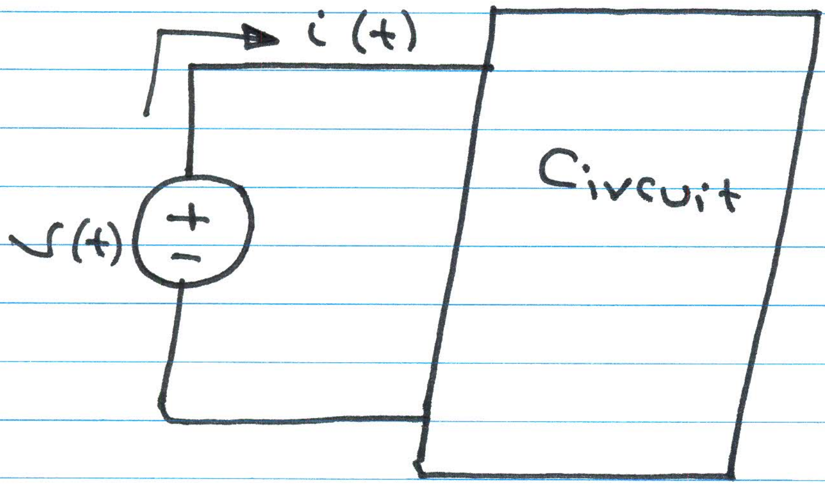
Sinusoidal Steady State
Power Calculation

Instantaneous Power : $p(t)$



$$v(t) = V_m \cos(\omega t + \theta_v)$$

$$i(t) = I_m \cos(\omega t + \phi_i)$$

$$p(t) = v(t)\, i(t)$$

$$p(t) = V_m I_m \cos(\omega t + \theta_v) \cos(\omega t + \phi_i)$$

$$\cos\alpha \cos\beta = \frac{1}{2}\left[\cos(\alpha - \beta) + \cos(\alpha + \beta)\right]$$

$$\therefore p(t) = \frac{V_m I_m}{2}\left[\cos(\theta_v - \phi_i) + \cos(2\omega t + \theta_v + \phi_i)\right]$$

Constant

Twice the excitation frequency

Example

$v(t) = 4\cos(\omega t + 60^\circ)$ V

$Z(j\omega) = 2\angle 30^\circ$ Ω

Find $p(t)$.

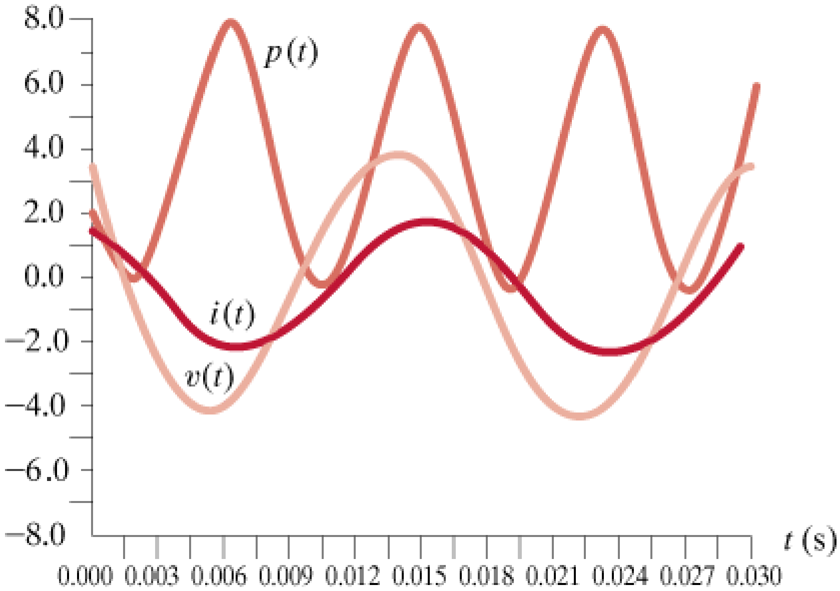
$$\vec{I} = \frac{\vec{V}}{Z} = \frac{4\angle 60^\circ}{2\angle 30^\circ} = 2\angle 30^\circ \text{ A}$$

$\therefore i(t) = 2\cos(\omega t + 30^\circ)$ A

$p(t) = v(t)\, i(t)$

$p(t) = 4\cos 30^\circ + 4\cos(2\omega t + 90^\circ)$

$p(t) = 3.46 + 4\cos(2\omega t + 90^\circ)$


8.0
6.0
4.0
2.0
0.0
−2.0
−4.0
−6.0
−8.0
p(t)
i(t)
v(t)
t (s)
0.000 0.003 0.006 0.009 0.012 0.015 0.018 0.021 0.024 0.027 0.030

Average Power : Real Power

$$P_{av} = \frac{1}{T}\int_0^T p(t)\,dt$$

$$P_{av} = \frac{1}{2} V_m I_m \cos(\theta_v - \phi_i)$$

$$\theta_v - \phi_i = \theta_z$$

$$\therefore \; P_{av} = \frac{1}{2} V_m I_m \cos\theta_z$$

1) For Resistor

$$\theta_v - \phi_i = 0 \longrightarrow \theta_z = 0$$

$$\therefore \; P_{av} = \frac{1}{2} V_m I_m = \frac{V_m^2}{2R} = \frac{I_m^2 R}{2}$$

2) For Inductor

$$\theta_v - \phi_i = 90^\circ$$

$$P_{av} = 0$$

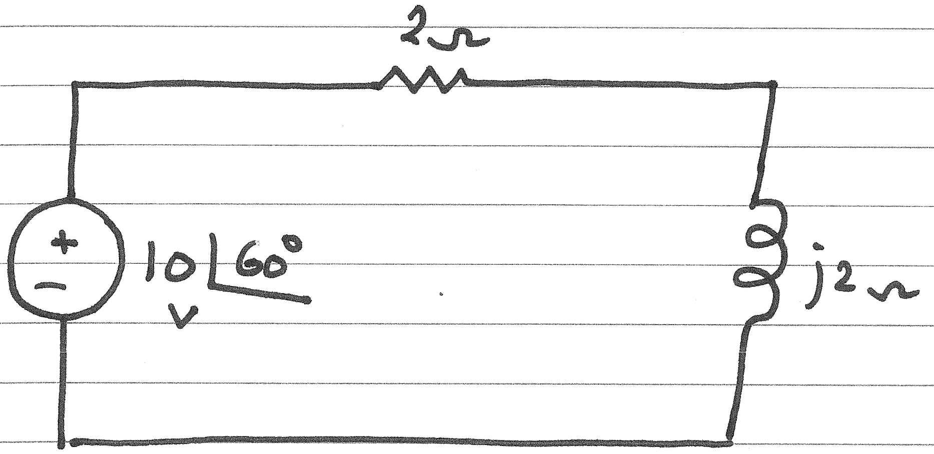
3) For Capacitor

$$\theta_v - \phi_i = -90^\circ$$

$\therefore \quad P_{av} = 0$

$\therefore$ Reactive impedances absorb no average power

Example



Find the average power absorbed by each elemen.

$$\vec{I} = \frac{10\angle 60^\circ}{2+j2} = 3.53\angle 15^\circ \; A$$

$$P_{av} = \frac{V_m I_m}{2} \cos(\theta_v - \phi_i)$$

$$P_{av_{j2\Omega}} = 0$$

$$P_{av_{2\Omega}} = \frac{I_m^2 R}{2} = \frac{(3.53)^2 \cdot 2}{2} = 12.5\,W$$

To Calculate the average power supplied by the source

$$P_{av_{V_s}} = \frac{V_m I_m}{2} \cos(\theta_v - \phi_i)$$

$I_m = 3.53\,A$

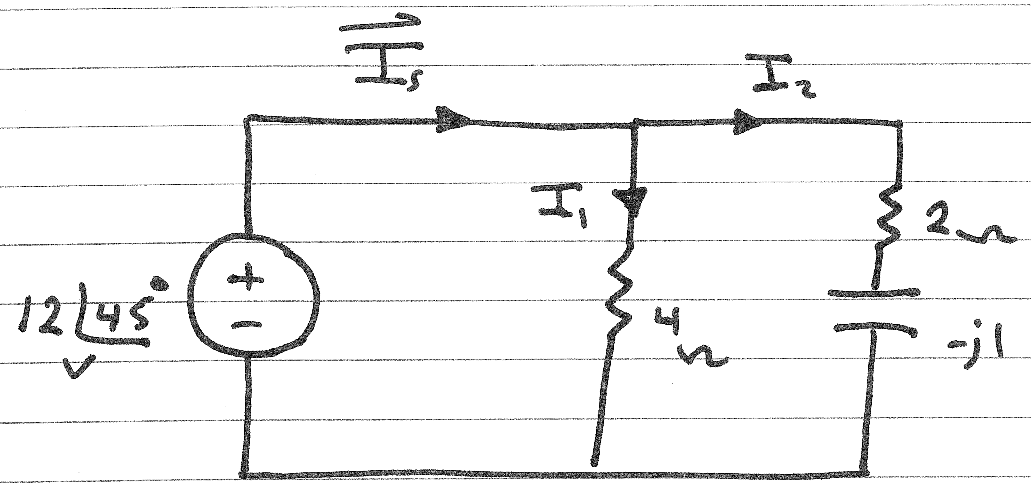
$V_m = 10\,V$

$\theta_v = 60^\circ$, $\phi_i = 15^\circ$

$$\therefore\ P_{av_{V_s}} = \frac{(10)(3.53)}{2} \cos(60 - 15^\circ)$$

$$= 12.5 \text{ Watt}$$

Example



Determine the average power absorbed by each resistor.

Determine the total average power absorbed and the average power supplied by the source.

$$\vec{I_1} = \frac{12\angle 45^\circ}{4\Omega} = 3\angle 45^\circ \text{ A}$$

$$\vec{I_2} = \frac{12\angle 45^\circ}{2-j1} = 5.36\angle 71.57^\circ \text{ A}$$

$$\vec{I_s} = \vec{I_1} + \vec{I_2} = 8.15\angle 62.1^\circ \text{ A}$$

1) $P_{4\Omega} = \frac{I_{1m}^2 \cdot 4}{2} = 18\text{W}$

2) $P_{2\Omega} = \frac{I_{2m}^2 \cdot 2}{2} = 28.7\text{ W}$

$\therefore$ Total Average power absorbed = 46.7 W

$$P_{V_s} = \frac{V_m I_m}{2} \cos(\theta_v - \phi_i)$$

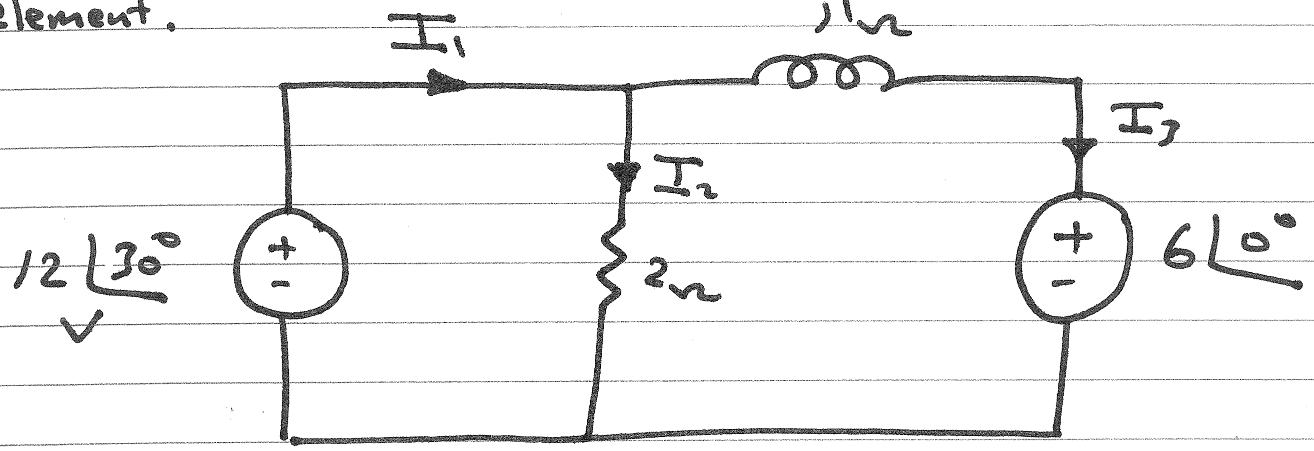
$$P_{V_s} = \frac{(12)(8.16)}{2} \cos(45 - 62.1)$$

$$P_{V_s} = 46.7 \text{ W}$$

$$\therefore \quad P_{V_s} = P_{4\Omega} + P_{2\Omega} + P_{-j1}$$

Example

Determine average power absorbed or supplied by each element.



$$\vec{I_2} = \frac{12\angle 30^\circ}{2} = 6\angle 30^\circ \ A$$

$$\vec{I_3} = \frac{12\angle 30^\circ - 6\angle 30^\circ}{j1} = 7.43\angle -36.19^\circ \ A$$

$$\vec{I_1} = \vec{I_2} + \vec{I_3} = 11.29\angle -7.07^\circ \ A$$

$$P_{2\Omega} = \frac{I_{2m}^2 \cdot 2}{2} = 36\ W$$

$$P_{12\angle 30^\circ} = \frac{V_m I_m}{2} \cos(\theta_v - \phi_i)$$

$$P_{12\angle 30^\circ} = \frac{(12)(11.29)}{2} \cos\left(30 - (-7.07)\right)$$

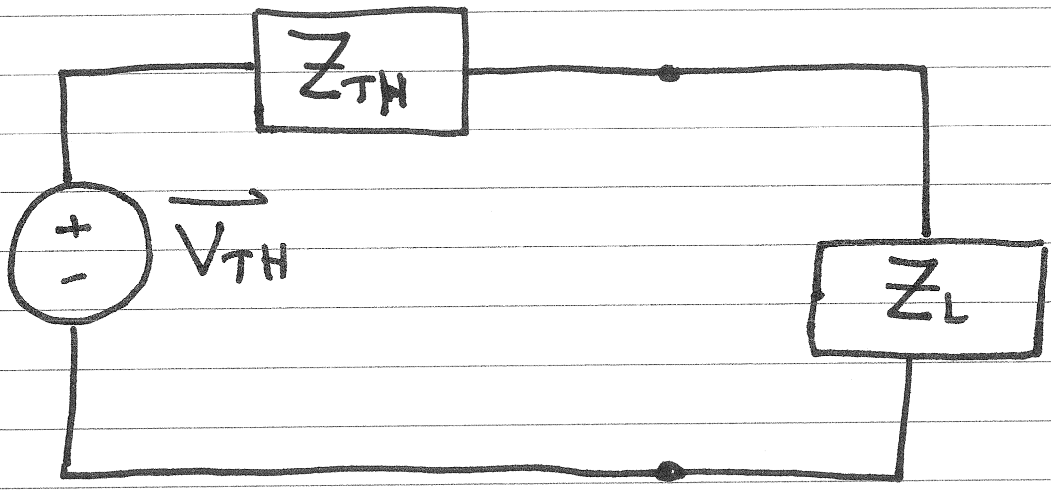
$$P_{12\angle 30^\circ} = 54\ W \quad \text{Supply}$$

$$P_{6\angle 0^\circ} = \frac{V_m I_m}{2} \cos(\theta_v - \phi_i)$$

$$P_{6\angle 0^\circ} = \frac{(6)(7.43)}{2} \cos(0 - (-36.19^\circ))$$

$$P_{6\angle 0^\circ} = 18\ W \quad \text{absorbed}$$

Maximum Average power Transfer



$$Z_{TH} = R_{TH} + j\,X_{TH}$$

$$Z_L = R_L + j\,X_L$$

$$P_L = \frac{I_{Lm}^2 \cdot R_L}{2}$$

$$\vec{I} = \frac{\vec{V}_{TH}}{Z_{TH} + Z_L}$$

$$\vec{I} = \frac{\vec{V}_{TH}}{(R_{TH} + R_L) + j(X_{TH} + X_L)}$$

$$P_L = \frac{I_m^2 R_L}{2}$$

$$P_L = \frac{1}{2} \frac{V_{TH}^2 \cdot R_L}{(R_{TH}+R_L)^2 + (X_{TH}+X_L)^2}$$

$$\frac{\partial P_L}{\partial R_L} = 0 \quad ; \quad \frac{\partial P_L}{\partial X_L} = 0$$

$$\frac{\partial P_L}{\partial X_L} = \frac{-2V_{TH}^2 R_L (X_L + X_{TH})}{2\left[(R_L+R_{TH})^2 + (X_L+X_{TH})^2\right]^2}$$

$$\text{For } \frac{\partial P_L}{\partial X_L} = 0 \longrightarrow X_L = -X_{TH}$$

$$\frac{\partial P_L}{\partial R_L} = \frac{V_{TH}^2\left[(R_L+R_{TH})^2 + (X_L+X_{TH})^2 - 2R_L(R_L+R_{TH})\right]}{2\left[(R_L+R_{TH})^2 + (X_L+X_{TH})^2\right]^2}$$

$$\text{For } \frac{\partial P_L}{\partial R_L} = 0 \longrightarrow R_L = \sqrt{R_{TH}^2 + (X_L+X_{TH})^2}$$

$$X_L = -X_{TH}$$

$$\therefore R_L = R_{TH}$$

$$\therefore Z_L = Z_{TH}^*$$

$$P_{L,max} = \frac{1}{8} \frac{V_{TH}^2}{R_L}$$

$\therefore$ For maximum average power Transfer

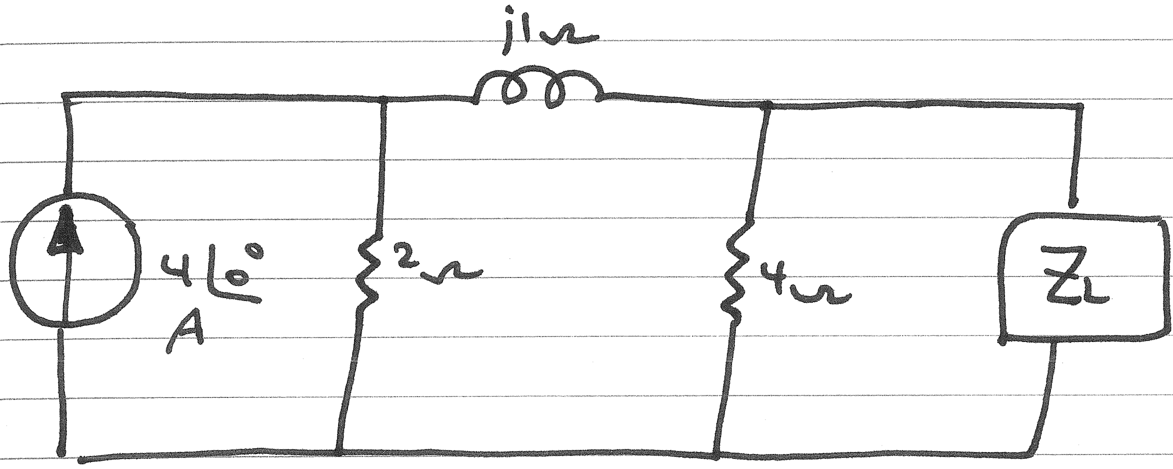
$$\overline{Z}_L = Z_{TH}^*$$

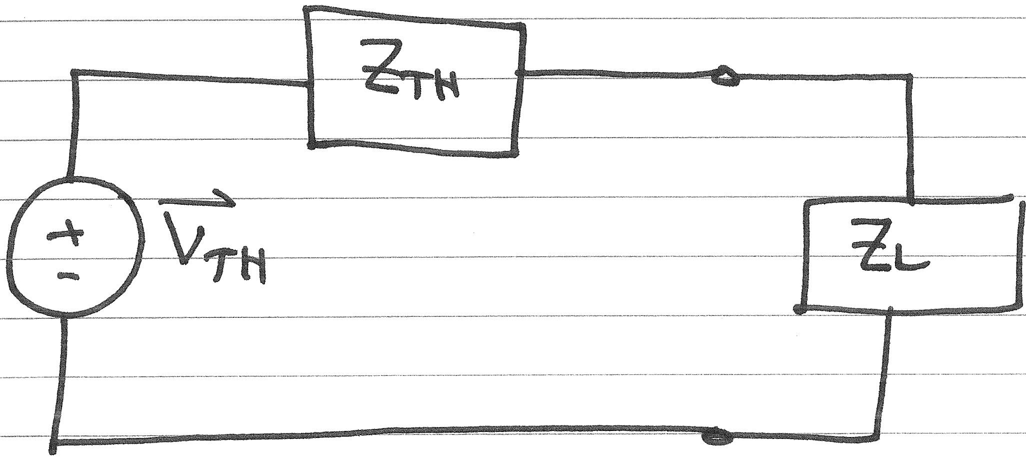
$$P_{L,max} = \frac{1}{8}\frac{V_{TH}^2}{R_{TH}}$$

Example

Find Z_L for maximum average power transfer.

Compute the maximum average power supplied to the load.





$$\vec{V}_{TH} = 4\angle 0^\circ \, \frac{2}{2 + j1 + 4} \cdot 4 = 5.28\angle -9.46^\circ \ \text{V}$$

$$Z_{TH} = 4\,\Omega \parallel (2+j1)\,\Omega$$

$$Z_{TH} = (1.4 + j\,0.43)\,\Omega$$

$$\therefore Z_L = (1.4 - j\,0.43)\,\Omega$$

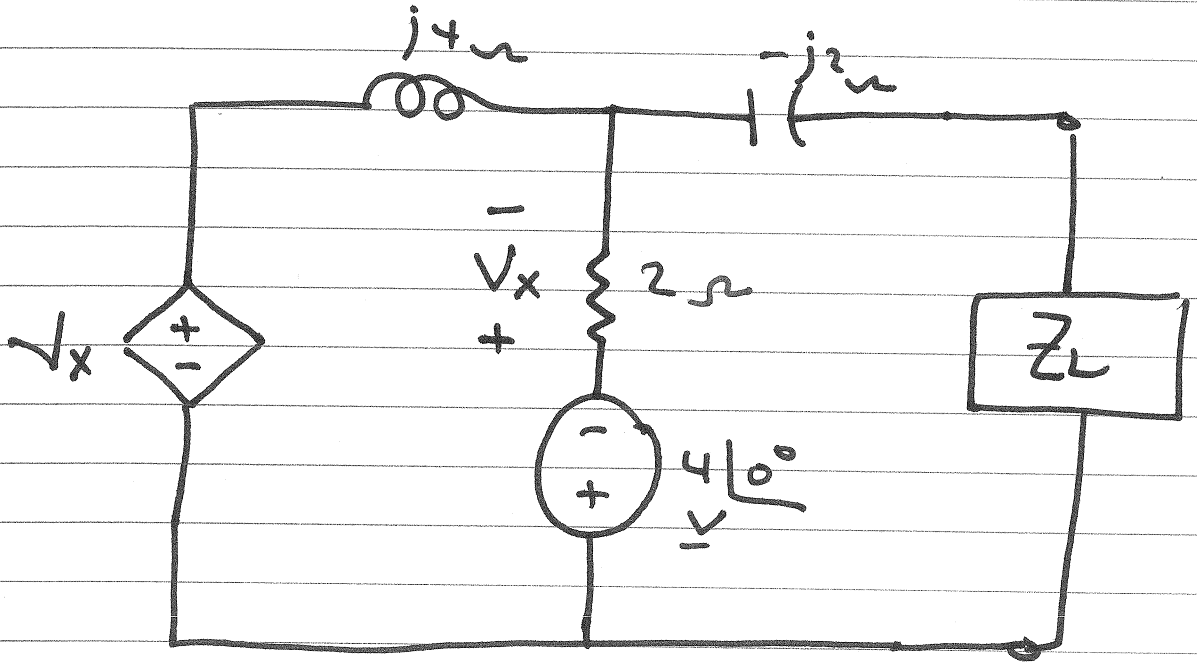
$$P_{,max} = \frac{1}{8}\,\frac{V_{TH}^2}{R_{TH}}$$

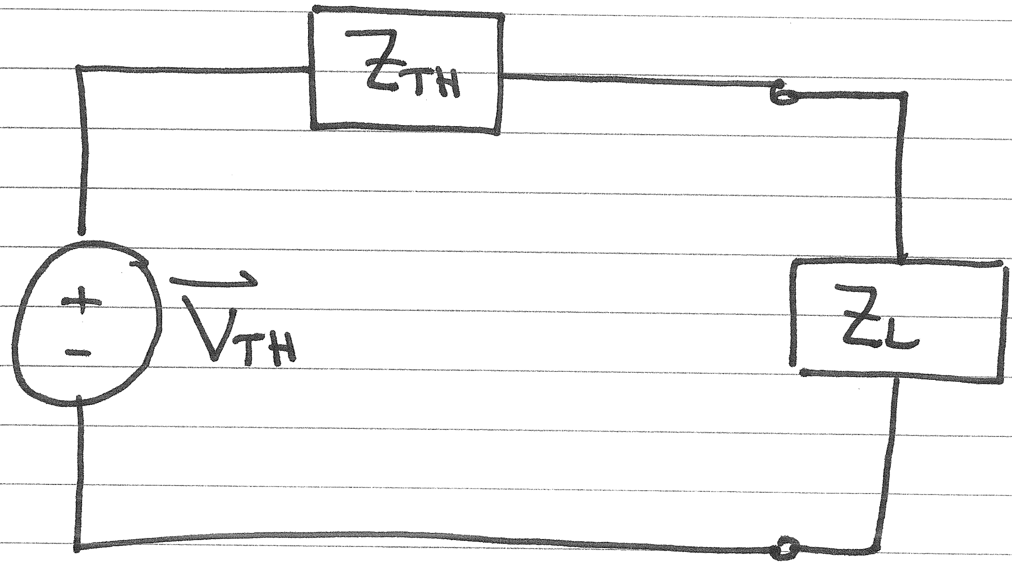
$$P_{,max} = 2.489\ W$$

Example

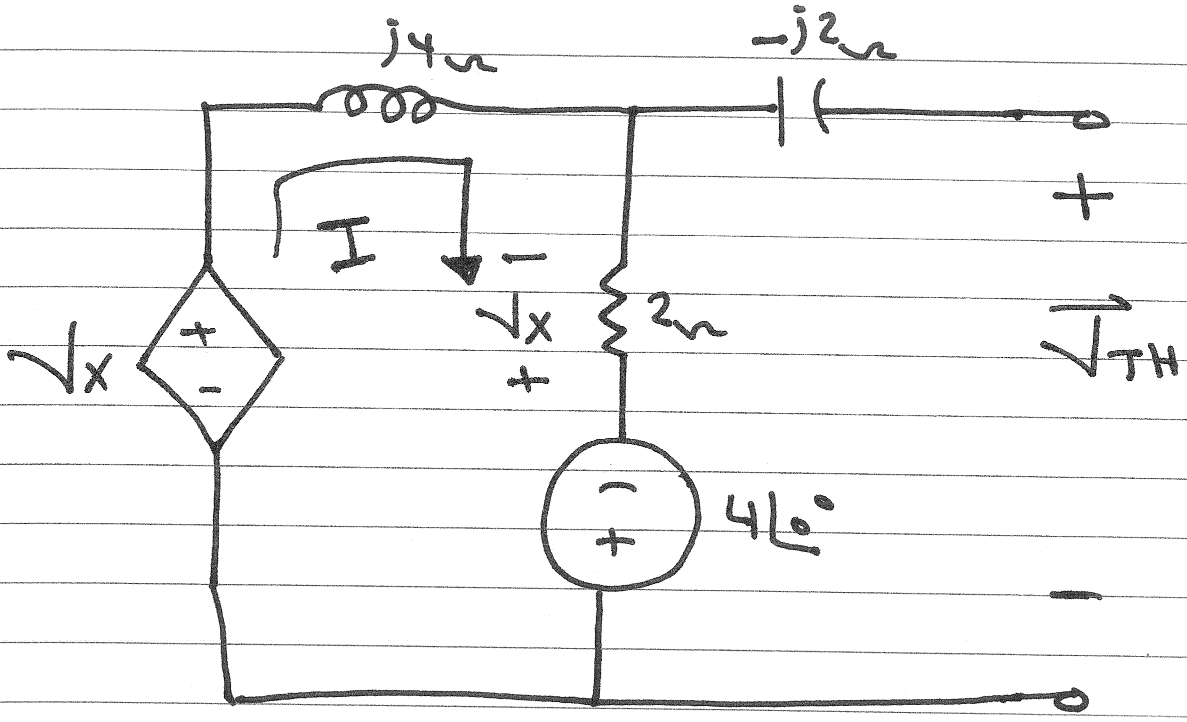
Find Z_L for maximum average power transfer

Compute the maximum average power supplied to Z_L





$$Z_L = Z_{TH}^{*}$$



$$\vec{V}_{TH} = 2\vec{I} - 4\angle 0^\circ$$

$$\vec{I} = \frac{\vec{V}_x + 4\angle 0^\circ}{2 + j4}$$

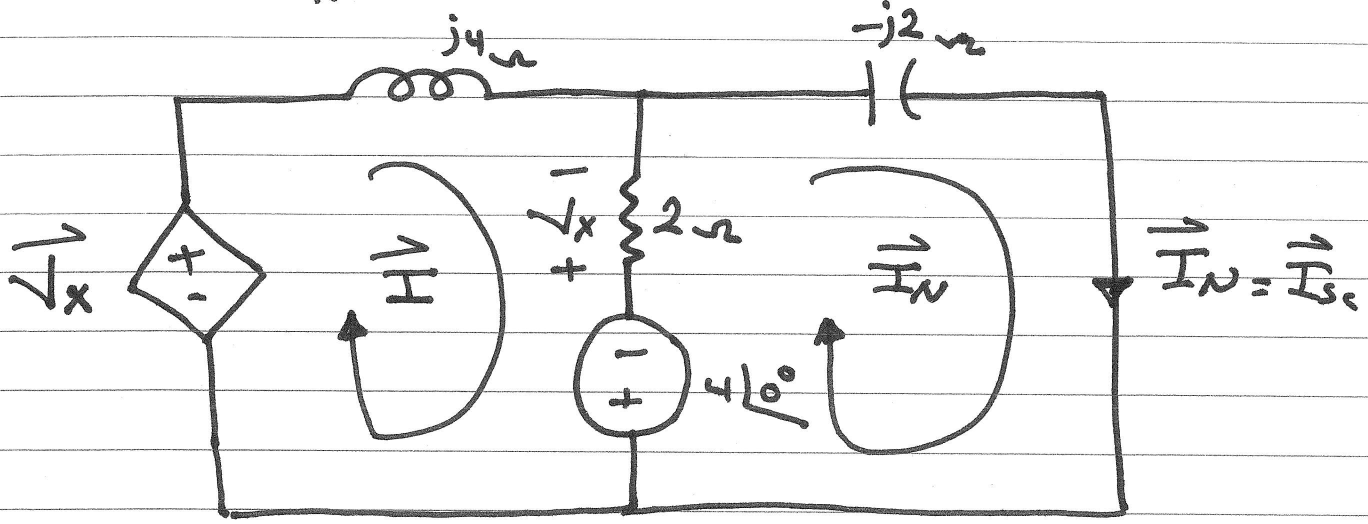
$$\vec{V}_x = -2\vec{I}$$

$$\vec{I} = 0.707\angle -45^\circ \ A$$

$$\therefore \vec{V}_{TH} = (-3 - j1)\,V$$

$$\vec{V}_{TH} = 3.16\angle 198.43^\circ \ V$$

$$Z_{TH} = \frac{\vec{V}_{TH}}{\vec{I}_N}$$



KVL for mesh 1 :

$$\vec{V}_x + 4\angle 0° = (2+j4)\vec{I} - 2\vec{I}_N$$

$$\vec{V}_x = 2(\vec{I}_N - \vec{I})$$

KVL for mesh 2 :

$$-4\angle 0° = -2\vec{I} + (2-j2)\vec{I}_N$$

Solving for $\vec{I}_N$

$$\vec{I}_N = (-1-j2)\ A$$

$$\vec{I}_N = 2.24\angle 243.43°\ A$$

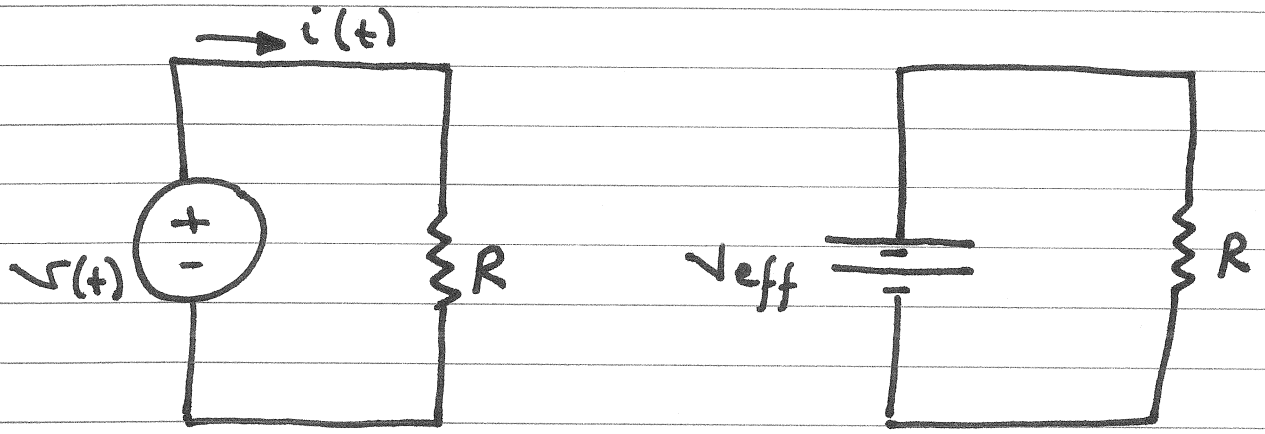
$$\therefore Z_{TH} = \frac{\vec{V}_{TH}}{\vec{I}_N} = 1.41\angle -45°\ \Omega = (1-j1)\ \Omega$$

$$\therefore Z_L = Z_{TH}^* = 1.41\angle +45°\ \Omega = (1+j1)\ \Omega$$

$$\therefore P_{L,max} = \frac{V_{TH}^2}{8R_{TH}} = 1.25\ W$$

Effective or RMS value

The effective value of a periodic voltage (current) is the dc voltage (current) that delivers the same average power to a resistor as the periodic voltage (current).



let $v(t) = V_m \cos(\omega t + \theta_v)$

$$\therefore \quad P_1 = \frac{V_m^2}{2R}$$

$$P_2 = \frac{V_{eff}^2}{R}$$

$$P_1 = P_2$$

$$\therefore \quad \frac{V_m^2}{2R} = \frac{V_{eff}^2}{R}$$

$$\therefore V_{eff} = \frac{V_m}{\sqrt{2}}$$

RMS : Root Mean Square

let $v(t) = V_m \cos(\omega t + \theta_v)$

$$V_{RMS} = \sqrt{\frac{1}{T}\int_0^T V_m^2 \cos^2(\omega t + \theta_v)\,dt}$$

$$V_{RMS} = V_m \sqrt{\frac{1}{T}\int_0^T \cos^2(\omega t + \theta_v)\,dt}$$

$$V_{RMS} = V_m \sqrt{\frac{1}{T}\int_0^T \frac{1}{2}\left(1 + \cos 2(\omega t + \theta_v)\right)dt}$$

$$V_{RMS} = V_m \frac{1}{\sqrt{2}}$$

$$P_{av} = \frac{V_m I_m}{2} \cos(\theta_v - \phi_i)$$

$$P_{av} = V_{rms} I_{rms} \cos(\theta_v - \phi_i)$$

For a resistor

$$P_{av} = V_{rms} I_{rms} \cos(\theta_v - \phi_i)$$

$$V_{rms} = R I_{rms} \quad ; \quad \theta_v - \phi_i = 0$$

$$\therefore \quad P_{av} = \frac{V_{rms}^2}{R}$$

$$\therefore \quad P_{av} = I_{rms}^2 R$$

Apparent power and Power factor

$$P_{av} = V_{rms} I_{rms} \cos(\theta_v - \phi_i)$$

$$P_{apparent} = V_{rms} I_{rms}$$

$P_{apparent}$ measured in VA

PF = Power factor

$$PF = \cos(\theta_v - \phi_i)$$

$$\therefore \quad P_{av} = P_a \cdot PF$$

1) For Resistor

$$\theta_v - \phi_i = 0$$

$\therefore$ PF = 1

2) For inductor

$$\theta_v - \phi_i = +90^\circ$$

$\therefore$ PF = 0

3) For Capacitor

$$\theta_v - \phi_i = -90^\circ$$

$\therefore$ PF = 0

4) For Inductive Load

$$90^\circ > \theta_v - \phi_i > 0$$

$1 > PF > 0$ Lagging Power factor