 f_1(t) + c_2 f_2(t)$

$$F(s) = L\{f(t)\} = L\{c_1 f_1(t) + c_2 f_2(t)\}$$

$$= \int_0^\infty e^{-st} (c_1 f_1(t) + c_2 f_2(t))\, dt$$

$$= c_1 \int_0^\infty e^{-st} f_1(t)\, dt + c_2 \int_0^\infty e^{-st} f_2(t)\, dt$$

$$= c_1 L\{f_1(t)\} + c_2 L\{f_2(t)\}$$

$$= c_1 F_1(s) + c_2 F_2(s)$$

Hence, $L^{-1}\left(c_1 F_1(s) + c_2 F_2(s)\right) = c_1 L^{-1}(F_1(s)) + c_2 L^{-1}(F_2(s))$

$$= c_1 f_1(t) + c_2 f_2(t)$$

Exp $L\{2 + 3t^4\} = L\{2\} + 3L\{t^4\}$

$$= \frac{2}{s} + 3\,\frac{4!}{s^5}$$

(5) $f(t) = e^{at}$, $a \in \mathbb{R}$

$$F(s) = L\{f(t)\} = L\{e^{at}\} = \int_0^\infty e^{-st} e^{at}\, dt$$

$$= \int_0^\infty e^{(a-s)t}\, dt = \lim_{b\to\infty} \int_0^b e^{-(s-a)t}\, dt$$

$$= \lim_{b\to\infty} \frac{-1}{s-a} e^{-(s-a)t}\Big|_0^b$$

$$= \frac{1}{s-a} \lim_{b\to\infty}\left[-e^{-(s-a)b} + 1\right]$$

$$= \frac{1}{s-a}$$

Hence, $L^{-1}\left(\frac{1}{s-a}\right) = L^{-1}\left(L\{e^{at}\}\right) = e^{at} = f(t)$

Exp $L\{e^{2t}\} = \frac{1}{s-2}$

$$L\{e^{-et}\} = \frac{1}{s--e} = \frac{1}{s+e}$$

$$L\left\{\frac{1}{e^{5t}}\right\} = L\{e^{-5t}\} = \frac{1}{s+5}$$

$$L^{-1}\left(\frac{2}{s-\sqrt{\pi}}\right) = 2L^{-1}\left(L\{e^{\sqrt{\pi}t}\}\right) = 2e^{\sqrt{\pi}t}$$

$$L^{-1}\left(\frac{s-2}{s^2-4}\right) = L^{-1}\left(\frac{s-2}{(s-2)(s+2)}\right) = L^{-1}\left(\frac{1}{s+2}\right) = L^{-1}\left(L\{e^{-2t}\}\right) = e^{-2t}$$

⑥ $f(t) = \sin at$

$$F(s) = L\{f(t)\} = L\{\sin at\} = \int_0^\infty e^{-st} \sin at \, dt$$

$$= \lim_{b\to\infty} \int_0^b e^{-st} \sin at \, dt$$

$$\begin{array}{lll} e^{-st} & \xrightarrow{(+)} & \sin at \\ -se^{-st} & \xrightarrow{(-)} & \frac{-1}{a}\cos at \\ s^2 e^{-st} & \xrightarrow{\int} & \frac{-1}{a^2}\sin at \end{array}$$

$$= \lim_{b\to\infty} \left[-\frac{1}{a} e^{-st} \cos at - \frac{s}{a^2} e^{-st} \sin at \right]_0^b - \frac{s^2}{a^2} \int_0^\infty e^{-st} \sin at \, dt$$

$$= \left[(0 - 0) - \left(\frac{-1}{a} - 0\right) \right] - \frac{s^2}{a^2} F(s)$$

$$F(s) + \frac{s^2}{a^2} F(s) = \frac{1}{a}$$

$$F(s)\left(1 + \frac{s^2}{a^2}\right) = \frac{1}{a} \Rightarrow F(s)\left(\frac{s^2+a^2}{a^2}\right) = \frac{1}{a}$$

$$F(s) = \frac{a}{s^2+a^2}$$

Hence, $L^{-1}\left(\frac{a}{s^2+a^2}\right) = L^{-1}\left(L\{\sin at\}\right) = \sin at$

Exp $L\{\sin 3t\} = \frac{3}{s^2+9}$

$L\{\sin \sqrt{2} t\} = \frac{\sqrt{2}}{s^2+2}$

$$L^{-1}\left(\frac{1}{s^2+4}\right) = \frac{1}{2} L^{-1}\left(\frac{2}{s^2+4}\right) = \frac{1}{2} L^{-1}\left(L\{\sin 2t\}\right) = \frac{1}{2}\sin 2t$$

$$L^{-1}\left(\frac{3}{s^2+7}\right) = \frac{3}{\sqrt{7}} L^{-1}\left(\frac{\sqrt{7}}{s^2+7}\right) = \frac{3}{\sqrt{7}} L^{-1}\left(L\{\sin\sqrt{7}t\}\right) = \frac{3}{\sqrt{7}}\sin\sqrt{7}t$$

(7) $f(t) = \cos at$

$$F(s) = L\{f(t)\} = L\{\cos at\} = \int_0^\infty e^{-st} \cos at \, dt = \frac{s}{s^2+a^2}$$

we do same work as in (6)

Hence, $L^{-1}\left(\frac{s}{s^2+a^2}\right) = L^{-1}\left(L\{\cos at\}\right) = \cos at$

Exp $L\{\cos 4t\} = \frac{s}{s^2+16}$

$L\{3\cos 2t\} = 3 \frac{s}{s^2+4} = \frac{3s}{s^2+4}$

$L\{\cos e\} = \frac{\cos e}{s}$ since $\cos e$ is number

$$L^{-1}\left(\frac{6s}{s^2+1}\right) = 6 L^{-1}\left(L\{\cos t\}\right) = 6\cos t$$

$$L^{-1}\left(\frac{2s-4}{s^2+3}\right) = L^{-1}\left(\frac{2s}{s^2+3}\right) - L^{-1}\left(\frac{4}{s^2+3}\right)$$

$$= 2L^{-1}\left(\frac{s}{s^2+3}\right) - \frac{4}{\sqrt{3}} L^{-1}\left(\frac{\sqrt{3}}{s^2+3}\right)$$

$$= 2L^{-1}\left(L\{\cos\sqrt{3}\,t\}\right) - \frac{4}{\sqrt{3}} L^{-1}\left(L\{\sin\sqrt{3}\,t\}\right)$$

$$= 2\cos\sqrt{3}\,t - \frac{4}{\sqrt{3}}\sin\sqrt{3}\,t$$

Exp $L\{2\sin 3t - 10t^2 + 5e^{-3t} + \pi + 2\cos\sqrt{7}\,t\}$

$$= 2\left(\frac{3}{s^2+9}\right) - 10\left(\frac{2!}{s^3}\right) + 5\left(\frac{1}{s+3}\right) + \frac{\pi}{s} + 2\left(\frac{s}{s^2+7}\right)$$

Exp Find Laplace Inverse of $F(s) = \dfrac{1}{s^2 - 5s + 6}$

$$L^{-1}(F(s)) = L^{-1}\left(\frac{1}{s^2-5s+6}\right) = L^{-1}\left(\frac{1}{(s-2)(s-3)}\right)$$

$$= L^{-1}\left(\frac{A}{s-2} + \frac{B}{s-3}\right)$$

using Partial Fraction

$$A = \frac{1}{\boxed{2}-3} = -1$$

$$B = \frac{1}{\boxed{3}-2} = 1$$

$$= L^{-1}\left(\frac{-1}{s-2} + \frac{1}{s-3}\right)$$

$$= -L^{-1}\left(L\{e^{2t}\}\right) + L^{-1}\left(L\{e^{3t}\}\right)$$

$$= -e^{2t} + e^{3t} = f(t)$$

Exp Find Laplace Inverse of $G(s) = \dfrac{15-s}{s^2+5s}$

$$g(t) = L^{-1}(G(s)) = L^{-1}\left(\frac{15-s}{s^2+5s}\right) = L^{-1}\left(\frac{15-s}{s(s+5)}\right)$$

$$= L^{-1}\left(\frac{A}{s} + \frac{B}{s+5}\right)$$

using Partial Fraction

$$A = \frac{15-\boxed{0}}{\boxed{0}+5} = 3$$

$$B = \frac{15-\boxed{-5}}{\boxed{-5}} = \frac{+20}{-5} = -4$$

$$= L^{-1}\left(\frac{3}{s} + \frac{-4}{s+5}\right)$$

$$= L^{-1}\left(\frac{3}{s}\right) + -4\, L^{-1}\left(\frac{1}{s--5}\right)$$

$$= 3 + -4\, e^{-5t}$$

⑧ $f(t) = \sinh at$

$$F(s) = L\{f(t)\} = L\{\sinh at\} = L\left\{\frac{e^{at} - e^{-at}}{2}\right\}$$

$$= \frac{1}{2}\left[L\{e^{at}\} - L\{e^{-at}\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right]$$

$$= \frac{1}{2}\;\frac{\cancel{s}+a-\cancel{s}+a}{s^2-a^2}$$

$$= \frac{a}{s^2-a^2}$$

Hence, $L^{-1}\left(\frac{a}{s^2-a^2}\right) = L^{-1}\left(L\{\sinh at\}\right) = \sinh at$

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⑨ $f(t) = \cosh at$

$$F(s) = L\{f(t)\} = L\{\cosh at\} = L\left\{\frac{e^{at} + e^{-at}}{2}\right\}$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right] = \frac{1}{2}\;\frac{s+\cancel{a}+s-\cancel{a}}{s^2-a^2} = \frac{s}{s^2-a^2}$$

Hence, $L^{-1}\left(\frac{s}{s^2-a^2}\right) = L^{-1}\left(L\{\cosh at\}\right) = \cosh at$

Exp Find $h(t)$ if $H(s) = \frac{2s-12}{s^2-6}$

$$h(t) = L^{-1}(H(s)) = L^{-1}\left(\frac{2s-12}{s^2-6}\right) = 2L^{-1}\left(\frac{s}{s^2-6}\right) - \frac{12}{\sqrt{6}}L^{-1}\left(\frac{\sqrt{6}}{s^2-6}\right)$$

$$= 2\cosh\sqrt{6}\,t - \frac{12}{\sqrt{6}}\sinh\sqrt{6}\,t$$

Exp Find Laplace Transform of

(A) $f(t) = 2\sinh 7t \Rightarrow F(s) = (2)\dfrac{7}{s^2-49} = \dfrac{14}{s^2-49}$

(B) $r(t) = 1 - 3\cosh t \Rightarrow R(s) = \dfrac{1}{s} - \dfrac{3s}{s^2-1}$

(C) $h(t) = \begin{cases} 1 & , \quad 0 \le t \le 1 \\ 5 & , \quad t = 1 \\ 0 & , \quad t > 1 \end{cases}$

$$H(s) = L\{h(t)\} = \int_0^\infty e^{-st}\, h(t)\, dt$$

$$= \int_0^1 e^{-st}(1)\, dt + \int_1^1 e^{-st}(5)\, dt + \int_1^\infty e^{-st}(0)\, dt$$

$$= -\frac{1}{s} e^{-st}\Big|_0^1 + 0 + 0$$

$$= -\frac{1}{s}\left(e^{-s} - e^{0}\right)$$

$$= \frac{1}{s} - \frac{e^{-s}}{s} = \frac{1-e^{-s}}{s}$$

(D) $s(t) = -2 + t^3 - e^{-t} + 2\sin\frac{t}{2} - \cos 8t + \sinh\sqrt{10}\,t$

$$S(s) = L\{s(t)\} = L\{-2\} + L\{t^3\} - L\{e^{-t}\} + 2L\{\sin\tfrac{t}{2}\} - L\{\cos 8t\} + L\{\sinh\sqrt{10}\,t\}$$

$$= \frac{-2}{s} + \frac{3!}{s^4} - \frac{1}{s+1} + (2)\frac{\frac{1}{2}}{s^2+\frac{1}{4}} - \frac{s}{s^2+64} + \frac{\sqrt{10}}{s^2-10}$$

Summary

$$L\{f(t)\} = F(s) \Rightarrow L^{-1}(F(s)) = f(t)$$

$$L\{c\} = \frac{c}{s} \Rightarrow L^{-1}\left(\frac{c}{s}\right) = c$$

$$L\{t^n\} = \frac{n!}{s^{n+1}} \Rightarrow L^{-1}\left(\frac{n!}{s^{n+1}}\right) = t^n$$

$$L\{e^{at}\} = \frac{1}{s-a} \Rightarrow L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$$

$$L\{\sin at\} = \frac{a}{s^2+a^2} \Rightarrow L^{-1}\left(\frac{a}{s^2+a^2}\right) = \sin at$$

$$L\{\sinh at\} = \frac{a}{s^2-a^2} \Rightarrow L^{-1}\left(\frac{a}{s^2-a^2}\right) = \sinh at$$

$$L\{\cos at\} = \frac{s}{s^2+a^2} \Rightarrow L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$$

$$L\{\cosh at\} = \frac{s}{s^2-a^2} \Rightarrow L^{-1}\left(\frac{s}{s^2-a^2}\right) = \cosh at$$

$$L\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 L\{f_1(t)\} + c_2 L\{f_2(t)\} = c_1 F_1(s) + c_2 F_2(s)$$

$$L\{y'\} = s\,Y(s) - y_0$$

$$L\{y''\} = s^2\,Y(s) - s\,y_0 - y_0'$$

$$L\{y'''\} = s^3\,Y(s) - s^2 y_0 - s\,y_0' - y_0''$$

6.2 Solving IVP's Using LT

To solve Initial Value Problems using Laplace Transform we still need to find $L\{y'\}$, $L\{y''\}$, ..., $L\{y^{(n)}\}$

Th 6.2.1

Assume $f(t)$ is cont. and $f'(t)$ is PC on $0 \le t \le b$. Then

$$L\{f'(t)\} = s\,L\{f(t)\} - f(0)$$

That is
$$F'(s) = s\,F(s) - f_0$$

Proof

$$L\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt = \lim_{b\to\infty} \int_0^b e^{-st} f'(t)\,dt$$

Since f' is PC on $0 \le t \le b \Rightarrow$
f' is cont. on the sub-intervals $0 < t_1 < t_2 < \dots < t_n = b$

$$L\{f'(t)\} = \lim_{b\to\infty} \left[\int_0^{t_1} e^{-st} f'(t)\,dt + \int_{t_1}^{t_2} e^{-st} f'(t)\,dt + \dots + \int_{t_{n-1}}^{b=t_n} e^{-st} f'(t)\,dt \right]$$

$u = e^{-st}$ $\quad dv = f'(t)$
$du = -se^{-st}$ $\quad v = f(t)$

$$= \lim_{b\to\infty} \left[e^{-st} f(t)\Big|_0^{t_1} + e^{-st} f(t)\Big|_{t_1}^{t_2} + \dots + e^{-st} f(t)\Big|_{t_{n-1}}^{b} \right.$$

$$\left. + s\left(\int_0^{t_1} e^{-st} f(t)\,dt + \int_{t_1}^{t_2} e^{-st} f(t)\,dt + \dots + \int_{t_{n-1}}^{b} e^{-st} f(t)\,dt \right) \right]$$

since f is cont. on $[0, b] \Rightarrow$

$$L\{f'(t)\} = \lim_{b\to\infty} \left[e^{-st} f(t)\Big|_0^b + s\int_0^b e^{-st} f(t)\,dt \right]$$

$$= \lim_{b\to\infty} \left(e^{-sb} f(b) - e^{0} f(0) \right) + s\int_0^\infty e^{-st} f(t)\,dt$$

$$L\{f'(t)\} = 0 - f(0) + s\,L\{f(t)\}$$
$$= s\,F(s) - f_0$$

Exp Show that

$$L\{y''\} = s^2 L\{y\} - s\,y(0) - y'(0)$$

Since $L\{y'\} = s\,L\{y\} - y(0)$

$$\Rightarrow \quad L\{y''\} = s\,L\{y'\} - y'(0)$$
$$= s\big(s\,L\{y\} - y(0)\big) - y'(0)$$
$$= s^2 L\{y\} - s\,y(0) - y'(0)$$

Similarly one can show that

$$L\{y^{(n)}\} = s^n L\{y\} - s^{n-1} y(0) - s^{n-2} y'(0) - \cdots - s\,y^{(n-2)}(0) - y^{(n-1)}(0)$$

Exp ① $L\{y'\} = s\,L\{y\} - y(0)$

② $L\{y''\} = s^2 L\{y\} - s\,y(0) - y'(0)$

③ $L\{y'''\} = s^3 L\{y\} - s^2 y(0) - s\,y'(0) - y''(0)$

④ $L\{y^{(4)}\} = s^4 L\{y\} - s^3 y(0) - s^2 y'(0) - s\,y''(0) - y'''(0)$

$\vdots$

Note $L\{y\} = Y(s)$, $L\{y'\} = Y'(s)$, ...

Exp Use Laplace Transform to solve the IVP:

$$y'' - y' - 2y = 0 \quad , \quad y(0) = 1 \quad , \quad y'(0) = 0$$

$$L\{y''\} - L\{y'\} - L\{2y\} = L\{0\}$$

$$\left(s^2 L\{y\} - s\,y(0) - y'(0)\right) - \left(s\,L\{y\} - y(0)\right) - 2L\{y\} = \frac{0}{s}$$

$$(s^2 - s - 2)\,L\{y\} - s + 1 = 0$$

$$Y(s) = \frac{s-1}{s^2 - s - 2}$$

The unknow is $y(t)$ and not $Y(s)$ so we take inverse

$$y(t) = L^{-1}\left(Y(s)\right) = L^{-1}\left(\frac{s-1}{s^2-s-2}\right) = L^{-1}\left(\frac{s-1}{(s-2)(s+1)}\right)$$

$$= L^{-1}\left(\frac{A}{s-2} + \frac{B}{s+1}\right) \quad \text{where} \quad A = \frac{\boxed{2} - 1}{\boxed{2} + 1} = \frac{1}{3}$$

$$B = \frac{\boxed{-1} - 1}{\boxed{-1} - 2} = \frac{2}{3}$$

$$= L^{-1}\left(\frac{\frac{1}{3}}{s-2}\right) + L^{-1}\left(\frac{\frac{2}{3}}{s+1}\right)$$

$$= \frac{1}{3} e^{2t} + \frac{2}{3} e^{-t}$$

Note that $r^2 - r - 2 = 0$

$$(r-2)(r+1) = 0$$

$$r_1 = 2 \ , \ r_2 = -1$$

$$y_1 = e^{2t} \ , \ y_2 = e^{-t}$$

$$y(t) = c_1 e^{2t} + c_2 e^{-t}$$

$$= \frac{1}{3} e^{2t} + \frac{2}{3} e^{-t}$$

To find c_1 and c_2 $\Rightarrow$

$$y(0) = c_1 + c_2 = 1$$

$$y'(0) = 2c_1 - c_2 = 0$$

$$c_1 = \frac{1}{3}$$

$$c_2 = \frac{2}{3}$$

Exp Use LT to solve the IVP:

$$y'' + y = \sin 2t \quad , \quad y(0) = 2 \quad , \quad y'(0) = 1$$

$$L\{y''\} + L\{y\} = L\{\sin 2t\}$$

$$s^2 L\{y\} - s\,y(0) - y'(0) + L\{y\} = \frac{2}{s^2+4}$$

$$L\{y\}(s^2+1) - 2s - 1 = \frac{2}{s^2+4}$$

$$Y(s)(s^2+1) = 2s + 1 + \frac{2}{s^2+4} = \frac{2s^3 + s^2 + 8s + 4 + 2}{s^2+4}$$

$$Y(s) = \frac{2s^3 + s^2 + 8s + 6}{(s^2+1)(s^2+4)}$$

$$y(t) = L^{-1}\left(\frac{2s^3 + s^2 + 8s + 6}{(s^2+1)(s^2+4)}\right) = L^{-1}\left(\frac{as+b}{s^2+1} + \frac{cs+d}{s^2+4}\right)$$

using partial fraction $\Rightarrow a = 2,\ b = \frac{5}{3},\ c = 0,\ d = \frac{-2}{3}$

$$= L^{-1}\left(\frac{2s + \frac{5}{3}}{s^2+1}\right) + L^{-1}\left(\frac{-\frac{2}{3}}{s^2+4}\right)$$

$$= 2\,L^{-1}\left(\frac{s}{s^2+1}\right) + \frac{5}{3}L^{-1}\left(\frac{1}{s^2+1}\right) - \frac{1}{3}L^{-1}\left(\frac{2}{s^2+4}\right)$$

$$= \underbrace{2\cos t + \frac{5}{3}\sin t}_{y_h(t)} - \underbrace{\frac{1}{3}\sin 2t}_{y_p(t)}$$

$y_h(t): \Rightarrow r^2 + 1 = 0 \quad \Rightarrow r_{1,2} = \pm i \quad \Rightarrow y_1 = \cos t\ ,\ y_2 = \sin t$

$\Rightarrow y_h(t) = c_1 \cos t + c_2 \sin t$

$y_p(t) = A \sin 2t + B \cos 2t \quad \Rightarrow$ substitute $y_p\ ,\ y_p''$ in the DE $\Rightarrow A = -\frac{1}{3}$

$B = 0$

$= -\frac{1}{3}\sin 2t$

Exp Use Laplace Transform to solve the IVP:

$$y^{(4)} - y = 0 \ , \ y(0) = y'(0) = y'''(0) = 0 \ , \ y''(0) = 1$$

$$L\{y^{(4)}\} - L\{y\} = L\{0\}$$

$$s^4 L\{y\} - s^3 y(0) - s^2 y'(0) - s\, y''(0) - y'''(0) - L\{y\} = \frac{0}{s}$$

$$(s^4 - 1) L\{y\} - s^2 = 0$$

$$L\{y\} = \frac{s^2}{s^4 - 1}$$

$$y(t) = L^{-1}\left(\frac{s^2}{s^4 - 1}\right) = L^{-1}\left(\frac{s^2}{(s^2-1)(s^2+1)}\right)$$

$$= L^{-1}\left(\frac{as+b}{s^2-1} + \frac{cs+d}{s^2+1}\right)$$

using partial fraction $\Rightarrow$
$a = 0$
$b = \frac{1}{2}$
$c = 0$
$d = \frac{1}{2}$

$$= L^{-1}\left(\frac{\frac{1}{2}}{s^2-1}\right) + L^{-1}\left(\frac{\frac{1}{2}}{s^2+1}\right)$$

$$= \frac{1}{2}\sinh t + \frac{1}{2}\sin t$$

$$= \frac{1}{2}\left(\frac{e^t - e^{-t}}{2}\right) + \frac{1}{2}\sin t$$

$$= \frac{1}{4}e^t - \frac{1}{4}e^{-t} + \frac{1}{2}\sin t$$

Note that $r^4 - 1 = 0 \Rightarrow (r^2-1)(r^2+1) = 0 \Rightarrow (r-1)(r+1)(r^2+1) = 0$

$\Rightarrow r_1 = 1 \ , \ r_2 = -1 \ , \ r_{3,4} = \pm i$

$\Rightarrow y_1 = e^t \ , \ y_2 = e^{-t} \ , \ y_3 = \cos t \ , \ y_4 = \sin t$

$y(t) = c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t \Rightarrow c_1 = \frac{1}{4} \ , \ c_2 = \frac{-1}{4} \ , \ c_3 = 0 \ , \ c_4 = \frac{1}{2}$

Ⓐ Remark

To find Laplace Transform for product of two functions (one is exponensial), we use shifting.

1st shifting

$$L\{e^{at} f(t)\} = F(s-a)$$

where $F(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

Hence, $e^{at} f(t) = L^{-1}(F(s-a))$

Proof

$$L\{e^{at} f(t)\} = \int_0^\infty e^{-st} e^{at} f(t)\,dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)\,dt$$

$$= F(s-a)$$

Exp Find ① $L\{e^{2t} \sin t\} = F(s-2) = \dfrac{1}{(s-2)^2+1}$

where $F(s) = L\{f(t)\} = L\{\sin t\} = \dfrac{1}{s^2+1}$

$$= \frac{1}{s^2-4s+4+1}$$

$$= \frac{1}{s^2-4s+5}$$

② $L\{e^{-3t} \cos et\} = F(s+3) = \dfrac{s+3}{(s+3)^2+e^2}$

where $F(s) = L\{f(t)\} = L\{\cos et\}$

$$= \frac{s}{s^2+e^2}$$

(3) $L\{e^t t^2\} = F(s-1) = \frac{2}{(s-1)^3}$

where $F(s) = L\{f(t)\} = L\{t^2\} = \frac{2!}{s^3} = \frac{2}{s^3}$

(4) $L^{-1}\left(\frac{s}{(s-2)^2+9}\right) = L^{-1}\left(\frac{s-2+2}{(s-2)^2+9}\right)$

$= L^{-1}\left(\frac{s-2}{(s-2)^2+3^2}\right) + \frac{2}{3} L^{-1}\left(\frac{3}{(s-2)^2+3^2}\right)$

$= e^{2t}\cos 3t + \frac{2}{3} e^{2t}\sin 3t$

(5) $L^{-1}\left(\frac{2s-1}{s^2+2s+5}\right) = L^{-1}\left(\frac{2s-1}{(s+1)^2+4}\right) = L^{-1}\left(\frac{2(s+1)-3}{(s+1)^2+4}\right)$

$= 2L^{-1}\left(\frac{s+1}{(s+1)^2+4}\right) - \frac{3}{2} L^{-1}\left(\frac{2}{(s+1)^2+4}\right)$

$= 2e^{-t}\cos 2t - \frac{3}{2} e^{-t}\sin 2t$

(6) $L^{-1}\left(\frac{-10}{(s+1)^3}\right) = -5\,L^{-1}\left(\frac{2!}{(s+1)^3}\right) = -5e^{-t}t^2$

(7) $L^{-1}\left(\frac{s}{s^2+s-1}\right) = L^{-1}\left(\frac{s}{(s+\frac{1}{2})^2-\frac{5}{4}}\right) = L^{-1}\left(\frac{s+\frac{1}{2}-\frac{1}{2}}{(s+\frac{1}{2})^2-\frac{5}{4}}\right)$

$= L^{-1}\left(\frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2-\frac{5}{4}}\right) - \frac{1}{\sqrt{5}} L^{-1}\left(\frac{\frac{\sqrt{5}}{2}}{(s+\frac{1}{2})^2-\frac{5}{4}}\right)$

$= e^{-\frac{1}{2}t}\cosh\frac{\sqrt{5}}{2}t - \frac{1}{\sqrt{5}} e^{-\frac{1}{2}t}\sinh\frac{\sqrt{5}}{2}t$

8 inverse transform of $H(s) = \dfrac{4s-10}{s^2-6s+10}$

$$h(t) = L^{-1}(H(s)) = L^{-1}\left(\frac{4s-10}{(s-3)^2+1}\right) = L^{-1}\left(\frac{4(s-3)+2}{(s-3)^2+1}\right)$$

$$= 4L^{-1}\left(\frac{s-3}{(s-3)^2+1}\right) + 2L^{-1}\left(\frac{1}{(s-3)^2+1}\right)$$

$$= 4e^{3t}\cos t + 2e^{3t}\sin t$$

Exp Solve the following IVP using Laplace Transform:

$$y'' - 8y' + 25y = 0 \quad , \quad y(0) = 0 \; , \; y'(0) = 6$$

$$L\{y''\} - 8L\{y'\} + 25L\{y\} = L\{0\}$$

$$s^2L\{y\} - s\,y(0) - y'(0) - 8\left(sL\{y\} - y(0)\right) + 25L\{y\} = 0$$

$$(s^2 - 8s + 25)L\{y\} - 6 = 0$$

$$L\{y\} = \frac{6}{s^2 - 8s + 25}$$

$$y(t) = L^{-1}(L\{y\}) = L^{-1}\left(\frac{6}{(s-4)^2+9}\right) = 2L^{-1}\left(\frac{3}{(s-4)^2+9}\right)$$

$$= 2e^{4t}\sin 3t$$

(B)
Remark
To find Laplace Transform for product of two functions (one is poly. t^n), we use derivatives.

Exp show that
$$L\{t^n f(t)\} = (-1)^n F^{(n)}(s)$$

where $F(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

Proof
$$F(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$F'(s) = \frac{dF}{ds} = -\int_0^\infty t\, e^{-st} f(t)\,dt = (-1)^1 L\{t^1 f(t)\}$$

$$F''(s) = \frac{d^2F}{ds^2} = \int_0^\infty t^2 e^{-st} f(t)\,dt = (-1)^2 L\{t^2 f(t)\}$$

$$\vdots$$

$$F^{(n)}(s) = \frac{d^nF}{ds^n} = (-1)^n L\{t^n f(t)\}$$

$$\frac{1}{(-1)^n} F^{(n)}(s) = L\{t^n f(t)\}$$

$$(-1)^n F^{(n)}(s) = L\{t^n f(t)\}$$

Exp Find Laplace Transform of

① $h(t) = t \sin t$

$$H(s) = L\{t \sin t\} = (-1)^1 F'(s) = (-1)(-1)(s^2+1)^{-2}(2s) = \frac{2s}{(s^2+1)^2}$$

where $F(s) = L\{f(t)\} = L\{\sin t\} = \dfrac{1}{s^2+1} = (s^2+1)^{-1}$

---

② $h(t) = L\{t^2 \sin t\}$

$$H(s) = L\{t^2 \sin t\} = (-1)^2 F''(s) = F''(s) = \frac{6s^2-2}{(s^2+1)^3}$$

where $F(s) = L\{f(t)\} = L\{\sin t\} = \dfrac{1}{s^2+1} = (s^2+1)^{-1}$

$$F'(s) = (-1)(s^2+1)^{-2}(2s)$$
$$= (-2s)(s^2+1)^{-2}$$

$$F''(s) = (-2s)(-2)(s^2+1)^{-3}(2s) + (s^2+1)^{-2}(-2)$$

$$= (s^2+1)^{-2}\left(8s^2(s^2+1)^{-1} + (-2)\right)$$

$$= \frac{-2}{(s^2+1)^2}\left(1 - \frac{4s^2}{s^2+1}\right)$$

$$= \left(\frac{-2}{(s^2+1)^2}\right)\left(\frac{s^2+1-4s^2}{s^2+1}\right)$$

$$= \frac{6s^2-2}{(s^2+1)^3}$$

③ $h(t) = t^3 e^{4t}$

solution 1 $H(s) = L\{h(t)\} = L\{t^3 e^{4t}\} = (-1)^3 F'''(s)$

where $F(s) = L\{f(t)\}$
$= L\{e^{4t}\}$
$= \dfrac{1}{s-4}$
$= (s-4)^{-1}$

$F'(s) = -(s-4)^{-2}$
$F''(s) = 2(s-4)^{-3}$
$F'''(s) = -6(s-4)^{-4}$
$= \dfrac{-6}{(s-4)^4}$

$H(s) = (-1)^3 F'''(s)$
$= -F'''(s)$
$= \dfrac{6}{(s-4)^4}$

solution 2 $H(s) = L\{h(t)\} = L\{e^{4t} t^3\} = F(s-4)$

where $F(s) = L\{f(t)\} = L\{t^3\}$
$= \dfrac{3!}{s^4}$

$= \dfrac{6}{(s-4)^4}$

④ $h(t) = 2t\, e^{-t} \cosh t$

$H(s) = L\{h(t)\} = L\{2t\, e^{-t}\left(\dfrac{e^{t}+e^{-t}}{2}\right)\} = L\{t e^{-t}(e^{t}+e^{-t})\}$

$= L\{t + t e^{-2t}\} = L\{t\} + L\{t e^{-2t}\}$

$= \dfrac{1}{s^2} + (-1)^1 F'(s)$

where $F(s) = L\{e^{-2t}\}$
$= \dfrac{1}{s+2}$

$F'(s) = \dfrac{-1}{(s+2)^2}$

$= \dfrac{1}{s^2} + \dfrac{1}{(s+2)^2}$

Exp Find $L\{t\, e^{t} \cos 3t\}$

$$L\{t\, e^{t}\cos 3t\} = (-1)^{1} F'(s) = -F'(s) = \frac{(s-1)^2-9}{[(s-1)^2+9]^2}$$

where $F(s) = L\{e^{t} \cos 3t\}$

$$= G(s-1) \quad \text{where } G(s) = L\{\cos 3t\}$$

$$= \frac{s-1}{(s-1)^2+9} \qquad = \frac{s}{s^2+9}$$

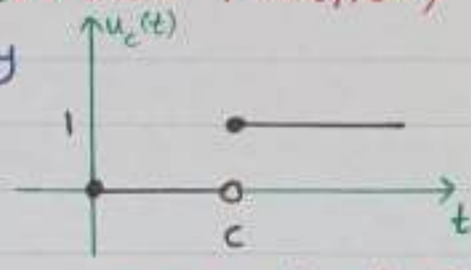
$$F'(s) = \frac{((s-1)^2+9)(1) - (s-1)(2)(s-1)}{[(s-1)^2+9]^2}$$

$$= \frac{9-(s-1)^2}{((s-1)^2+9)^2}$$

6.3 Step Functions

Def The unit step function (Heaviside Function) $u_c(t)$, $c>0$, is defined by

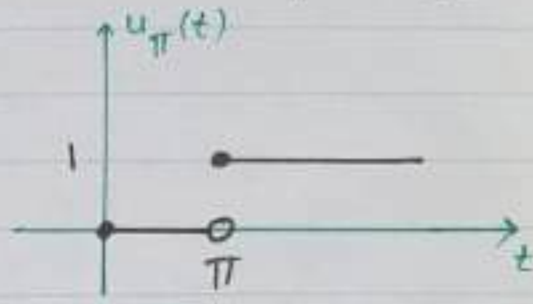
$$u_c(t) = \begin{cases} 0 , & 0 \le t < c \\ 1 , & c \le t \end{cases}$$



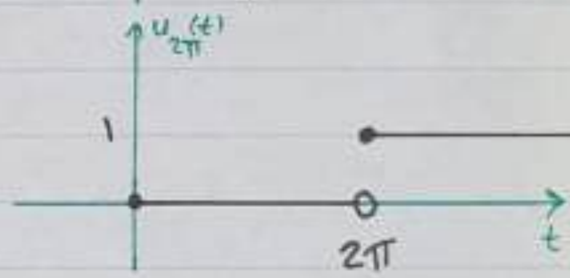
$u_c(t)$ is discont at c

Exp sketch ① $u_\pi(t)$ ② $u_{2\pi}(t)$ ③ $h(t) = u_\pi(t) - u_{2\pi}(t)$

① $u_\pi(t) = \begin{cases} 0 , & 0 \le t < \pi \\ 1 , & \pi \le t \end{cases}$

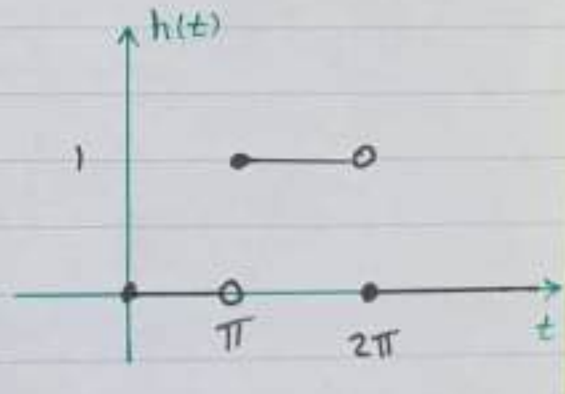


② $u_{2\pi}(t) = \begin{cases} 0 , & 0 \le t < 2\pi \\ 1 , & 2\pi \le t \end{cases}$



③ $h(t) = u_\pi(t) - u_{2\pi}(t)$

$$= \begin{cases} 0 , & 0 \le t < \pi \\ 1 , & \pi \le t < 2\pi \\ 0 , & 2\pi \le t \end{cases}$$



Exp Find $f(2)$ if $f(t) = t\, u_2(t) - 3u_3(t) + 2u_1(t) + t^3 u_5(t)$

$$f(2) = (2)u_2(2) - 3u_3(2) + 2u_1(2) + (8)u_5(2)$$

$$= (2)(1) - 3(0) + 2(1) + (8)(0)$$

$$= 4$$

Exp Express the following function interms of $u_c(t)$

$$f(t)=\begin{cases} 2 , & 0 \le t < 1 \\ -1 , & 1 \le t < 2 \\ 2 , & 2 \le t < 3 \\ -1 , & 3 \le t < 4 \\ 0 , & 4 \le t \end{cases}$$

$$f(t) = 2 + (-1-2)\, u_1(t) + (2--1)\, u_2(t) + (-1-2)\, u_3(t) + (0--1)\, u_4(t)$$

$$f(t) = 2 - 3u_1(t) + 3u_2(t) - 3u_3(t) + u_4(t)$$

$$f\left(\tfrac{5}{2}\right) = 2 - 3u_1\left(\tfrac{5}{2}\right) + 3u_2\left(\tfrac{5}{2}\right) - 3u_3\left(\tfrac{5}{2}\right) + u_4\left(\tfrac{5}{2}\right)$$

2 $\quad = 2 - 3(1) + 3(1) - 3(0) + (0)$

$= 2$

Exp show that $L\{u_c(t)\} = \frac{e^{-cs}}{s}$

$$u_c(t) = \begin{cases} 0 , & 0 \le t < c \\ 1 , & c \le t \end{cases}$$

$$L\{u_c(t)\} = \int_0^\infty e^{-st} u_c(t)\, dt$$

$$= \int_0^c e^{-st}(0)\, dt + \int_c^\infty e^{-st}(1)\, dt$$

$$= \lim_{b\to\infty} \int_c^b e^{-st}\, dt = \lim_{b\to\infty} \frac{-1}{s} e^{-st} \Big|_c^b$$

$$= \frac{-1}{s} \lim_{b\to\infty} \left[e^{-sb} - e^{-cs} \right] = \frac{-1}{s}\left(0 - e^{-cs}\right) = \frac{e^{-cs}}{s}$$

Hence, $L^{-1}\left(\frac{e^{-cs}}{s}\right) = u_c(t)$

Exp Find

① $L\{u_3(t)\} = \dfrac{e^{-3s}}{s}$

② $L\{1 - u_e(t)\} = L\{1\} - L\{u_e(t)\} = \dfrac{1}{s} - \dfrac{e^{-es}}{s}$

③ $L^{-1}\left(\dfrac{4e^{-2s}}{7s}\right) = \dfrac{4}{7} L^{-1}\left(\dfrac{e^{-2s}}{s}\right) = \dfrac{4}{7} u_2(t)$

④ $L^{-1}\left(\dfrac{se^{-\pi s} + 6s}{s^2}\right) = L^{-1}\left(\dfrac{e^{-\pi s}}{s} + \dfrac{6}{s}\right) = L^{-1}\left(\dfrac{e^{-\pi s}}{s}\right) + L^{-1}\left(\dfrac{6}{s}\right)$

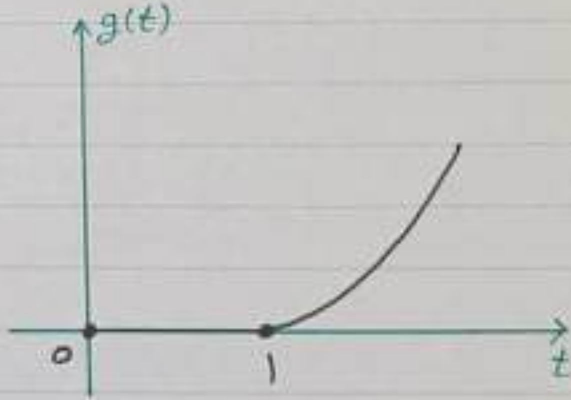
$= u_\pi(t) + 6$

Exp Let $f(t) = t^2$, $t \geq 0$.
sketch the graph of $g(t) = f(t-1)\,u_1(t)$

$g(t) = (t-1)^2\, u_1(t) = \begin{cases} 0, & 0 \leq t < 1 \\ (t-1)^2, & 1 \leq t \end{cases}$

Note that $g(t)$ is cont.

on $[0, \infty)$



© **Remark:** To find Laplace Transform for product of two functions: one is $u_c(t)$ $\Rightarrow$ first we shift the other one by c to the right.

Exp show that

$$L\{u_c(t)\, f(t-c)\} = e^{-cs}\, L\{f(t)\}$$
$$= e^{-cs}\, F(s)$$

Hence, $L^{-1}\left(e^{-cs}F(s)\right) = u_c(t)\, f(t-c)$

Proof

$$L\{u_c(t) f(t-c)\} = \int_0^\infty e^{-st}\, u_c(t)\, f(t-c)\, dt$$

$$= \int_0^c e^{-st}(0)\, f(t-c)\, dt + \int_c^\infty e^{-st}(1)\, f(t-c)\, dt$$

$$= \int_c^\infty e^{-st}\, f(t-c)\, dt \qquad u = t-c,\ du = dt$$

$$= \int_0^\infty e^{-s(u+c)}\, f(u)\, du \qquad t=c \Rightarrow u=0,\ t\to\infty \Rightarrow u\to\infty$$

$$= e^{-cs}\int_0^\infty e^{-su}\, f(u)\, du$$

$$= e^{-cs}\, L\{f(u)\}$$

$$= e^{-cs}\, F(s)$$

Exp Find ① $L\{u_2(t)\,(t-2)\}$

$f(t)=t$

$f(t-2)=t-2$

$$= e^{-2s}\, L\{t\}$$

$$= e^{-2s}\,\frac{1}{s^2} = \frac{e^{-2s}}{s^2}$$

Note that $L^{-1}\left(\frac{e^{-2s}}{s^2}\right) = u_2(t)\,(t-2)$

---

② $L\{u_2(t)\,(t-1)\}$

$f(t)=t$

$f(t-2)=t-2$

$$= L\{u_2(t)\left((t-2)+1\right)\}$$

$$= L\{u_2(t)(t-2) + u_2(t)\}$$

$$= L\{u_2(t)(t-2)\} + L\{u_2(t)\}$$

$$= \frac{e^{-2s}}{s^2} + \frac{e^{-2s}}{s}$$

---

③ $L\{u_2(t)\,t^2\}$ $\quad f(t)=t^2$

$f(t)=t^2$

$f(t-2)=(t-2)^2$

$$= L\{u_2(t)\,(t-2+2)^2\}$$

$$= L\{u_2(t)\left((t-2)^2 + 4(t-2) + 4\right)\}$$

$$= L\{u_2(t)(t-2)^2\} + 4L\{u_2(t)(t-2)\} + 4L\{u_2(t)\}$$

$$= e^{-2s}L\{t^2\} + 4\,e^{-2s}L\{t\} + 4\,\frac{e^{-2s}}{s}$$

$$= e^{-2s}\left(\frac{2}{s^3}\right) + 4\,e^{-2s}\left(\frac{1}{s^2}\right) + 4\,\frac{e^{-2s}}{s}$$

Note that we can apply Remark (B) to (3) $\Rightarrow$

$$L\{t^2 u_2(t)\} = (-1)^2 F''(s) = F''(s)$$

where $F(s) = L\{u_2(t)\} = \frac{e^{-2s}}{s}$

$$F'(s) = \frac{(s)(-2)e^{-2s} - e^{-2s}}{s^2} = -2e^{-2s}\left(\frac{1}{s}\right) - \left(\frac{1}{s^2}\right)e^{-2s}$$

$$F''(s) = -2e^{-2s}\left(\frac{-1}{s^2}\right) + \left(\frac{1}{s}\right)(4)e^{-2s} - \left(\frac{1}{s^2}\right)(-2)e^{-2s} - e^{-2s}(-2)\left(\frac{1}{s^3}\right)$$

$$= \frac{2}{s^3}e^{-2s} + 4e^{-2s}\left(\frac{1}{s^2}\right) + 4\frac{e^{-2s}}{s}$$

---

(4) $$L^{-1}\left(\frac{3+e^{-7s}}{s^4}\right) = 3L^{-1}\left(\frac{1}{s^4}\right) + L^{-1}\left(\frac{e^{-7s}}{s^4}\right)$$

$$= \frac{3}{3!}L^{-1}\left(\frac{3!}{s^4}\right) + \frac{1}{3!}L^{-1}\left(\frac{3!\,e^{-7s}}{s^4}\right)$$

$$= \frac{1}{2}t^3 + \frac{1}{6}u_7(t)(t-7)^3$$

---

(5) $F(s)$ if $f(t) = \begin{cases} \sin t & , 0 \le t < \frac{\pi}{4} \\ \sin t + \cos(t - \frac{\pi}{4}) & , \frac{\pi}{4} \le t \end{cases}$

$$f(t) = \sin t + \left(\sin t + \cos(t - \tfrac{\pi}{4}) - \sin t\right) u_{\frac{\pi}{4}}(t)$$
$$= \sin t + u_{\frac{\pi}{4}}(t)\cos(t - \tfrac{\pi}{4})$$

$$F(s) = \frac{1}{s^2+1} + e^{-\frac{\pi}{4}s}L\{\cos t\} = \frac{1}{s^2+1} + \frac{s\,e^{-\frac{\pi}{4}s}}{s^2+1}$$

$\sin(A+B) = \sin A \cos B + \sin B \cos A$
$\cos(A+B) = \cos A \cos B - \sin A \sin B$

Exp Find Laplace Transform of $f(t) = u_\pi(t) \sin t$

$$F(s) = L\{f(t)\} = L\{u_\pi(t) \sin t\} = L\{u_\pi(t) \sin(t-\pi+\pi)\}$$
$$= L\{u_\pi(t) \left(\sin(t-\pi)\cos\pi + \sin(\pi)\cos(t-\pi)\right)\}$$
$$= L\{u_\pi(t)\left(-\sin(t-\pi)+0\right)\}$$
$$= -L\{u_\pi(t)\sin(t-\pi)\}$$
$$= -e^{-\pi s} L\{\sin t\} = -e^{-\pi s}\frac{1}{s^2+1}$$

Exp Find $L\{u_3(t)e^{4t}\}$

S1

$$L\{u_3(t)e^{4(t-3+3)}\}$$
$$= L\{u_3(t)e^{4(t-3)}e^{12}\}$$
$$= e^{12} L\{u_3(t)e^{4(t-3)}\}$$
$$= e^{12}e^{-3s}L\{e^{4t}\}$$
$$= e^{-3(s-4)}\frac{1}{s-4}$$
$$= \frac{e^{-3(s-4)}}{s-4}$$

S2

$$L\{u_3(t)e^{4t}\}$$ Apply (A) Remark
$$= F(s-4)$$

where $F(s) = L\{u_3(t)\}$
$$= \frac{e^{-3s}}{s}$$

Hence, $F(s-4) = \dfrac{e^{-3(s-4)}}{s-4}$

Exp Find Laplace Inverse of $\frac{2(s-1)e^{-2s}}{s^2-2s+2}$

$$\mathcal{L}^{-1}\left(\frac{2(s-1)e^{-2s}}{s^2-2s+2}\right)$$

$$\mathcal{L}^{-1}\left(\frac{2(s-1)e^{-2s}}{(s-1)^2+1}\right) = 2\mathcal{L}^{-1}\left(\frac{(s-1)e^{-2s}}{(s-1)^2+1}\right)$$

$$= 2\,u_2(t)\,e^{t-2}\cos(t-2)$$

Exp Find $F(s)$ if $f(t)=\begin{cases}3, & 0\le t<2\\ t^2-4t+4, & 2\le t\end{cases}$

$$f(t) = 3 + (t^2-4t+4-3)\,u_2(t)$$

$$= 3 + (t^2-4t+1)\,u_2(t)$$

$$= 3 + \left((t-2)^2-3\right)u_2(t)$$

$$= 3 + u_2(t)(t-2)^2 - 3u_2(t)$$

$$F(s) = L\{3\} + L\{u_2(t)(t-2)^2\} - 3L\{u_2(t)\}$$

$$= \frac{3}{s} + e^{-2s}L\{t^2\} - 3\frac{e^{-2s}}{s}$$

$$= \frac{3}{s} + e^{-2s}\frac{2}{s^3} - 3\frac{e^{-2s}}{s}$$

$$= \frac{3(1-e^{-2s})}{s} + \frac{2e^{-2s}}{s^3}$$

6.4 Solving IVP's with Step Functions

In this section we will find cont. and diff. solution for a given IVP with discont. step functions.

Exp Solve the IVP:

$$y'' + 4y = 6 + t\,u_3(t) \quad , \quad y(0)=0 \;, \; y'(0)=0$$

$$L\{y''\} + L\{4y\} = L\{6\} + L\{u_3(t)\,(t-3+3)\}$$

$$s^2 L\{y\} - s\,y(0) - y'(0) + 4L\{y\} = \frac{6}{s} + L\{u_3(t)(t-3)\} + 3L\{u_3(t)\}$$

$$(s^2+4)\,L\{y\} = \frac{6}{s} + e^{-3s} L\{t\} + 3\,\frac{e^{-3s}}{s}$$

$$(s^2+4)\,L\{y\} = \frac{6}{s} + e^{-3s}\left(\frac{1}{s^2}\right) + \frac{3e^{-3s}}{s}$$

$$L\{y\} = \frac{6}{s(s^2+4)} + \frac{e^{-3s}}{s^2(s^2+4)} + \frac{3e^{-3s}}{s(s^2+4)}$$

$$y(t) = 6L^{-1}\left(\frac{1}{s(s^2+4)}\right) + L^{-1}\left(\frac{e^{-3s}}{s^2(s^2+4)}\right) + 3L^{-1}\left(\frac{e^{-3s}}{s(s^2+4)}\right)$$

$$= 6L^{-1}\left(\frac{A}{s} + \frac{Bs+C}{s^2+4}\right) + L^{-1}\left[e^{-3s}\left(\frac{D}{s} + \frac{E}{s^2} + \frac{Fs+G}{s^2+4}\right)\right] +$$

$$3L^{-1}\left[e^{-3s}\left(\frac{A}{s} + \frac{Bs+C}{s^2+4}\right)\right]$$

Using Partial Fraction: $A = \frac{1}{4}$, $B = -\frac{1}{4}$, $C = 0$ | $D = 0$, $E = \frac{1}{4}$, $F = 0$, $G = -\frac{1}{4}$

$$y(t) = 6L^{-1}\left(\frac{\frac{1}{4}}{s} - \frac{\frac{1}{4}s}{s^2+4}\right) + L^{-1}\left[e^{-3s}\left(\frac{\frac{1}{4}}{s^2} - \frac{\frac{1}{4}}{s^2+4}\right)\right] + 3L^{-1}\left[e^{-3s}\left(\frac{\frac{1}{4}}{s} - \frac{\frac{1}{4}s}{s^2+4}\right)\right]$$

$$y(t) = \frac{6}{4} L^{-1}\left(\frac{1}{s}\right) - \frac{6}{4} L^{-1}\left(\frac{s}{s^2+4}\right) + \frac{1}{4} L^{-1}\left(\frac{e^{-3s}}{s^2}\right) - \frac{1}{8} L^{-1}\left(\frac{2e^{-3s}}{s^2+4}\right)$$

$$+ \frac{3}{4} L^{-1}\left(\frac{e^{-3s}}{s}\right) - \frac{3}{4} L^{-1}\left(\frac{se^{-3s}}{s^2+4}\right)$$

$$= \frac{3}{2}(1) - \frac{3}{2}\cos 2t + \frac{1}{4} u_3(t)(t-3) - \frac{1}{8} u_3(t)\sin 2(t-3)$$

$$+ \frac{3}{4} u_3(t) - \frac{3}{4} u_3(t)\cos 2(t-3)$$

$$y(t) = \frac{3}{2}(1-\cos 2t) + \frac{1}{4} u_3(t)\left[t - \cancel{3} - \frac{1}{2}\sin 2(t-3) + \cancel{3} - 3\cos 2(t-3)\right]$$

$$= \frac{3}{2}(1-\cos 2t) + \frac{1}{4} u_3(t)\left[t - \frac{1}{2}\sin 2(t-3) - 3\cos 2(t-3)\right]$$

Note that y and y' are cont. at $t=3$

Exp Solve the IVP:

$$y'' + y = \begin{cases} 1, & 0 \le t < 1 \\ t, & 1 \le t \end{cases} \qquad y(0)=0, \quad y'(0)=0$$

$$y'' + y = 1 + (t-1)u_1(t)$$

$$L\{y''\} + L\{y\} = L\{1\} + L\{u_1(t)(t-1)\}$$

$$s^2 L\{y\} - sy(0) - y'(0) + L\{y\} = \frac{1}{s} + e^{-s} L\{t\}$$

$$(s^2+1) L\{y\} = \frac{1}{s} + e^{-s}\frac{1}{s^2}$$

$$L\{y\} = \frac{1}{s(s^2+1)} + \frac{e^{-s}}{s^2(s^2+1)}$$

$$y(t) = L^{-1}\left(\frac{1}{s(s^2+1)}\right) + L^{-1}\left(\frac{e^{-s}}{s^2(s^2+1)}\right)$$

$$y(t) = L^{-1}\left(\frac{A}{s} + \frac{Bs+C}{s^2+1}\right) + L^{-1}\left[e^{-s}\left(\frac{D}{s} + \frac{E}{s^2} + \frac{Fs+G}{s^2+1}\right)\right]$$

Using Partial Fraction $\Rightarrow A = 1,\ B = -1,\ C = 0$

$\Rightarrow D = 0,\ E = 1,\ F = 0,\ G = -1$

$$y(t) = L^{-1}\left(\frac{1}{s}\right) - L^{-1}\left(\frac{s}{s^2+1}\right) + L^{-1}\left(\frac{e^{-s}}{s^2}\right) + L^{-1}\left(\frac{e^{-s}}{s^2+1}\right)$$

$$= 1 - \cos t + u_1(t)(t-1) + u_1(t)\sin(t-1)$$

Exp Solve the IVP: $y'' = e^t u_1(t)$, $y(0)=1$, $y'(0)=0$

$$L\{y''\} = L\{e^t u_1(t)\} = L\{u_1(t)e^{t-1+1}\} = e\,L\{u_1(t)e^{t-1}\}$$

$$s^2L\{y\} - s\,y(0) - y'(0) = e\,e^{-s}\,L\{e^t\} = e\,e^{-s}\frac{1}{s-1}$$

$$s^2L\{y\} - s = e\,\frac{e^{-s}}{s-1}$$

$$L\{y\} = \frac{1}{s} + e\,\frac{e^{-s}}{s^2(s-1)}$$

Using Partial Fraction
$A = -1$
$B = -1$
$C = 1$

$$y(t) = L^{-1}\left(\frac{1}{s}\right) + e\,L^{-1}\left(\frac{e^{-s}}{s^2(s-1)}\right)$$

$$= 1 + e\,L^{-1}\left[e^{-s}\left(\frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1}\right)\right]$$

$$= 1 + e\,L^{-1}\left[e^{-s}\left(\frac{-1}{s} - \frac{1}{s^2} + \frac{1}{s-1}\right)\right]$$

$$= 1 - e\left[u_1(t) + u_1(t)(t-1) - u_1(t)e^{t-1}\right]$$

$$= 1 - e\,u_1(t)\left[t - e^{t-1}\right]$$

y and y' are cont. at $t=1$

Exp Solve the IVP: $2y'' + y' + 2y = g(t)$, $y(0) = y'(0) = 0$

where $g(t) = \begin{cases} 0 , & 0 \le t < 5 \\ 1 , & 5 \le t < 20 \\ 0 , & 20 \le t \end{cases}$

$$g(t) = 0 + (1-0)\,u_5(t) + (0-1)\,u_{20}(t) = u_5(t) - u_{20}(t)$$

$$2L\{y''\} + L\{y'\} + 2L\{y\} = L\{u_5(t)\} - L\{u_{20}(t)\}$$

$$2\left(s^2 L\{y\} - s\,y(0) - y'(0)\right) + \left(sL\{y\} - s\,y(0)\right) + 2L\{y\} = \frac{e^{-5s}}{s} - \frac{e^{-20s}}{s}$$

$$(2s^2 + s + 2)\,L\{y\} = \frac{e^{-5s}}{s} - \frac{e^{-20s}}{s}$$

$$L\{y\} = \frac{e^{-5s}}{s(2s^2+s+2)} - \frac{e^{-20s}}{s(2s^2+s+2)}$$

$$y(t) = L^{-1}\left(e^{-5s} H(s)\right) - L^{-1}\left(e^{-20s} H(s)\right)$$

$$= u_5(t)\,h(t-5) - u_{20}(t)\,h(t-20)$$

$$= u_5(t)\left[\frac{1}{2} - \frac{1}{2} e^{-\frac{1}{4}(t-5)} \cos\frac{\sqrt{15}}{4}(t-5) - \frac{1}{2\sqrt{15}} e^{-\frac{1}{4}(t-5)} \sin\frac{\sqrt{15}}{4}(t-5)\right]$$

$$- u_{20}(t)\left[\frac{1}{2} - \frac{1}{2} e^{-\frac{1}{4}(t-20)} \cos\frac{\sqrt{15}}{4}(t-20) - \frac{1}{2\sqrt{15}} e^{-\frac{1}{4}(t-20)} \sin\frac{\sqrt{15}}{4}(t-20)\right]$$

Note that y and y' are cont. at $t=5$ and $t=20$

$$H(s) = \frac{1}{s(2s^2+s+2)}$$

$$= \frac{A}{s} + \frac{Bs+C}{2s^2+s+2}$$

$$= \frac{\frac{1}{2}}{s} + \frac{-s-\frac{1}{2}}{2s^2+s+2}$$

Using Partial Fraction

$$h(t) = L^{-1}(H(s))$$

$$= L^{-1}\left(\frac{\frac{1}{2}}{s}\right) - L^{-1}\left(\frac{s+\frac{1}{2}}{2(s^2+\frac{s}{2}+1)}\right)$$

$$= \frac{1}{2} - \frac{1}{2} L^{-1}\left(\frac{s+\frac{1}{4}+\frac{1}{4}}{(s+\frac{1}{4})^2 + \frac{15}{16}}\right)$$

$$= \frac{1}{2} - \frac{1}{2} L^{-1}\left(\frac{s+\frac{1}{4}}{(s+\frac{1}{4})^2 + \frac{15}{16}}\right) - \frac{1}{2\sqrt{15}} L^{-1}\left(\frac{\frac{\sqrt{15}}{4}}{(s+\frac{1}{4})^2 + \frac{15}{16}}\right)$$

$$= \frac{1}{2} - \frac{1}{2} e^{-\frac{1}{4}t} \cos\frac{\sqrt{15}}{4}t - \frac{1}{2\sqrt{15}} e^{-\frac{1}{4}t} \sin\frac{\sqrt{15}}{4}t$$

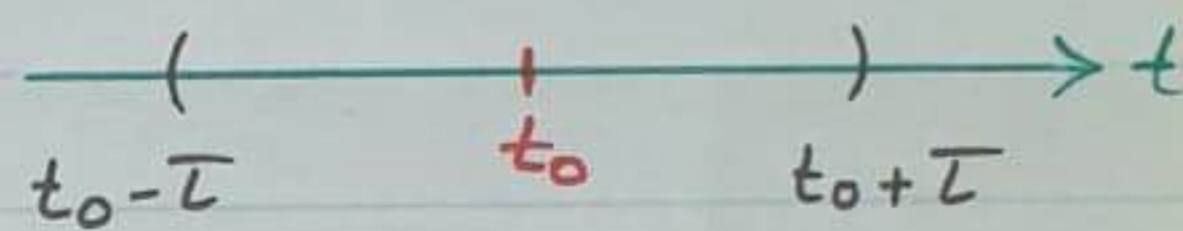
6.5 Impulse Functions

In some applications, it is necessary to deal with impulsive (اندفاعي) nature. For example, voltages or forces with large magnitude that act over short time intervals. Such applications lead to DE's of the form:

$$ay'' + by' + cy = d_\tau(t - t_0)$$

where the forcing function $d_\tau(t - t_0) = \begin{cases} \frac{1}{2\tau} , & t_0 - \tau < t < \tau + t_0 \\ 0 , & \text{otherwise} \end{cases}$

where $\tau > 0$ and t_0 is the center.



$$|t - t_0| < \tau$$

- When the center $t_0 = 0 \Rightarrow$ the forcing function becomes

$$d_\tau(t) = \begin{cases} \frac{1}{2\tau} , & -\tau < t < \tau \\ 0 , & \text{otherwise} \end{cases}$$

- To measure the strength of the forcing function $d_\tau(t)$ we use the Integral

$$I(\tau) = \int_{-\infty}^{\infty} d_\tau(t)\, dt = \int_{-\tau}^{\tau} \frac{1}{2\tau}\, dt = \frac{1}{2\tau}(\tau - -\tau) = 1$$

Exp* Note that $\lim_{\tau \to 0} d_\tau(t) = 0$

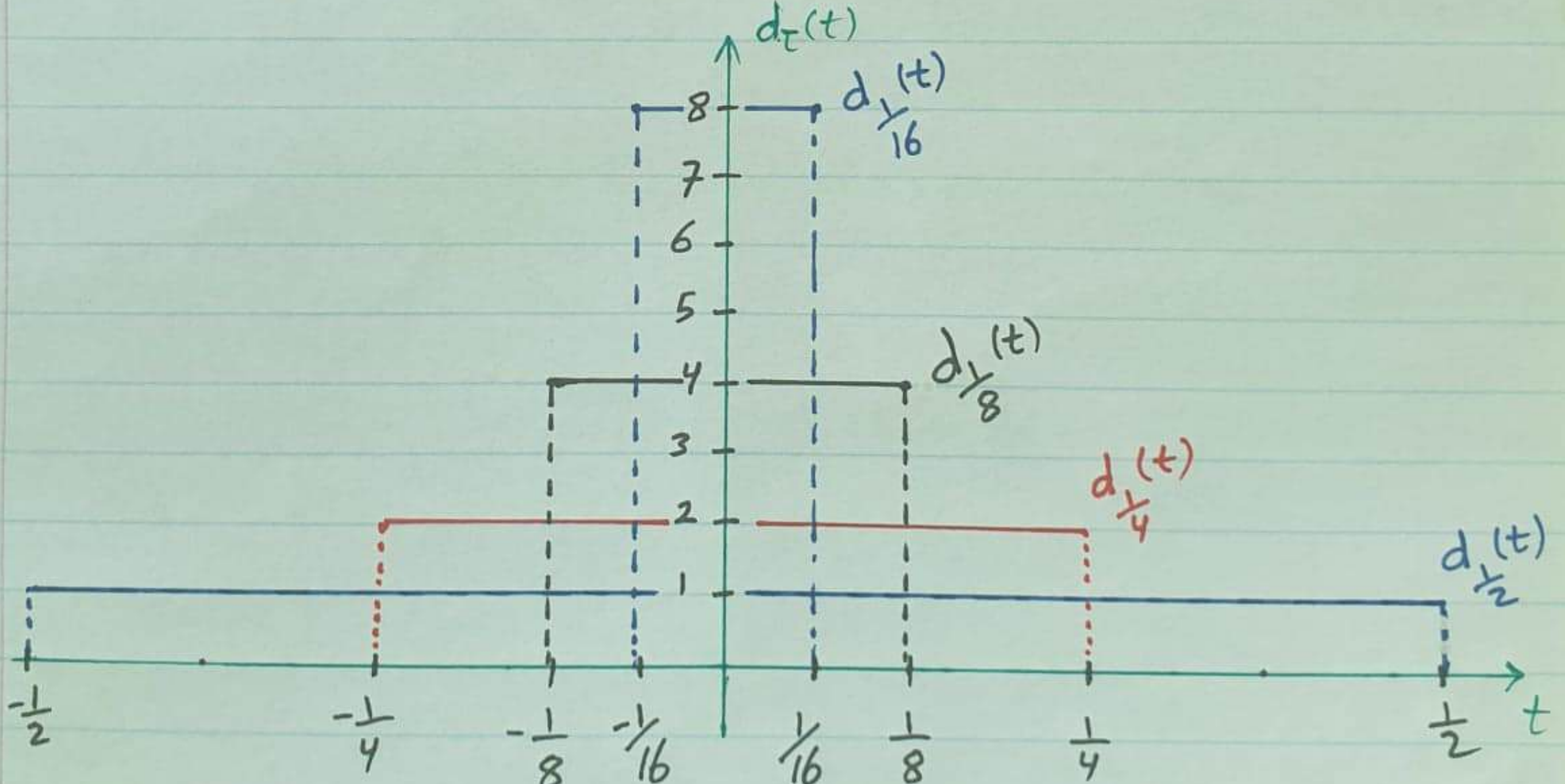
when $\tau = \frac{1}{2} \Rightarrow d_{\frac{1}{2}}(t) = \begin{cases} 1 & , -\frac{1}{2} < t < \frac{1}{2} \\ 0 & , \text{otherwise} \end{cases}$

$\tau = \frac{1}{4} \Rightarrow d_{\frac{1}{4}}(t) = \begin{cases} 2 & , -\frac{1}{4} < t < \frac{1}{4} \\ 0 & , \text{otherwise} \end{cases}$

$\tau = \frac{1}{8} \Rightarrow d_{\frac{1}{8}}(t) = \begin{cases} 4 & , -\frac{1}{8} < t < \frac{1}{8} \\ 0 & , \text{otherwise} \end{cases}$

$\tau = \frac{1}{16} \Rightarrow d_{\frac{1}{16}}(t) = \begin{cases} 8 & , -\frac{1}{16} < t < \frac{1}{16} \\ 0 & , \text{otherwise} \end{cases}$

$\vdots$



Def The Unit Impulse Function δ at point t_0 is defined by

$$\delta(t-t_0)=\begin{cases}1 & \text{if } t=t_0\\ 0 & \text{if } t\neq t_0\end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty}\delta(t-t_0)\,dt=1$$

Special Case when $t_0=0 \Rightarrow$ The unit impulse function becomes

$$\delta(t)=\begin{cases}1 & \text{if } t=0\\ 0 & \text{if } t\neq 0\end{cases}$$

Remark

$$\delta(t)=\lim_{\tau\to 0} d_\tau(t) \quad \text{when } t\neq 0$$

Hence, $\delta(t-t_0)=\lim_{\tau\to 0} d_\tau(t-t_0)$ when $t\neq t_0$

Exp show that $L\{\delta(t-t_0)\}=e^{-t_0 s}$

Proof $L\{\delta(t-t_0)\}=L\{\lim_{\tau\to 0} d_\tau(t-t_0)\}$ by Remark above

$$=\lim_{\tau\to 0}\int_0^{\infty} e^{-st}\, d_\tau(t-t_0)\,dt = \lim_{\tau\to 0}\int_{t_0-\tau}^{t_0+\tau} e^{-st}\left(\frac{1}{2\tau}\right)dt$$

$$=\lim_{\tau\to 0}\frac{1}{2\tau}\,\frac{-1}{s}\left[e^{-st}\Big|_{t_0-\tau}^{t_0+\tau}\right]$$

$$=\lim_{\tau\to 0}\frac{-1}{2s\tau}\left[e^{-s(t_0+\tau)}-e^{-s(t_0-\tau)}\right]$$

$$L\{\delta(t-t_0)\} = \lim_{\tau\to 0} \frac{1}{s\tau}\left[\frac{e^{s\tau} - e^{-s\tau}}{2}\right] e^{-t_0 s}$$

$$= e^{-t_0 s} \lim_{\tau\to 0} \frac{\sinh s\tau}{s\tau}$$

$$= e^{-t_0 s} \lim_{\tau\to 0} \frac{s\cosh s\tau}{s}$$

$$= e^{-t_0 s}$$

Exp Find

① $L\{\delta(t-\pi)\} = e^{-\pi s}$

② $L\{\delta(t-1)\} = e^{-s}$

③ $L\{\delta(t)\} = L\{\delta(t-0)\} = e^{-0s} = e^{0} = 1$

④ $L^{-1}(e^{-3s}) = \delta(t-3)$

⑤ $L^{-1}(4) = 4L^{-1}(1) = 4\,\delta(t)$

⑥ $L\{4\} = \frac{4}{s}$

Exp Find

$$L^{-1}\left(L^{-1}\left(\frac{7}{s}\right)\right) = L^{-1}(7) = 7L^{-1}(1) = 7\,\delta(t)$$

Exp Solve this IVP: $2y'' + y' + 2y = \delta(t-5)$, $y(0) = y'(0) = 0$

$$2L\{y''\} + L\{y'\} + 2L\{y\} = L\{\delta(t-5)\}$$

$$2\left(s^2 Y(s) - sy(0) - y'(0)\right) + \left(sY(s) - sy(0)\right) + 2Y(s) = e^{-5s}$$

$$(2s^2 + s + 2)\,Y(s) = e^{-5s}$$

$$Y(s) = \frac{e^{-5s}}{2s^2+s+2}$$

$$y(t) = L^{-1}\left(e^{-5s} H(s)\right)$$

$$= u_5(t)\, h(t-5)$$

$$= \frac{2}{\sqrt{15}}\, u_5(t)\, e^{-\frac{(t-5)}{4}} \sin \frac{\sqrt{15}}{4}(t-5)$$

$$H(s) = \frac{1}{2s^2+s+2}$$

$$h(t) = L^{-1}\left(\frac{1}{2(s^2 + \frac{s}{2} + 1)}\right)$$

$$= \frac{1}{2} L^{-1}\left(\frac{1}{(s+\frac{1}{4})^2 + \frac{15}{16}}\right)$$

$$= \frac{1}{2}\,\frac{4}{\sqrt{15}}\, L^{-1}\left(\frac{\frac{\sqrt{15}}{4}}{(s+\frac{1}{4})^2 + \frac{15}{16}}\right)$$

$$= \frac{2}{\sqrt{15}}\, e^{-\frac{t}{4}} \sin \frac{\sqrt{15}}{4} t$$

Remark D To find Laplace Transform for product of two functions: one is the unit impulse function $\delta(t - t_0)$
$\Rightarrow$ we use the following result

Exp show that $L\{\delta(t-t_0)f(t)\} = e^{-t_0 s} f(t_0)$
where f is cont.

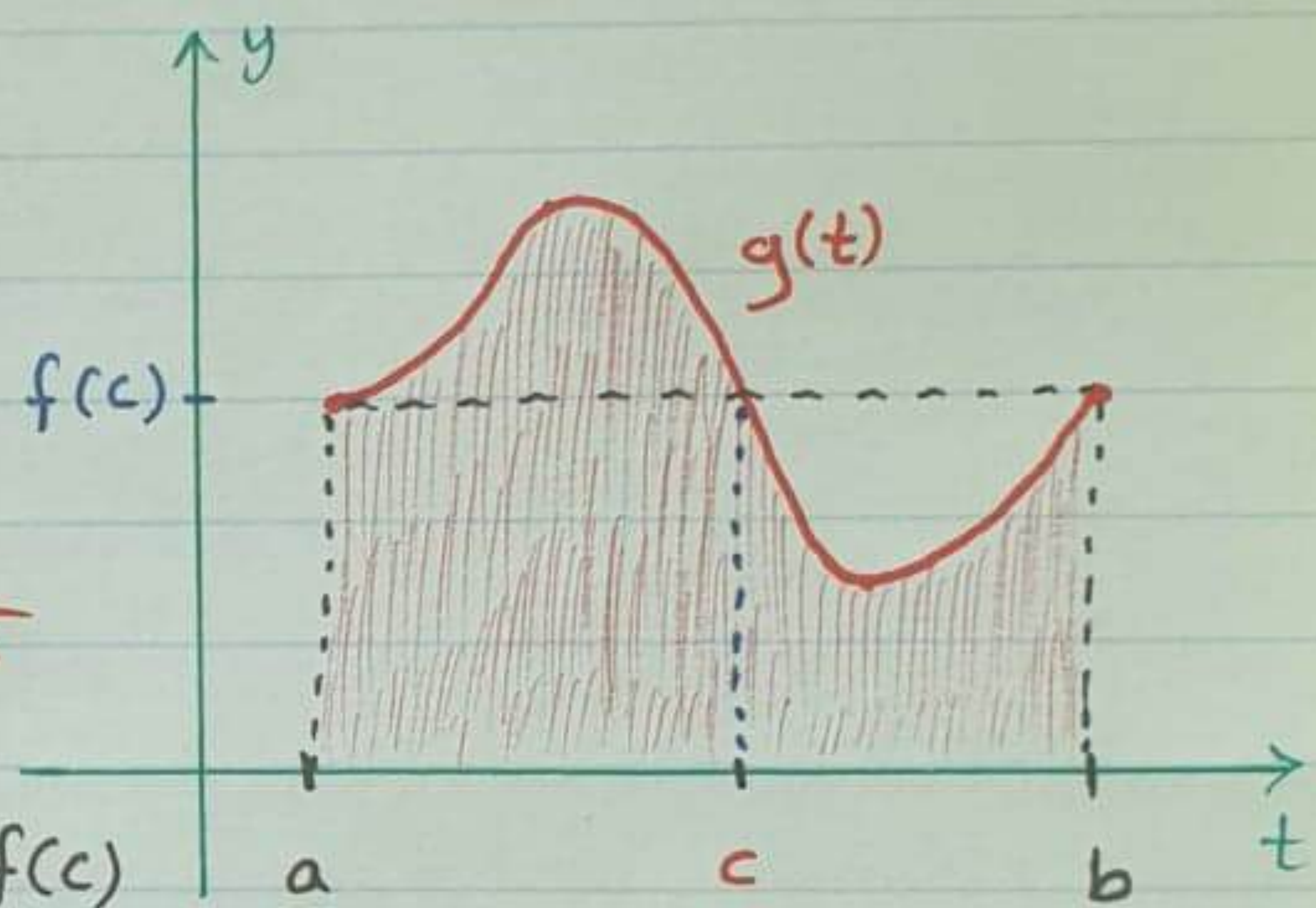
$$L\{\delta(t-t_0)f(t)\} = \int_0^\infty e^{-st}\,\delta(t-t_0)f(t)\,dt$$

$$= \lim_{\tau\to 0}\int_0^\infty e^{-st}\, d_\tau(t-t_0)\, f(t)\,dt$$

where the forcing function

$$d_\tau(t-t_0) = \begin{cases}\frac{1}{2\tau}, & t_0-\tau < t < \tau+t_0\\ 0, & \text{Otherwise}\end{cases}$$

$$= \lim_{\tau\to 0}\int_{t_0-\tau}^{t_0+\tau} e^{-st}\,\frac{1}{2\tau}\, f(t)\,dt$$



$$= \lim_{\tau\to 0}\frac{1}{2\tau}\, g(c)\,(b-a)$$

$t_0-\tau < c < t_0+\tau$

$$= \lim_{\tau\to 0}\frac{1}{2\tau}\, e^{-cs}\big(t_0+\tau-(t_0-\tau)\big)f(c)$$

$$= \lim_{\tau\to 0}\frac{1}{2\tau}\,(2\tau)\, e^{-cs} f(c)$$

$$= \lim_{\tau\to 0} e^{-cs} f(c)$$

$$= e^{-t_0 s} f(t_0)$$

since as $\tau\to 0 \Rightarrow c\to t_0$

g satisfies MVT for integrals on $[a,b] \Rightarrow$

$$\int_a^b g(t)\,dt = f(c)(b-a)$$

for some $c\in(a,b)$

Hence, $g(t) = e^{-st}f(t)$ satisfies MVT on $(t_0-\tau,\ t_0+\tau)$ since g is cont. function.

Exp ① $L\{\delta(t-\pi)\cos t\} = e^{-\pi s}\cos\pi = -e^{-\pi s}$

② $L\{\delta(t-\ln 2)\, e^{t}\} = e^{-(\ln 2)s}\, e^{\ln 2} = 2e^{-(\ln 2)s}$

③ $L\{\delta(t)\sin t\} = L\{\delta(t-0)\sin t\}$

$= e^{-0s}\sin 0 = (1)(0) = 0$

Exp Solve the IVP: $y'' + y = \delta(t-\pi)\cos t + u_2(t) + \delta(t)$

$y(0) = y'(0) = 0$

$L\{y''\} + L\{y\} = L\{\delta(t-\pi)\cos t\} + L\{u_2(t)\} + L\{\delta(t)\}$

$s^2 Y(s) - s y(0) - y'(0) + Y(s) = e^{-\pi s}\cos\pi + \dfrac{e^{-2s}}{s} + 1$

$(s^2+1)Y(s) = -e^{-\pi s} + \dfrac{e^{-2s}}{s} + 1$

$Y(s) = -\dfrac{e^{-\pi s}}{s^2+1} + \dfrac{e^{-2s}}{s(s^2+1)} + \dfrac{1}{s^2+1}$

$y(t) = -L^{-1}\left(\dfrac{e^{-\pi s}}{s^2+1}\right) + L^{-1}\left(e^{-2s}H(s)\right) + L^{-1}\left(\dfrac{1}{s^2+1}\right)$

$= -u_\pi(t)\sin(t-\pi) + u_2(t)\,h(t-2) + \sin t$

$= -u_\pi(t)\sin(t-\pi) + u_2(t)\left(1-\cos(t-2)\right) + \sin t$

$H(s) = \dfrac{1}{s(s^2+1)}$

$= \dfrac{A}{s} + \dfrac{Bs+C}{s^2+1}$

$= \dfrac{1}{s} - \dfrac{s}{s^2+1}$

$h(t) = L^{-1}\left(\dfrac{1}{s}\right) - L^{-1}\left(\dfrac{s}{s^2+1}\right)$

$= 1 - \cos t$

$y(t) = -u_\pi(t)\sin(t-\pi) - u_2(t)\cos(t-2) + u_2(t) + \sin t$

6.6 Convolution Integral (*)

Exp show that $L\{f(t)\,g(t)\} \neq L\{f(t)\}\,L\{g(t)\}$

$$f(t) = 3 \Rightarrow F(s) = L\{f(t)\} = L\{3\} = \frac{3}{s}$$

$$g(t) = \sin t \Rightarrow G(s) = L\{g(t)\} = L\{\sin t\} = \frac{1}{s^2+1}$$

$$f(t)\,g(t) = 3\sin t \Rightarrow L\{f(t)g(t)\} = L\{3\sin t\} = \frac{3}{s^2+1}$$

Hence, $L\{f(t)g(t)\} = \dfrac{3}{s^2+1} \neq \left(\dfrac{3}{s}\right)\left(\dfrac{1}{s^2+1}\right) = F(s)G(s)$

Def $f(t)$ convolution $g(t)$ is defined by

$$f(t) * g(t) = \int_0^t f(t-\tau)\,g(\tau)\,d\tau \quad \text{--- (1)}$$

Remark The integral in (1) is also called the convolution integral of f and g

---

Exp Show that $f * g = g * f$

$$f(t) * g(t) = \int_0^t f(t-\tau)\,g(\tau)\,d\tau$$

$u = t - \tau$
$du = -d\tau$

$$= -\int_t^0 f(u)\,g(t-u)\,du$$

$\tau = 0 \Rightarrow u = t$
$\tau = t \Rightarrow u = 0$

$\tau = t - u$

$$= \int_0^t g(t-u)\,f(u)\,du = g(t) * f(t)$$

Th If $h(t) = f(t) * g(t)$ then

$$H(s) = L\{h(t)\} = L\{f(t) * g(t)\}$$
$$= L\{f(t)\}\ L\{g(t)\}$$
$$= F(s)\ G(s)$$

Hence, $L^{-1}\Big(F(s)\,G(s)\Big) = L^{-1}\Big(L\{f(t)\}\ L\{g(t)\}\Big)$

$$= f(t) * g(t)$$

Exp Find Laplace Transform of

① $h(t) = e^{-t} * t^3$

$$H(s) = L\{e^{-t} * t^3\} = L\{e^{-t}\}\ L\{t^3\} = \left(\frac{1}{s+1}\right)\left(\frac{3!}{s^4}\right)$$
$$= \frac{6}{s^4(s+1)}$$

② $r(t) = 2 * u_2(t)$

$$R(s) = L[2 * u_2(t)\} = L\{2\}\ L\{u_2(t)\}$$
$$= \left(\frac{2}{s}\right)\left(\frac{e^{-2s}}{s}\right) = \frac{2e^{-2s}}{s^2}$$

③ $m(t) = \int_0^t (t-\tau)\sin 2\tau\ d\tau$

$m(t) = t * \sin 2t \Rightarrow M(s) = L\{t * \sin 2t\}$

$$M(s) = L\{t\}\ L\{\sin 2t\} = \left(\frac{1}{s^2}\right)\left(\frac{2}{s^2+4}\right) = \frac{2}{s^2(s^2+4)}$$

④ $s(t) = \cos t * t^2$

$$S(s) = L\{\cos t * t^2\} = L\{\cos t\}\, L\{t^2\} = \left(\frac{s}{s^2+1}\right)\left(\frac{2}{s^3}\right)$$

$$= \frac{2s}{s^3(s^2+1)}$$

⑤ $f(t) = \int_0^t t\, e^{\tau}\, d\tau$

(5.1)

$$f(t) = \int_0^t (t - \tau + \tau)\, e^{\tau}\, d\tau = \int_0^t (t-\tau) e^{\tau}\, d\tau + \int_0^t \tau e^{\tau}\, d\tau$$

$$= t * e^t + \int_0^t \tau e^{\tau}\, d\tau$$

$$\begin{array}{cc} \tau & e^{\tau} \\ 1 \xrightarrow{+} & e^{\tau} \\ 0 \xrightarrow{-} & e^{\tau} \end{array}$$

$$= t * e^t + \left(\tau e^{\tau} - e^{\tau}\right)\Big|_0^t$$

$$= t * e^t + t e^t - e^t - (0 - 1)$$

$$F(s) = L\{t * e^t\} + L\{t e^t\} - L\{e^t\} + L\{1\}$$

$$G(s) = L\{t\} = \frac{1}{s^2}$$

$$= L\{t\}\, L\{e^t\} + G(s-1) - \frac{1}{s-1} + \frac{1}{s}$$

$$= \left(\frac{1}{s^2}\right)\left(\frac{1}{s-1}\right) + \frac{1}{(s-1)^2} - \frac{1}{s-1} + \frac{1}{s} = \frac{1}{(s-1)^2} - \frac{1}{s^2}$$

(5.2) $f(t) = \int_0^t t\, e^{\tau}\, d\tau = t\left(e^{\tau}\Big|_0^t\right) = t\left(e^t - 1\right)$

$$= t e^t - t$$

$$F(s) = L\{t e^t\} - L\{t\} = G(s-1) - \frac{1}{s^2}$$

$$= \frac{1}{(s-1)^2} - \frac{1}{s^2}$$

Exp Find Inverse transform of $H(s) = \dfrac{2}{s^2(s-2)}$

(S1) $h(t) = L^{-1}(H(s)) = L^{-1}\left(\dfrac{2}{s^2(s-2)}\right)$

$$= L^{-1}\left(\frac{A}{s-2} + \frac{B}{s} + \frac{C}{s^2}\right)$$

$A = \frac{1}{2}$

$B = -\frac{1}{2}$

$C = -1$

$$= \frac{1}{2} L^{-1}\left(\frac{1}{s-2}\right) - \frac{1}{2} L^{-1}\left(\frac{1}{s}\right) - L^{-1}\left(\frac{1}{s^2}\right)$$

$$= \frac{1}{2} e^{2t} - \frac{1}{2} - t$$

(S2) $H(s) = 2\left(\dfrac{1}{s^2}\right)\left(\dfrac{1}{s-2}\right) = 2\, L\{t\}\, L\{e^{2t}\}$

$$h(t) = 2 L^{-1}\left(L\{t\}\, L\{e^{2t}\}\right) = 2t * e^{2t}$$

$$= 2\int_0^t (t-\tau)\, e^{2\tau}\, d\tau$$

$t-\tau$	$e^{2\tau}$
-1	$\frac{1}{2} e^{2\tau}$
0	$\frac{1}{4} e^{2\tau}$

$$= 2\left((t-\tau)\left(\tfrac{1}{2}\right) e^{2\tau} + \frac{1}{4} e^{2\tau} \Big|_0^t \right)$$

$$= 2\left(0 + \frac{1}{4} e^{2t} - \left(\frac{t}{2} + \frac{1}{4}\right)\right)$$

$$= \frac{1}{2} e^{2t} - t - \frac{1}{2}$$

Exp Solve the integral equation

$$\phi'(t) + 2\phi(t) + \int_0^t 2\,\phi(u)\,du = 0 \quad , \phi(0) = 1$$

$$\phi'(t) + 2\phi(t) + 2 * \phi(t) = 0$$

$$L\{\phi'(t)\} + 2\,L\{\phi(t)\} + L\{2 * \phi(t)\} = L\{0\}$$

$$s\,\Phi(s) - \phi(0) + 2\,\Phi(s) + L\{2\}\,L\{\phi(t)\} = \frac{0}{s}$$

$$(s+2)\,\Phi(s) - 1 + \frac{2}{s}\,\Phi(s) = 0$$

$$\left(s + 2 + \frac{2}{s}\right)\Phi(s) = 1$$

$$\left(\frac{s^2 + 2s + 2}{s}\right)\Phi(s) = 1$$

$$\Phi(s) = \frac{s}{s^2 + 2s + 2}$$

$$\phi(t) = L^{-1}\left(\frac{s}{s^2 + 2s + 2}\right)$$

$$= L^{-1}\left(\frac{s+1-1}{(s+1)^2 + 1}\right)$$

$$= L^{-1}\left(\frac{s+1}{(s+1)^2+1}\right) - L^{-1}\left(\frac{1}{(s+1)^2+1}\right)$$

$$= e^{-t}\cos t - e^{-t}\sin t$$

Exp show that $1 * \delta(t-2) = u_2(t)$

$$L\{1 * \delta(t-2)\} = L\{1\}\, L\{\delta(t-2)\}$$

$$= \left(\frac{1}{s}\right) e^{-2s}$$

Hence,

$$1 * \delta(t-2) = L^{-1}\left(\frac{e^{-2s}}{s}\right) = u_2(t)$$

Exp Solve the IVP: $y'' = \delta(t-\pi) - \delta(t-e)$, $y(0)=0$, $y'(0)=0$

$$L\{y''\} = L\{\delta(t-\pi)\} - L\{\delta(t-e)\}$$

$$s^2 Y(s) - sy(0) - y'(0) = e^{-\pi s} - e^{-es}$$

$$s^2 Y(s) = e^{-\pi s} - e^{-es}$$

$$Y(s) = \frac{e^{-\pi s}}{s^2} - \frac{e^{-es}}{s^2}$$

$$y(t) = L^{-1}\left(\frac{e^{-\pi s}}{s^2}\right) - L^{-1}\left(\frac{e^{-es}}{s^2}\right)$$

$$= u_\pi(t)\,(t-\pi) - u_e(t)\,(t-e)$$

Exp Let $f(t) = u_1(t)$ and $g(t) = t$. Find $(f*g)(t)$

$$L\{f*g\} = L\{u_1(t) * t\} = L\{u_1(t)\}\, L\{t\} = \frac{e^{-s}}{s}\left(\frac{1}{s^2}\right)$$

Hence,

$$f*g = L^{-1}\left(\frac{e^{-s}}{s^3}\right) = \frac{1}{2} L^{-1}\left(\frac{2e^{-s}}{s^3}\right) = \frac{1}{2} u_1(t)\,(t-1)^2$$

Exp Solve the integral equation:

$$h(t) + \int_0^t (t-\xi)\, h(\xi)\, d\xi = t + u_2(t)$$

$$h(t) + t * h(t) = t + u_2(t)$$

$$H(s) + \frac{1}{s^2} H(s) = \frac{1}{s^2} + \frac{e^{-2s}}{s}$$

$$H(s)\left(1 + \frac{1}{s^2}\right) = \frac{1}{s^2} + \frac{e^{-2s}}{s}$$

$$\left(\frac{s^2+1}{s^2}\right) H(s) = \frac{1}{s^2} + \frac{e^{-2s}}{s}$$

$$(s^2+1) H(s) = 1 + s\, e^{-2s}$$

$$H(s) = \frac{1}{s^2+1} + \frac{s\, e^{-2s}}{s^2+1}$$

$$h(t) = L^{-1}\left(\frac{1}{s^2+1}\right) + L^{-1}\left(\frac{s\, e^{-2s}}{s^2+1}\right)$$

$$= \sin t + u_2(t) \cos(t-2)$$

Exp Show that $L\{f(ct)\} = \frac{1}{c} F\left(\frac{s}{c}\right)$ where $F(s) = L\{f(t)\}$

$$L\{f(ct)\} = \int_0^\infty e^{-st} f(ct)\, dt = \int_0^\infty e^{-s\left(\frac{u}{c}\right)} f(u) \frac{du}{c}$$

$u = ct$

$du = c\, dt$

$\frac{du}{c} = dt$

$$= \frac{1}{c} \int_0^\infty e^{-\left(\frac{s}{c}\right) u} f(u)\, du$$

$t = \frac{u}{c}$

$t = 0 \Rightarrow u = 0$

$t \to \infty \Rightarrow u \to \infty$

$$= \frac{1}{c} F\left(\frac{s}{c}\right)$$