14.4 The Chain Rule

* Chain Rule for functions of single variable (section 3.6):

if $w = f(x)$ is diff function of x and
$x = g(t)$ is diff function of t, then w is diff function of t:

$$\frac{dw}{dt} = \frac{dw}{dx}\,\frac{dx}{dt}$$

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* Chain Rule for functions of two variables:

Th (1 indep. variable and 2 intermediate variables)

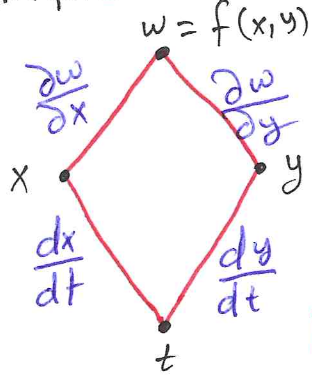
If $w = f(x,y)$ is diff and $x = x(t)$, $y = y(t)$ are diff functions of t, then the composite $w = f(x(t), y(t))$ is diff function of t:

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} \quad (*)$$

where $\frac{\partial w}{\partial x} = \frac{\partial f}{\partial x} = f_x$, $\frac{\partial w}{\partial y} = \frac{\partial f}{\partial y} = f_y$ } partial derivatives

$\frac{dx}{dt} = x'(t)$ and $\frac{dy}{dt} = y'(t)$ } ordinary derivatives

Branch diagram for (*):



Dep. variable

Intermediat variables

Indep. variable

Note that $\frac{dw}{dt}(t_0) = \frac{\partial w}{\partial x}(x_0,y_0)\,\frac{dx}{dt}(t_0) + \frac{\partial w}{\partial y}(x_0,y_0)\,\frac{dy}{dt}(t_0)$

Exp [a] Find $\frac{dw}{dt}$ for $w = x^2 + y^2$, $x = \cos t + \sin t$, $y = \cos t - \sin t$ using the chain Rule and without using the chain Rule.

[b] Find $\left.\frac{dw}{dt}\right|_{t=0}$

[a] • $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}\dfrac{dy}{dt}$

$$= (2x)(\cos t - \sin t) + (2y)(-\sin t - \cos t)$$
$$= 2(\cos t + \sin t)(\cos t - \sin t) - 2(\cos t - \sin t)(\cos t + \sin t)$$
$$= 2(\cos^2 t - \sin^2 t) - 2(\cos^2 t - \sin^2 t)$$
$$= 0$$

• $w = x^2 + y^2 = (\cos t + \sin t)^2 + (\cos t - \sin t)^2$

$$= \cos^2 t + \sin^2 t + \cancel{2\cos t \sin t} + \cos^2 t + \sin^2 t - \cancel{2\cos t \sin t}$$
$$= 2$$

$\dfrac{dw}{dt} = 0$

[b] $\dfrac{dw}{dt}(0) = 0$

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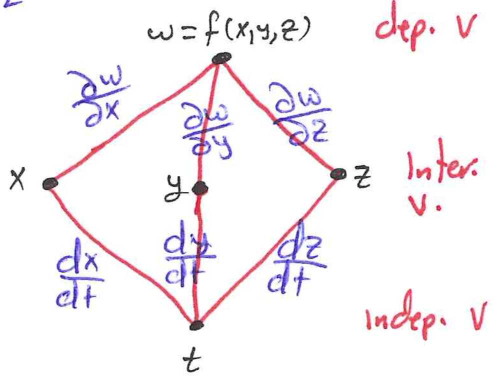
* Chain Rule for functions of three variables

Th (1 indep. variable and 3 intermediate variables)

If $w = f(x,y,z)$ is diff and x, y, z are diff functions of t, then w is diff function of t:

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

Branch diagram



Exp Find $\left.\dfrac{dw}{dt}\right|_{t=0}$ for

$w = \ln(x^2+y^2+z^2)$, $x = \cos t$

$y = \sin t$

$z = 4\sqrt{t}$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$= \left(\frac{2x}{x^2+y^2+z^2}\right)(-\sin t) + \left(\frac{2y}{x^2+y^2+z^2}\right)(\cos t) + \left(\frac{2z}{x^2+y^2+z^2}\right)\left(\frac{2}{\sqrt{t}}\right)$$

$$\frac{dw}{dt} = \frac{\cancel{-2\sin t \cos t}}{\cancel{x^2+y^2+z^2}} + \frac{\cancel{2\sin t \cos t}}{\cancel{x^2+y^2+z^2}} + \frac{16}{1+16t}$$

$$= \frac{16}{1+16t} \quad \Rightarrow \frac{dw}{dt}(0) = 16$$

Note that $w = \ln(x^2+y^2+z^2) = \ln(1+16t) \Rightarrow \frac{dw}{dt} = \frac{16}{1+16t}$

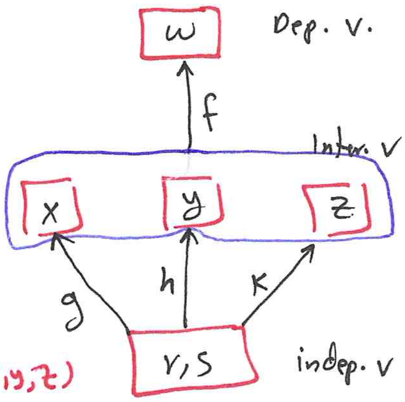
* Chain Rule for functions defined on surfaces:

Th (2 indep. variables and 3 intermediate variables)

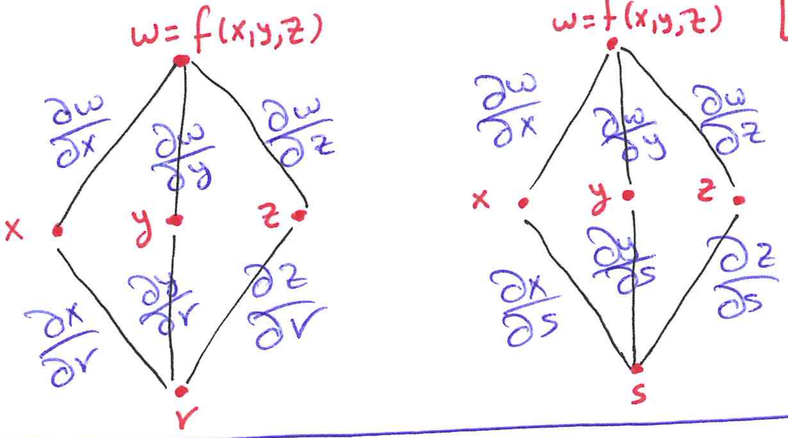
If $w = f(x,y,z)$, $x = g(r,s)$, $y = h(r,s)$, $z = K(r,s)$ are diff functions, then w has partial derivatives

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$



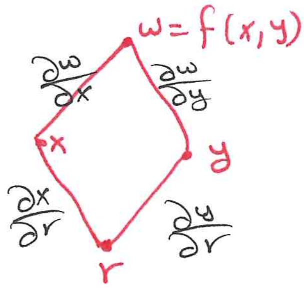
Branch diagrams:



Note that if $w = f(x,y)$, $x = g(r,s)$ and $y = h(r,s)$ then

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}$$



Q9 $w = xy + yz + xz$, $x = u+v$, $y = u-v$, $z = uv$

$(u, v) = (\frac{1}{2}, 1)$

[a] Find $\frac{\partial w}{\partial u}$, $\frac{\partial w}{\partial v}$

- $$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}$$

$$= (y+z) + (x+z) + (y+x)(v)$$

$$= (y+x)(1+v) + 2z \quad = 2u(1+v) + 2uv$$

$$= 2u + 4uv$$

- $$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}$$

$$= (y+z) + (x+z)(-1) + (y+x)(u)$$

$$= \cancel{u} - v - \cancel{u} - v + \underline{\underline{u^2}} - \cancel{uv} + \underline{\underline{u^2}} + \cancel{uv}$$

$$= 2u^2 - 2v$$

- $w = xy + yz + xz = (u+v)(u-v) + (u-v)(uv) + (u+v)(uv)$

$$= u^2 - v^2 + u^2v - \cancel{uv^2} + u^2v + \cancel{uv^2} = u^2 - v^2 + 2u^2v$$

$\frac{\partial w}{\partial v} = -2v + 2u^2$ and $\frac{\partial w}{\partial u} = 2u + 4uv$

[b] Find $\frac{\partial w}{\partial u}(\frac{1}{2}, 1)$ and $\frac{\partial w}{\partial v}(\frac{1}{2}, 1)$

$\frac{\partial w}{\partial u}(\frac{1}{2}, 1) = 1 + 2 = 3$ and $\frac{\partial w}{\partial v}(\frac{1}{2}, 1) = -2 + \frac{1}{2} = -\frac{3}{2}$

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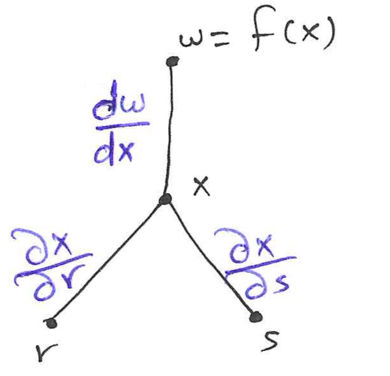
* If $w = f(x)$ and $x = g(r,s)$ then

$$\frac{\partial w}{\partial r} = \frac{dw}{dx}\frac{\partial x}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{dw}{dx}\frac{\partial x}{\partial s}$$



Branch diagram

Exp Find $\left.\frac{\partial z}{\partial u}\right|_{(u,v)=(\ln 2,1)}$ for $z = 5\tan^{-1}x$, $x = e^u + \ln v$

- $\frac{\partial z}{\partial u} = \frac{dz}{dx}\frac{\partial x}{\partial u} = \left(\frac{5}{1+x^2}\right)e^u = \frac{5e^u}{1+(e^u+\ln v)^2}$

$\frac{dz}{du}(\ln 2,1) = \frac{2(5)}{1+(2+0)^2} = \frac{2(5)}{1+4} = \frac{2(5)}{5} = 2$

- Find $\left.\frac{\partial z}{\partial v}\right|_{(u,v)=(\ln 2,1)}$

$\frac{\partial z}{\partial v} = \frac{dz}{dx}\frac{\partial x}{\partial v} = \left(\frac{5}{1+x^2}\right)\frac{1}{v} = \frac{5}{v+v(e^u+\ln v)^2}$

$\frac{dz}{dv}(\ln 2,1) = \frac{5}{1+(2+0)^2} = \frac{5}{1+4} = \frac{5}{5} = 1$

Th* (Implicit Differentiation)

If $F(x,y)$ is diff s.t $F(x,y)=0$ defines y as a diff function of x, then at any point where $F_y \neq 0$ we have $\frac{dy}{dx} = -\frac{F_x}{F_y}$

Exp Apply Th* to find $\frac{dy}{dx}(1,1)$ if $\underbrace{x^3-2y^2+xy}_{F(x,y)}=0$

$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{3x^2+y}{-4y+x} \Rightarrow \frac{dy}{dx}(1,1) = -\frac{4}{-3} = \frac{4}{3}$

or $3x^2 - 4y\frac{dy}{dx} + x\frac{dy}{dx} + y = 0 \Leftrightarrow \frac{dy}{dx} = \frac{-(3x^2+y)}{x-4y}$

If $F(x,y,z)=0$ and $z=f(x,y)$ are diff then

$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$ and $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$