Mustafa Jarrar: Lecture Notes in Discrete Mathematics.
Birzeit University, Palestine, 2015

Relations

8.1. Introduction to Relations

8.2 Properties of Relations

8.3 Equivalence Relations

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Acknowledgement:
This lecture is based on (but not limited to) to chapter 5 in "Discrete Mathematics with Applications by Susanna S. Epp (3rd Edition)".

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Relations

8.1 Introduction to Relations

In this lecture:

- Part 1: **What is a Relation**
- Part 2: Inverse of a Relation;
- Part 3: Directed Graphs;
- Part 4: n-ary Relations,
- Part 5: Relational Databases

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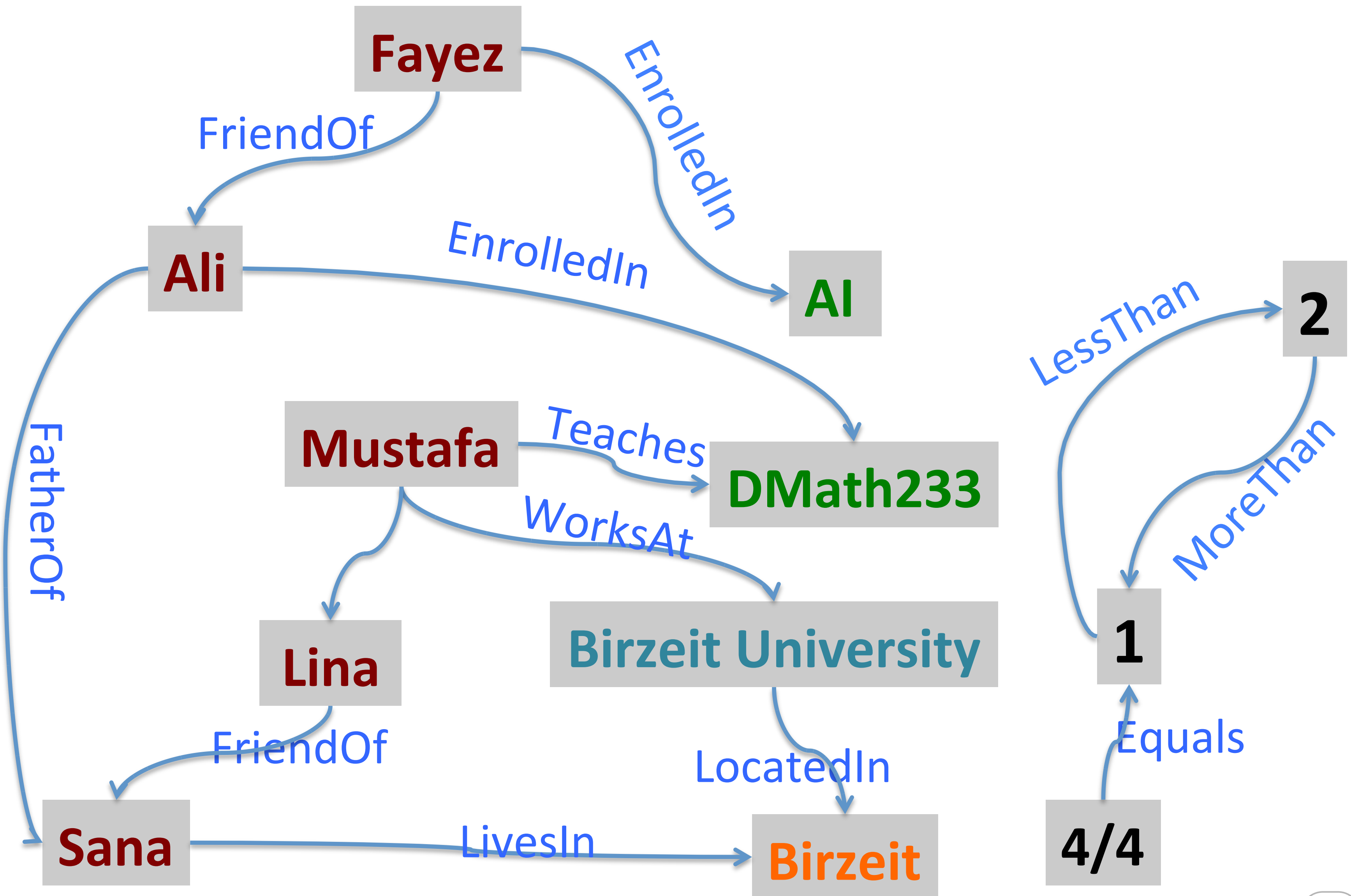
What is a Relation?



What is a Relation?

• Definition

Let A and B be sets. A **(binary) relation R from A to B** is a subset of $A \times B$. Given an ordered pair (x, y) in $A \times B$, **x is related to y by R,** written $x\ R\ y$, if, and only if, (x, y) is in R.

$$x\ R\ y \quad \Leftrightarrow \quad (x, y) \in R$$

$$x \not R\ y \quad \Leftrightarrow \quad (x, y) \notin R$$

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Example

The Less-than Relation for Real Numbers

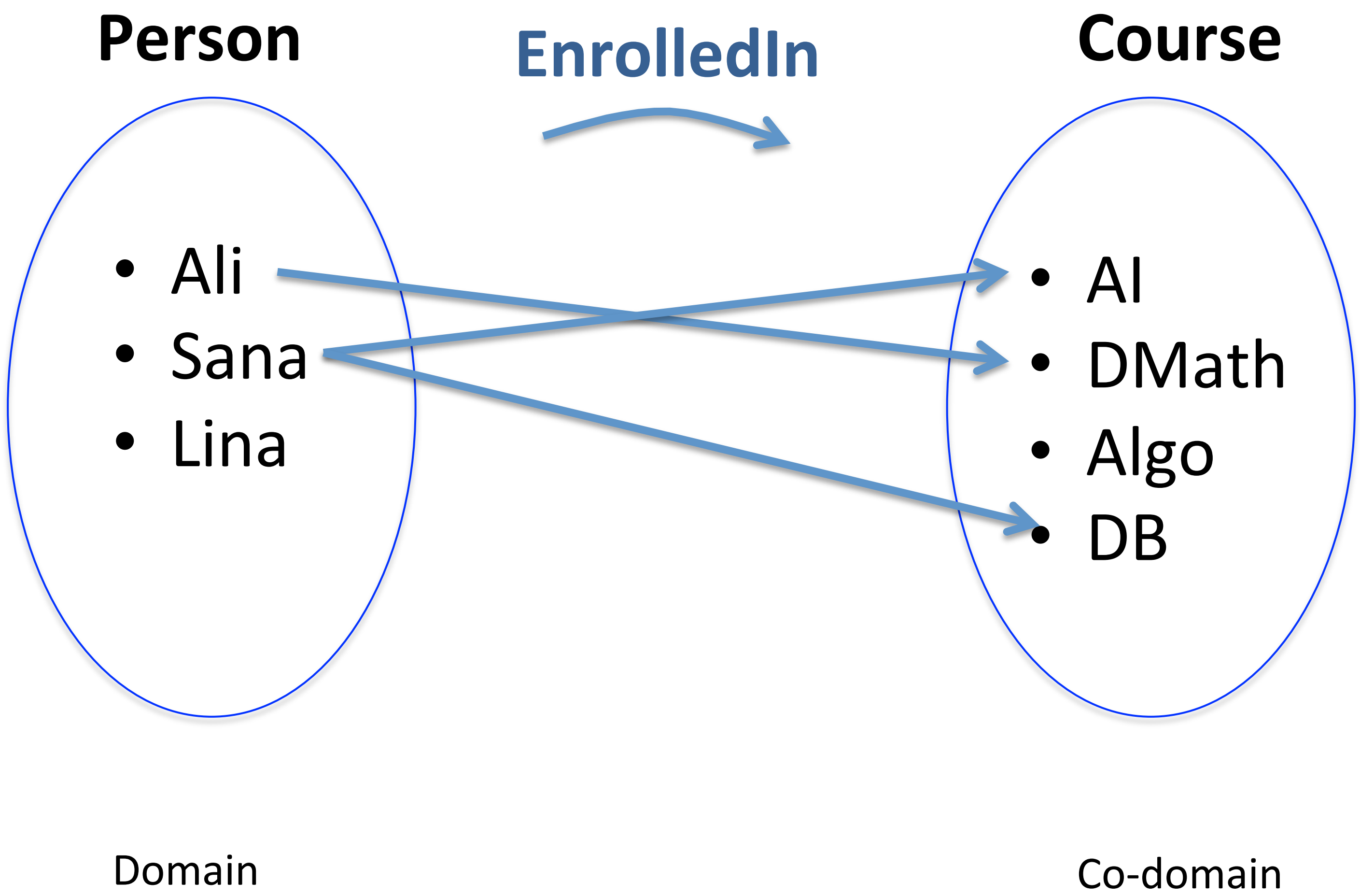
Define a relation L from **R** to **R** as follows: For all real numbers x and y,

$$x \; L \; y \Leftrightarrow x < y.$$

a. Is $57 \; L \; 53$? b. Is $(-17) \; L \; (-14)$? c. Is $143 \; L \; 143$? d. Is $(-35) \; L \; 1$?

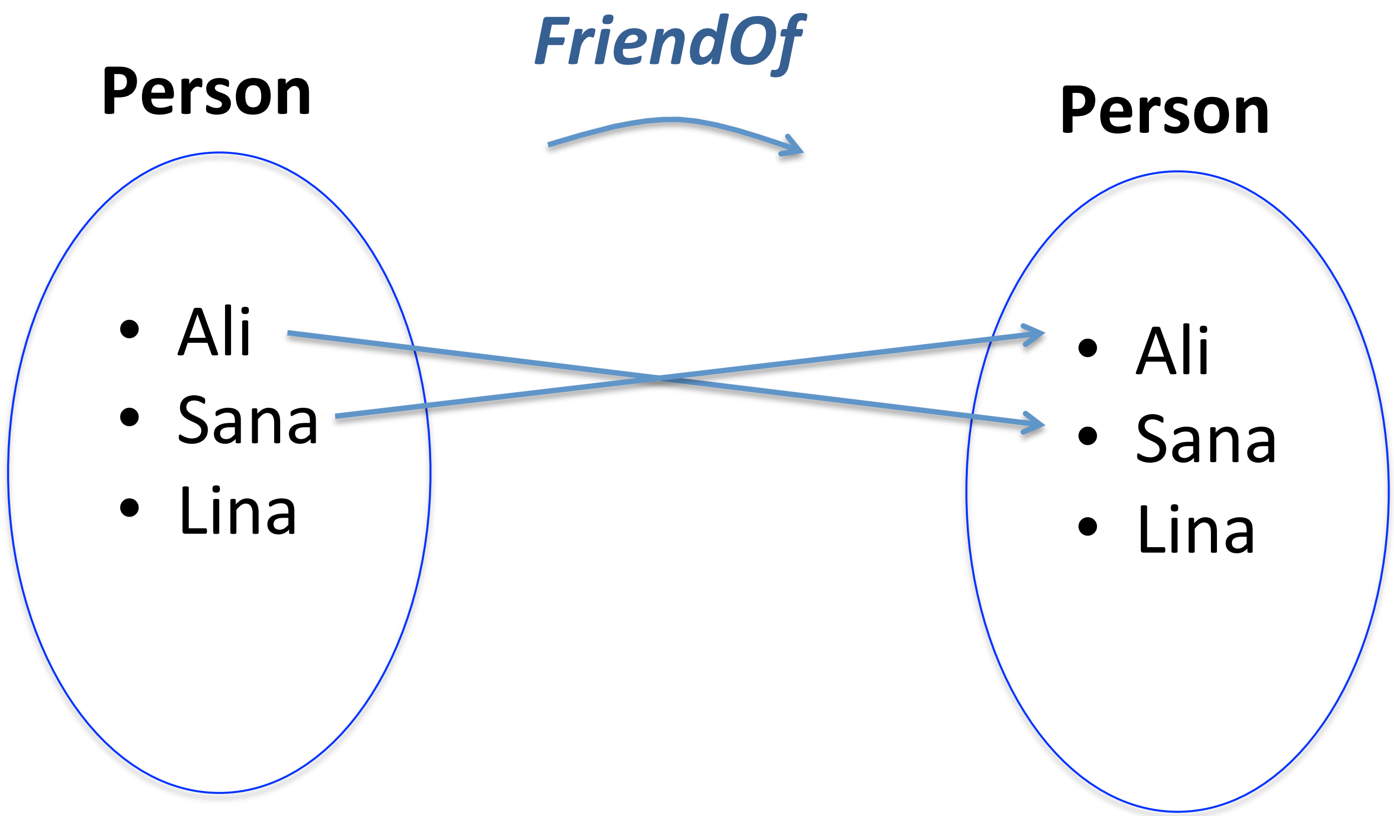
a. No, $57 > 53$ b. Yes, $-17 < -14$ c. No, $143 = 143$ d. Yes, $-35 < 1$

Example



Person
EnrolledIn
Course
Ali
Sana
Lina
AI
DMath
Algo
DB
Domain
Co-domain

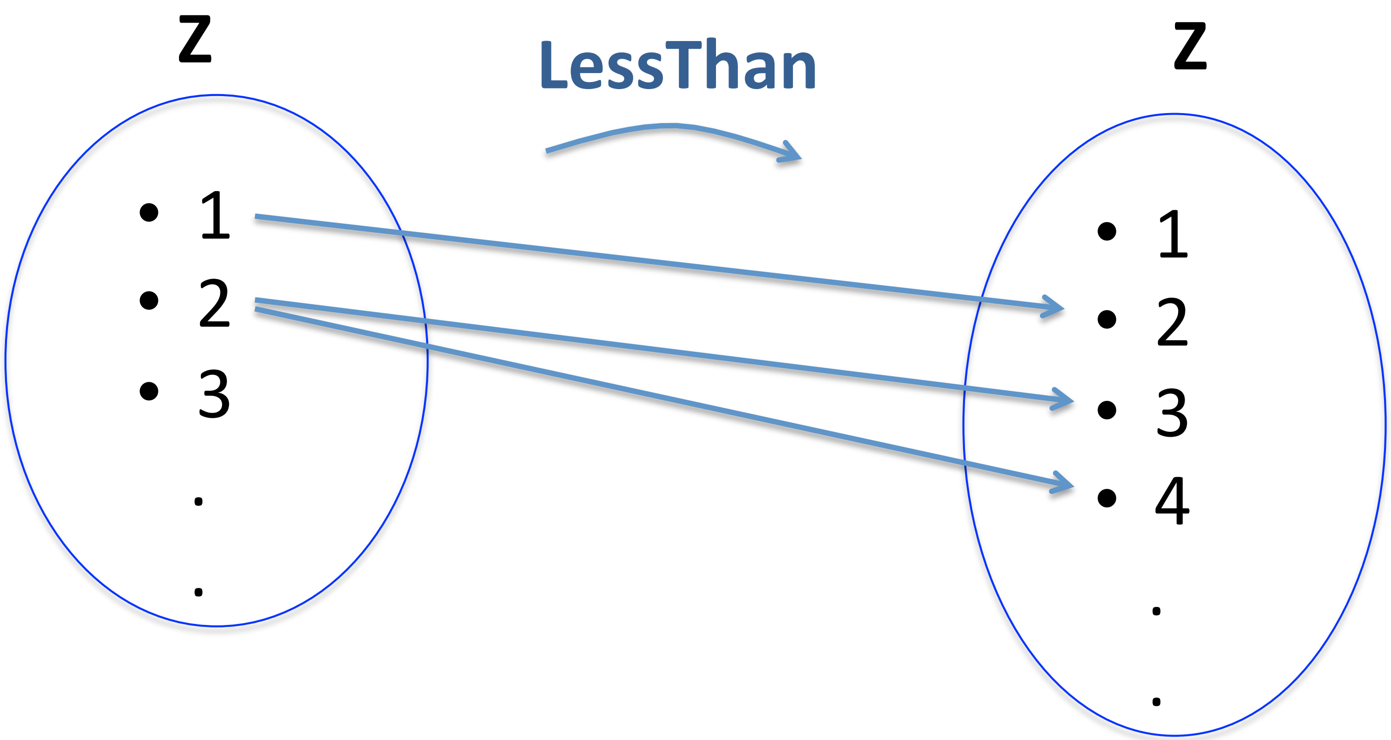
Example



$$FriendOf^{-1} = \{(\text{Ali}, \text{Sana}), (\text{Sana}, \text{Ali})\}$$

Inverse Relation

Example



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Example

Define a relation *E* from **Z** to **Z** as follows: For all $(m,n) \in \mathbf{Z} \times \mathbf{Z}$, $mEn \Leftrightarrow m-n$ is even.

a. Is 4 *E* 0? Is 2 *E* 6? Is 3 *E* (−3)? Is 5 *E* 2?

b. List five integers that are related by *E* to 1.

c. Prove that if *n* is any odd integer, then *n E* 1.

a. Yes,4 *E* 0 because 4−0=4 and 4 is even. Yes, 2 *E* 6 because 2−6=−4 and −4 is even. Yes, 3 *E* (−3) because 3−(−3)=6 and 6 is even. No, 5 *E* 2 because 5−2=3 and 3 is not even.

b. 1 because1−1=0 is even, 3 because 3−1=2 is even, 5 because 5−1=4 is even, −1 because −1−1=−2 is even, −3 because −3−1=−4 is even.

c. Suppose *n* is any odd integer. Then $n = 2k + 1$ for some integer *k*. By definition of *E* , *n E* 1 if, and only if, $n - 1$ is even. By substitution, $n - 1 = (2k + 1) - 1 = 2k$, and since *k* is an integer, $2k$ is even. Hence *n E* 1 *[as was to be shown]*.

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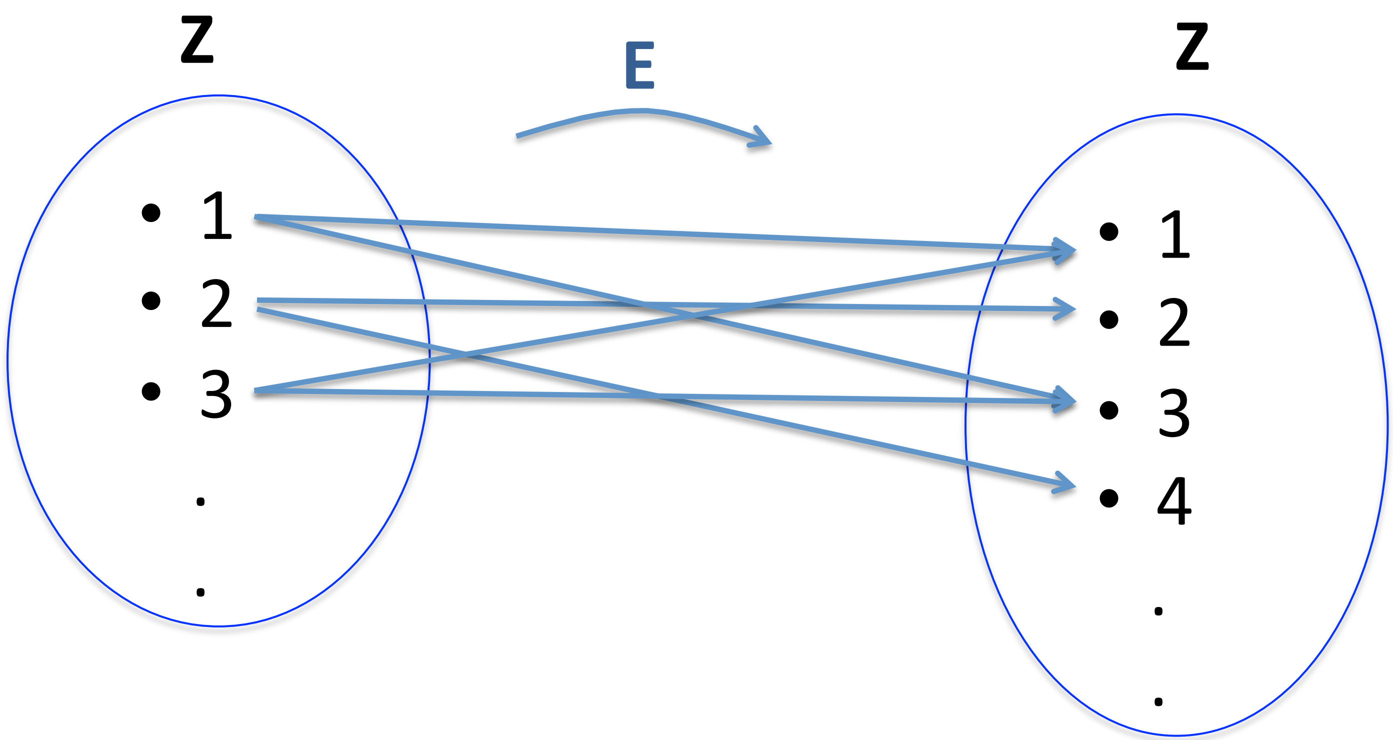
Example



Define a relation E from $\mathbf{Z}$ to $\mathbf{Z}$ as follows:

For all $(m, n) \in \mathbf{Z} \times \mathbf{Z}, \quad m\, E\, n \Leftrightarrow m - n$ is even.

Example: a relation on a Power Set

Let $X = \{a, b, c\}$. Then $P(X) = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$. Define a relation S from $P(X)$ to $\mathbf{Z}$ as follows: For all sets A and B in $P(X)$ (i.e., for all subsets A and B of X),

$$A \text{ S } B \Leftrightarrow A \text{ has at least as many elements as } B.$$

✓ a. Is $\{a,b\}$ S $\{b,c\}$? Yes, both sets have two elements.

✓ b. Is $\{a\}$ S $\varnothing$? Yes, $\{a\}$ has one element and $\varnothing$ has zero elements, and $1 \geq 0$.

✗ c. Is $\{b,c\}$ S $\{a,b,c\}$? No, $\{b, c\}$ has two elements and $\{a, b, c\}$ has three elements and $2 < 3$.

✓ d. Is $\{c\}$ S $\{a\}$? Yes, both sets have one element.

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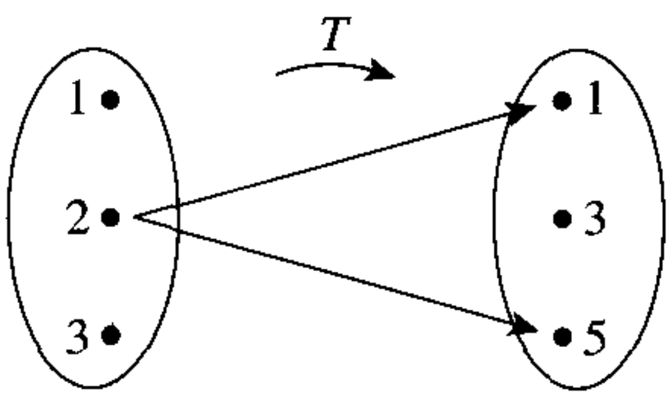
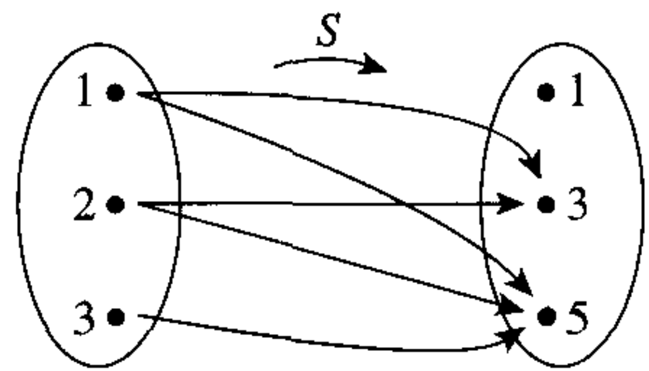
Example

Let A = {1, 2, 3} and B = {1, 3, 5}, define relations **S** and **T** from A to *B* as follows: For all $(x, y) \in A \times B$,

$(x, y) \in S \Leftrightarrow \quad x < y$ S is a "LessThan" relation.

$T = \{(2, 1), (2, 5)\}$.





Relations and Functions

• Definition

A **function F from a set A to a set B** is a relation from A to B that satisfies the following two properties:

1. For every element x in A, there is an element y in B such that $(x, y) \in F$.
2. For all elements x in A and y and z in B,

$$\text{if } (x, y) \in F \text{ and } (x, z) \in F, \text{ then } y = z.$$

If F is a function from A to B, we write

$$y = F(x) \quad \Leftrightarrow \quad (x, y) \in F.$$

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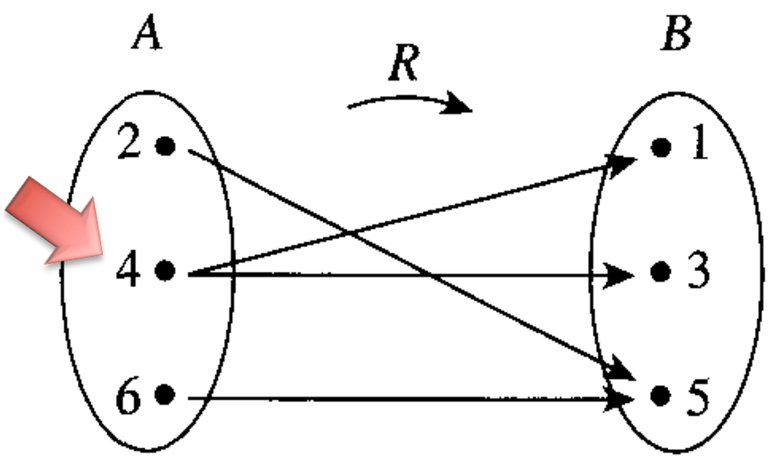
Example

Let A = {2 , 4, 6} and B = {1, 3 , 5}.
Is relation R a function from A to *B?*

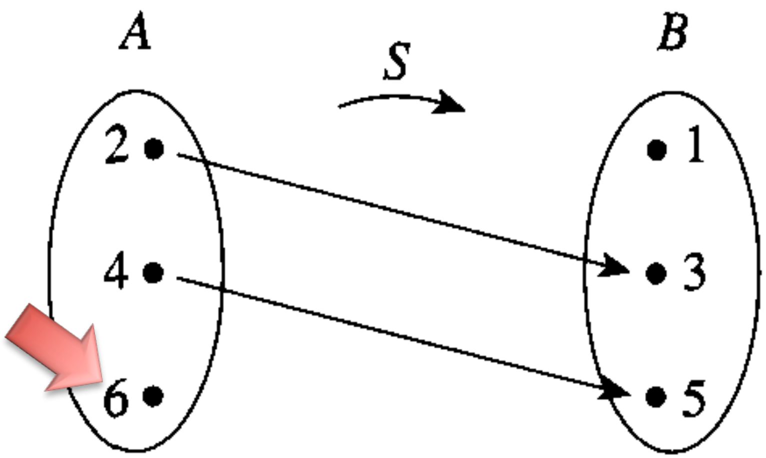
$$R = \{(2, 5), (4, 1), (4, 3), (6, 5)\}.$$ ✗



Example

Let A = {2 , 4, 6} and B = {1, 3 , 5}.
Is relation R a function from A to *B?*

For all $(x,y) \in A \times B$, $(x,y) \in S$ $y=x+1$. ✗



Relations

8.1 Introduction to Relations

In this lecture:

- ❑ Part 1: What is a Relation
- ❑ Part 2: **Inverse of a Relation**
- ❑ Part 3: Directed Graphs
- ❑ Part 4: *n*-ary Relations
- ❑ Part 5: Relational Databases

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Inverse Relation

Definition

Let R be a relation from A to B. Define the inverse relation R^{-1} from B to A as follows:

$$R^{-1} = \{(y,x) \in B \times A \mid (x,y) \in R\}.$$

For all $x \in A$ and $y \in B$, $(y,x) \in R^{-1} \Leftrightarrow (x,y) \in R$.

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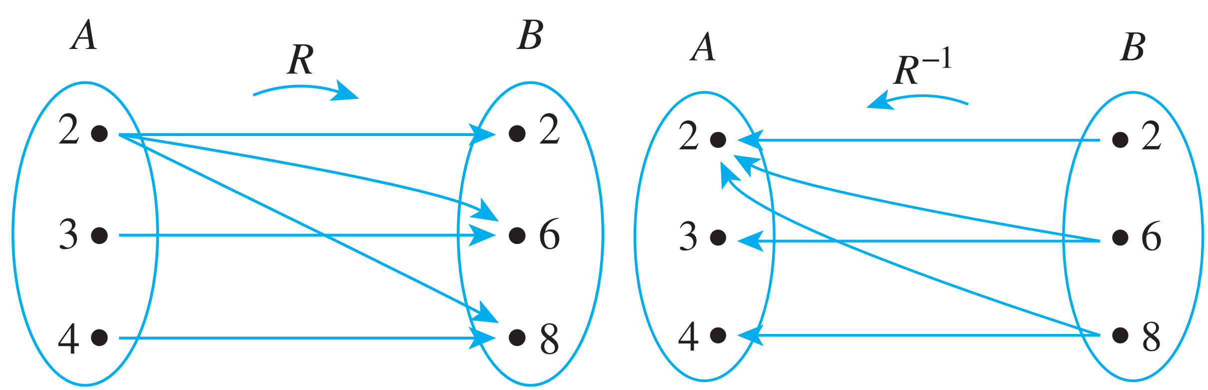
Example

Let $A = \{2,3,4\}$ and $B = \{2,6,8\}$ and let R be the "divides" relation from A to B: For all $(x, y) \in A \times B$,

$$x \, R \, y \Leftrightarrow x \mid y \qquad x \text{ divides } y.$$

a. State explicitly which ordered pairs are in R and $R-1$, and draw arrow diagrams for R
and R^{-1}

$$R = \{(2,2),(2,6),(2,8),(3,6),(4,8)\}$$
$$R^{-1} = \{(2,2),(6,2),(8,2),(6,3),(8,4)\}$$



b. Describe $R-1$ in words.

For all $(y, x) \in B \times A$, $y \, R^{-1} \, x \Leftrightarrow y$ is a multiple of x.

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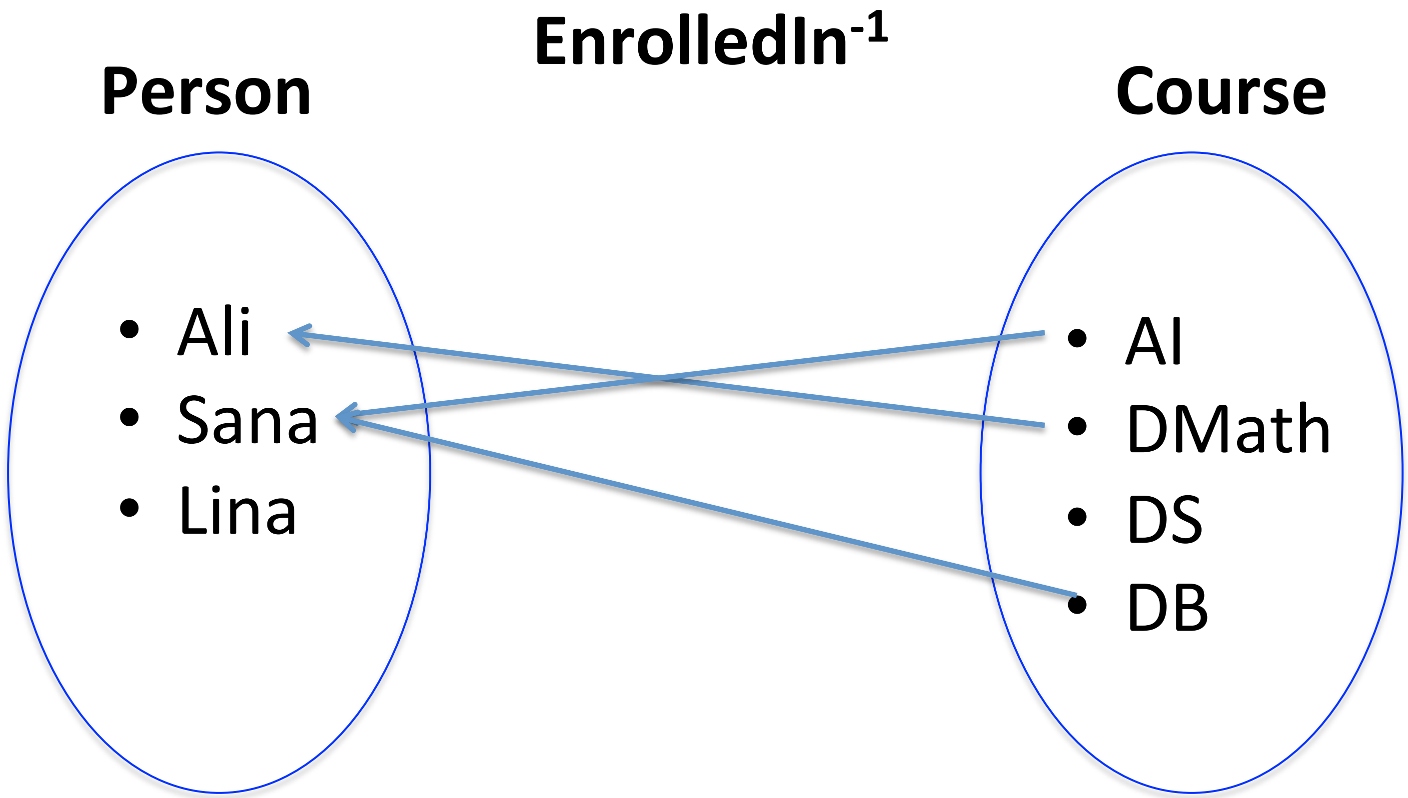
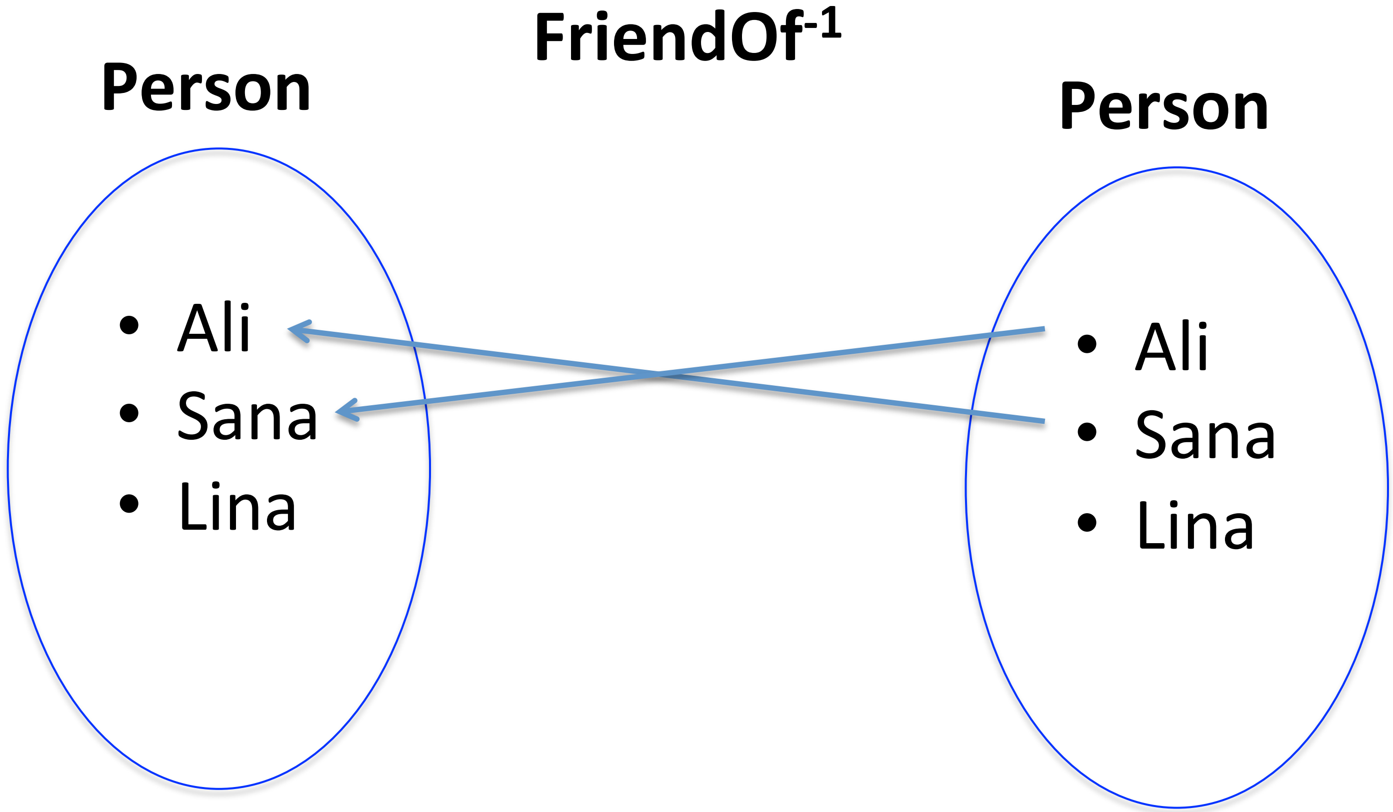
Example



Example



Inverse of Relations in Language

What would be the inverse of the following relations in English

$SonOf^{-1} = ?$ Parent of

$WifeOf^{-1} = ?$ Husband of

$WorksAt^{-1} = ?$ Employs

$EnrolledOf^{-1} = ?$ Enrolls

$PresidentOf^{-1} = ?$ Leadership of / Led by

$BrotherOf^{-1} = ?$ Brother of

Symmetric Relation

$SisterOf^{-1} = ?$ Sister of

….

Relations

8.1 Introduction to Relations

In this lecture:

- ❑ Part 1: What is a Relation
- ❑ Part 2: Inverse of a Relation
- ❑ Part 3: **Directed Graphs**
- ❑ Part 4: *n*-ary Relations
- ❑ Part 5: Relational Databases

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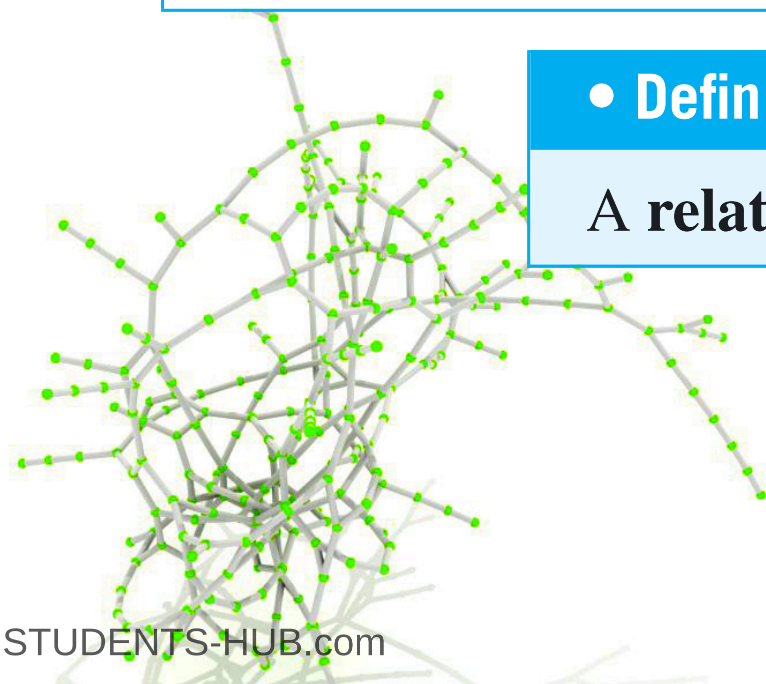
Directed Graph of a Relation

When a relation R is defined *on* a set A, the arrow diagram of the relation can be modified so that it becomes a **directed graph.**

For all points x and y in A,

$$\text{there is an arrow from } x \text{ to } y \quad \Leftrightarrow \quad x \; R \; y \quad \Leftrightarrow \quad (x, y) \in R.$$



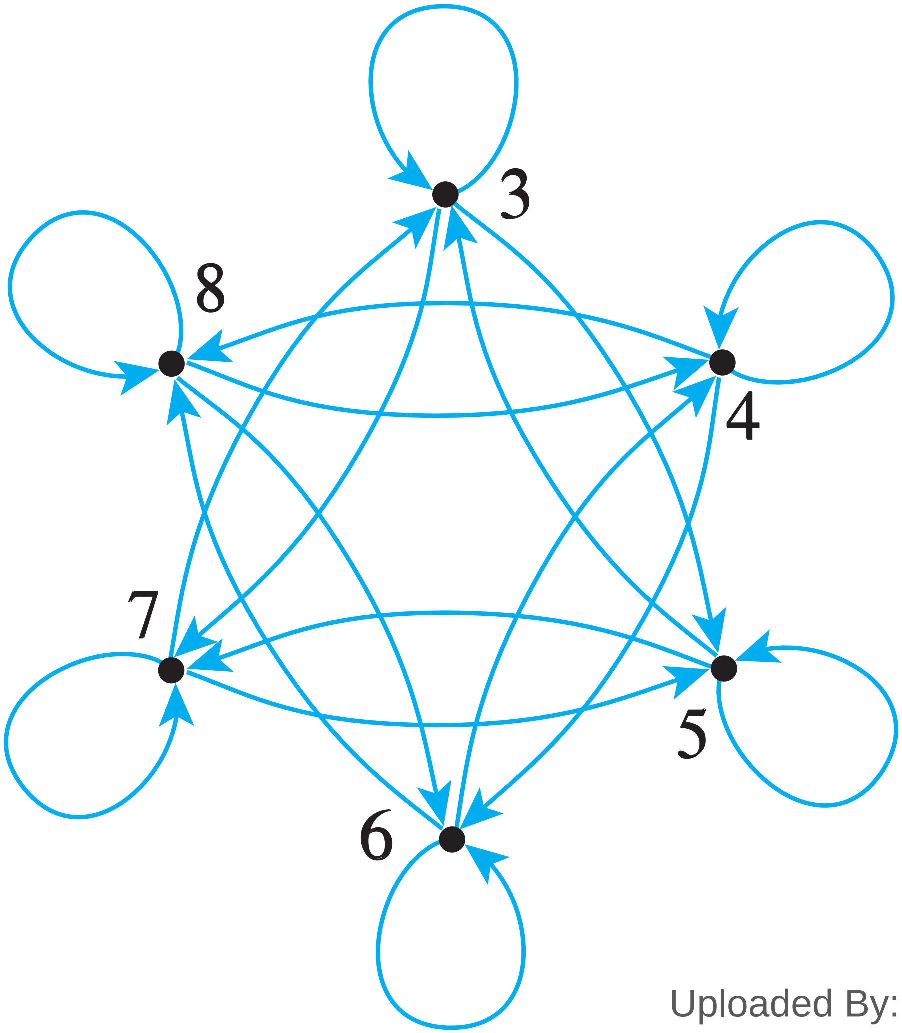
- **Definition**

A **relation on a set** A is a relation from A to A.

It is important to distinguish clearly between a relation and the set on which it is defined.

Example

Let $A = \{3, 4, 5, 6, 7, 8\}$ and define a relation R on A as follows: For all $x, y \in A$, $x\ R\ y \Leftrightarrow 2 \mid (x-y)$.



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Relations

8.1 Introduction to Relations

In this lecture:

- Part 1: What is a Relation
- Part 2: Inverse of a Relation
- Part 3: Directed Graphs
- Part 4: ***n*-ary Relations**
- Part 5: Relational Databases

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N-ary Relations

EnrolledIn(Ali, Dmath) EnrolledIn(Sami, DB)	Binary (2-ary)
Enrollment(Sami, DB, 99)	Ternary (3-ary)
Enrollment(Sami, DB, 99, 2014)	Quaternary (4-ary)
Enrollment(Sami, DB, 99, 2014,F)	5-ary
$R(a_1, a_2, a_3, \ldots.., a_n)$	*n*-ary

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N-ary Relations

- **Definition**

Given sets $A_1, A_2, \ldots, A_n$, an ***n*-ary relation** R on $A_1 \times A_2 \times \cdots \times A_n$ is a subset of $A_1 \times A_2 \times \cdots \times A_n$. The special cases of 2-ary, 3-ary, and 4-ary relations are called **binary, ternary,** and **quaternary relations,** respectively.

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Relations

8.1 Introduction to Relations

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- ❑ Part 5: **Relational Databases**

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Relational Databases

Let $A1$ be a set of positive integers, $A2$ a set of alphabetic character strings, $A3$ a set of numeric character strings, and $A4$ a set of alphabetic character strings. Define a quaternary relation R on $A1 \times A2 \times A3 \times A4$ as follows:

$(a1, a2, a3, a4) \in R \Leftrightarrow$ a patient with patient ID number $a1$, named $a2$, was admitted on date $a3$, with primary diagnosis $a4$.

Patient(ID, Name, Date, **Diagnosis)**

(011985, John Schmidt, 020710, asthma)

(574329, Tak Kurosawa, 114910, pneumonia)

(466581, Mary Lazars, 103910, appendicitis)

(008352, Joan Kaplan, 112409, gastritis)

(011985, John Schmidt, 021710, pneumonia)

(244388, Sarah Wu, 010310, broken leg)

(778400, Jamal Baskers, 122709, appendicitis)

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Relational Databases

R on $A1 \times A2 \times A3 \times A4$ as follows:

$(a1, a2, a3, a4) \in R \Leftrightarrow$ a patient with patient ID number $a1$, named $a2$, was admitted on date $a3$, with primary diagnosis $a4$.

Relation

Patient			
ID	**Name**	**Date**	**Diagnosis**
(011985,	John Schmidt,	020710,	asthma)
(574329,	Tak Kurosawa,	114910,	pneumonia)
(466581,	Mary Lazars,	103910,	appendicitis)
(008352,	Joan Kaplan,	112409,	gastritis)
(011985,	John Schmidt,	021710,	pneumonia)
(244388,	Sarah Wu,	010310,	broken leg)
(778400,	Jamal Baskers,	122709,	appendicitis)

Each row is called **tuple**

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Relational Databases

R on $A1 \times A2 \times A3 \times A4$ as follows:

$(a1, a2, a3, a4) \in R \Leftrightarrow$ a patient with patient ID number $a1$, named $a2$, was admitted on date $a3$, with primary diagnosis $a4$.

Relation

Each row is called **tuple**

Patient			
ID	Name	Date	Diagnosis
(0			
(5			
(4			
(0			
(0			
(2			
(778400,	Jamal Baskers,	122709,	appendicitis)

- Notice that **Tables** in this way are called **Relations**.
- Information stored in this way is called a "**Relational Database**"

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Relations

8.1. Introduction to Relations

8.2 Properties of Relations

8.3 Equivalence Relations

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Relations

8.2 Properties of Relations

In this lecture:

- ❑ Part 1: **Properties: Reflexivity, Symmetry, Transitivity**
- ❑ Part 2: Proving Properties of Relations
- ❑ Part 3: Transitive Closure

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Reflexivity, Symmetry, and Transitivity

Let R be a relation on a set A.

1. R is **reflexive** if, and only if, for all $x \in A$, $x\ R\ x$.
2. R is **symmetric** if, and only if, for all $x, y \in A$, ***if*** $x\ R\ y$ then $y\ R\ x$.
3. R is **transitive** if, and only if, for all $x, y, z \in A$, ***if*** $x\ R\ y$ and $y\ R\ z$ then $x\ R\ z$.

Because of the equivalence of the expressions $x\ R\ y$ and $(x, y) \in R$ for all x and y
n A, the reflexive, symmetric, and transitive properties can also be written as follows:

1. R is reflexive $\Leftrightarrow$ for all x in A, $(x, x) \in R$.
2. R is symmetric $\Leftrightarrow$ for all x and y in A, ***if*** $(x, y) \in R$ then $(y, x) \in R$.
3. R is transitive $\Leftrightarrow$ for all x, y and z in A, ***if*** $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$.

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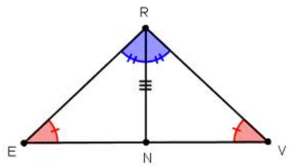


Reflexivity تناظر

R is reflexive $\Leftrightarrow$ for all x in A, $(x, x) \in R$.

R is Reflexive: Each element is related to itself.

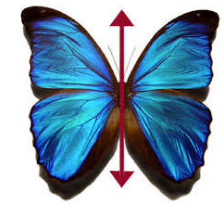
علاقة ثنائية على مجموعة ما ، وكل عنصر في المجموعة مرتبط بنفسه في إطار هذه العلاقة.

R is not reflexive: there is an element x in A such that $x\ R\ x$ *[that is, such that* $(x, x) \notin R$ *]* .

Examples:

Likes?	MemberOf?	BrotherOf?
LocatedIn?	PartOf?	SonOf?
Kills?	SubSetOf?	FatherOf?
FreindOf?	SameAS?	RelativeOf?

Symmetry تماثل



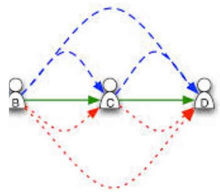
R is symmetric $\Leftrightarrow$ for all x and y in A, ***if*** $(x, y) \in R$ then $(y, x) \in R$.

R is Symmetric: If any one element is related to any other element, then the second is related to the first.

R **is not Symmetric:** there are elements x and y in A such that $x\ R\ y$ but $y \not{R}\ x$ *[that is, such that* $(x, y) \in$ *R but* $(y, x) \notin$ *R]*.

Examples:

Likes?	MemberOf?	BrotherOf?
LocatedIn?	PartOf?	SonOf?
Kills?	SubSetOf?	FatherOf?
FreindOf?	SameAS?	RelativeOf?



Transitivity تعدي

R is transitive $\Leftrightarrow$ for all x, y and z in A, ***if*** $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$.

R is Transitive: If any one element is related to a second and that second element is related to a third, then the first element is related to the third.

***R* is not transitive:** there are elements x,y and z in A such that xRy and yRz but $x \not R z$ *[that is, such that* $(x,y) \in R$ *and* $(y,z) \in R$ *but* $(x , z) \notin R$ *]* .

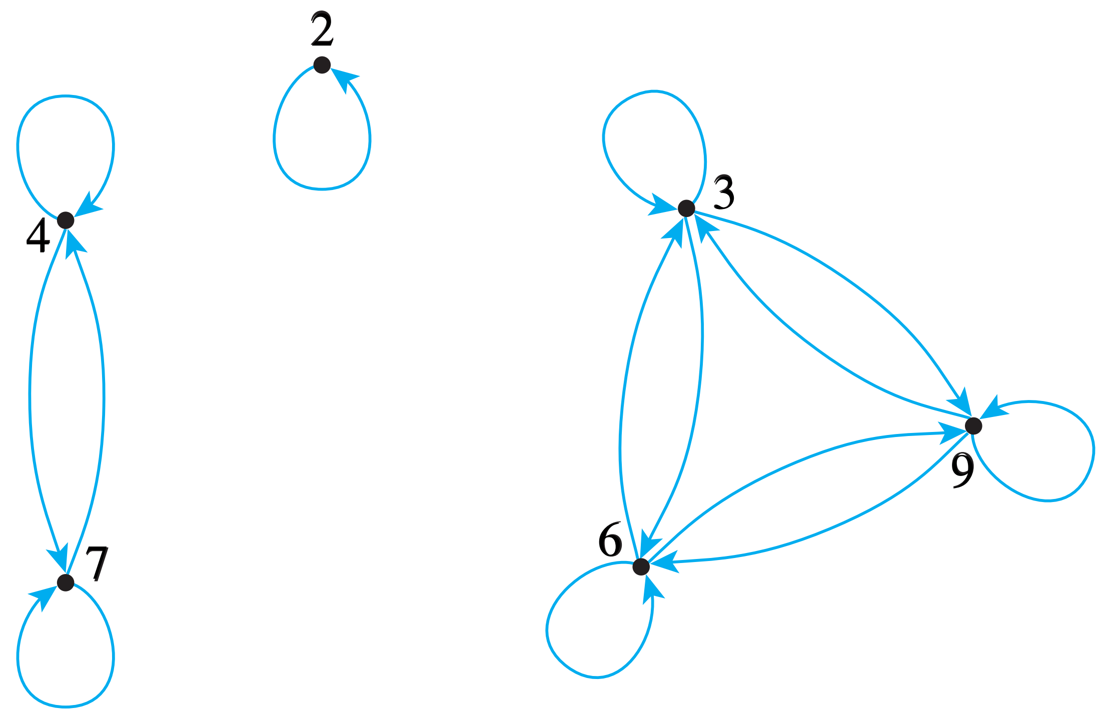
Examples:

Likes?	MemberOf?	BrotherOf?
LocatedIn?	PartOf?	SonOf?
Kills?	SubSetOf?	FatherOf?
FreindOf?	SameAS?	RelativeOf?

Example

Let $A = \{2,3,4,6,7,9\}$ and define a relation R on A as:

For all $x, y \in A$, $x\ R\ y \Leftrightarrow 3 \mid (x-y)$.



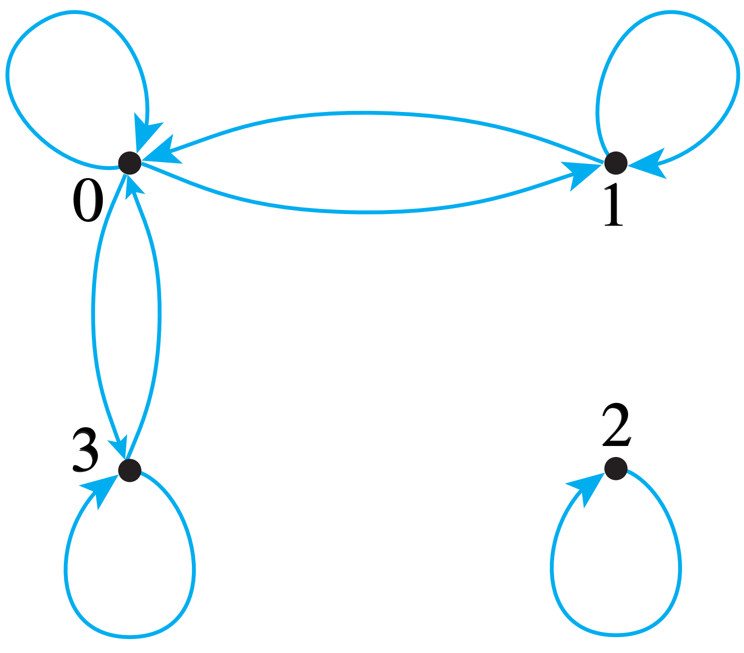
Is R Reflexive? Symmetric? Transitive?

Exercise

Let $A = \{0, 1, 2, 3\}$ and define relation R on A as:

$$R = \{(0,0), (0,1), (0,3), (1,0), (1,1), (2,2), (3,0), (3,3)\}$$

Is R Reflexive? ☑ *Symmetric?* ☑ *Transitive?* ☒

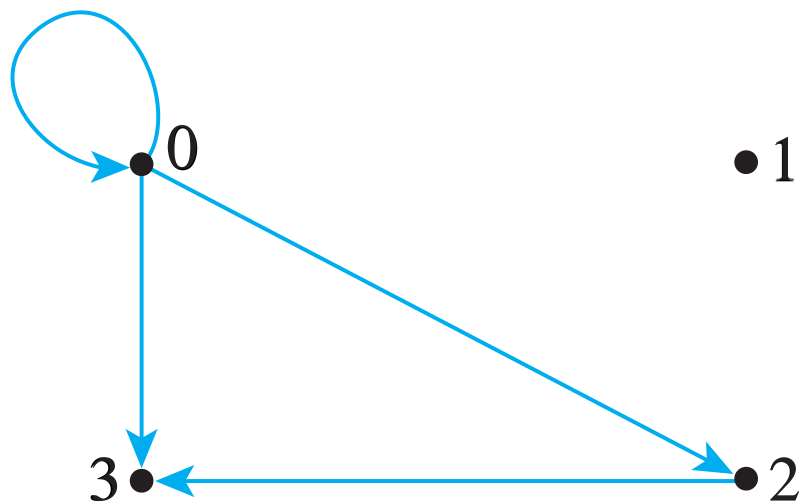


Exercise

Let $A = \{0, 1, 2, 3\}$ and define relation R on A as:

$R = \{(0, 0), (0, 2), (0, 3), (2, 3)\}$

Is R Reflexive? ☒ *Symmetric?* ☒ *Transitive?* ☑



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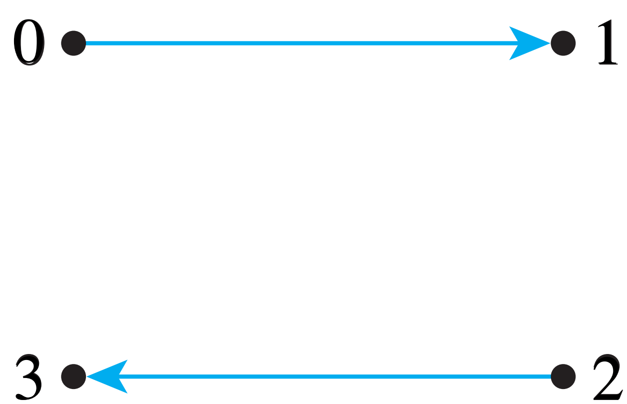
Exercise

Let $A = \{0, 1, 2, 3\}$ and define relation R on A as:

$R = \{(0,1),(2,3)\}$

Is R Reflexive? ☒ *Symmetric?* ☒ *Transitive?* ☑



***T* is transitive by default because it is *not* not transitive!**

Relations

8.2 Properties of Relations

In this lecture:

- Part 1: Properties: Reflexivity, Symmetry, Transitivity
- Part 2: **Proving Properties of Relations**
- Part 3: Transitive Closure

Proving Properties on Relations on Infinite Sets

Until Now we discussed relation on Finite Sets

Next, we discussed relation on infinite Sets

To prove a relation is reflexive, symmetric, or transitive, first write down what is to be proved, in **First Order Logic**.

For instance, for symmetry

$$\forall x, y \in A, \text{if } x \, R \, y \text{ then } y \, R \, x.$$

Then use **direct methods** of proving

Properties of Equality

Define a relation R on **R** (the set of all real numbers) as follows:
For all real numbers x and y. $x\ R\ y \Leftrightarrow x = y$.

Is R Reflexive? *Symmetric?* *Transitive?*

R is reflexive*: *R is reflexive if, and only if, the following statement is true: For all $x \in \mathbf{R}$, $x\ R\ x$. And since $x\ R\ x$ just means that $x = x$, this is the same as saying For all $x \in \mathbf{R}$, $x = x$. Which is true; every real number is equal to it

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Properties of Equality

Define a relation R on **R** (the set of all real numbers) as follows:
For all real numbers x and y. $x \, R \, y \Leftrightarrow x = y$.

Is R Reflexive? *Symmetric?* *Transitive?*

R is symmetric**:** R is symmetric if, and only if, the following statement is true:
For all $x,y \in$ **R**, **if** $x \, R \, y$ then $y \, R \, x$.

By definition of R, $x \, R \, y$ means that $x = y$ and $y \, R \, x$ means that $y = x$.
Hence R is symmetric if, and only if,
For all $x,y \in$ **R**, **if** $x=y$ then $y=x$.

This statement is true; if one number is equal to a second, then the second is equal to the first.

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Properties of Equality

Define a relation R on **R** (the set of all real numbers) as follows: For all real numbers x and y. $x \, R \, y \Leftrightarrow x = y$.

Is R Reflexive? Symmetric? Transitive?

***R is transitive*:** R is transitive if, and only if, the following statement is true: For all $x,y,z \in \mathbf{R}$, **if** $x \, R \, y$ and $y \, R \, z$ then $x \, R \, z$.

By definition of R, $x \, R \, y$ means that $x = y$, $y \, R \, z$ means that $y = z$, and $x \, R \, z$ means that $x = z$. Hence R is transitive iff the following statement is true: For all $x,y,z \in \mathbf{R}$, **if** $x=y$ and $y=z$ then $x=z$.

This statement is true: If one real number equals a second and the second equals a third, then the first equals the third.

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Properties of Less Than

Define a relation R on **R** (the set of all real numbers) as follows:
For all $x, y \in R$, $x \, R \, y \Leftrightarrow x < y$.

Is R Reflexive? *Symmetric?* *Transitive?*

***R is not reflexive*:** R is reflexive if, and only if, $\forall x \in \mathbf{R}, x \; R \; x$. By definition of R, this means that $\forall x \in \mathbf{R}, x < x$. But this is false: $\exists x \in \mathbf{R}$ such that $x \not< x$. As a counterexample, let $x = 0$ and note that $0 \not< 0$. Hence R is not reflexive.

***R is not symmetric*:** R is symmetric if, and only if, $\forall x, y \in \mathbf{R}$, if $x \; R \; y$ then $y \; R \; x$. By definition of R, this means that $\forall x, y \in \mathbf{R}$, if $x < y$ then $y < x$. But this is false: $\exists x, y \in \mathbf{R}$ such that $x < y$ and $y \not< x$. As a counterexample, let $x = 0$ and $y = 1$ and note that $0 < 1$ but $1 \not< 0$. Hence R is not symmetric.

***R is transitive*:** R is transitive if, and only if, for all $x, y, z \in \mathbf{R}$, if $x \; R \; y$ and $y \; R \; z$ then $x \; R \; z$. By definition of R, this means that for all $x, y, z \in \mathbf{R}$, if $x < y$ and $y < z$, then $x < z$. But this statement is true by the transitive law of order for real numbers (Appendix A, T18). Hence R is transitive. ■

Properties of Congruence Modulo 3

Define a relation T on $\mathbf{Z}$ (the set of all integers) as follows: For all integers m and n, $m\ T\ n \Leftrightarrow 3|(m-n)$.

Is R ***Reflexive?*** *Symmetric?* *Transitive?*

For all $m \in \mathbf{Z}$, $3|(m-m)$.

Suppose m is a particular but arbitrarily chosen integer. *[We must show that m T m.]*

Now, $m-m = 0$.

But $3 \mid 0$ since $0 = 3\cdot 0$.

Hence $3|(m-m)$.

Thus, by definition of T, $mT\ m$

[as was to be shown].

Properties of Congruence Modulo 3

Define a relation T on **Z** (the set of all integers) as follows: For all integers m and n,

$$m\ T\ n \Leftrightarrow 3 | (m - n).$$

Is R Reflexive? <u>*Symmetric?*</u> *Transitive?*

For all $m, n \in$ **Z**, **if** $3|(m-n)$ then $3|(n-m)$.

Suppose m and n are particular but arbitrarily chosen integers that satisfy the condition $m\ T\ n$.

[We must show that n T m.]
By definition of T, since $m\ T\ n$ then $3 \mid (m - n)$. By definition of "divides," this means that $m - n = 3k$, for some integer k.
Multiplying both sides by -1 gives $n - m = 3(-k)$. Since $-k$ is an integer, this equation shows that $3 \mid (n - m)$. Hence, by definition of T, $n\ T\ m$
[as was to be shown].

Properties of Congruence Modulo 3

Define a relation T on **Z** (the set of all integers) as follows: For all integers m and n,

$$m\ T\ n \Leftrightarrow 3|(m-n).$$

Is R Reflexive? *Symmetric?* *Transitive?*

For all $m, n \in \mathbf{Z}$, **if** $3|(m-n)$ and $3|(n-p)$ then $3|(m-p)$.

Suppose m, n, and p are particular but arbitrarily chosen integers that satisfy the condition $m\ T\ n$ and $n\ T\ p$. *[We must show that $m\ T\ p$.]* By definition of T, since m T n *and* $n\ T\ p$, *then* $3|(m-n)$ *and* $3|(n-p)$. *By* definition of "divides," this means that $m - n = 3r$ and $n - p = 3s$, for some integers r and s. Adding the two equations gives $(m-n)+(n-p)=3r+3s$, and simplifying gives that $m - p = 3(r + s)$. Since $r + s$ is an integer, this equation shows that $3|(m - p)$. Hence, by definition of T, $m\ T\ p$ *[as was to be shown]*.

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Relations

8.2 Properties of Relations

In this lecture:

- Part 1: Properties: Reflexivity, Symmetry, Transitivity
- Part 2: Proving Properties of Relations
- Part 3: **Transitive Closure**

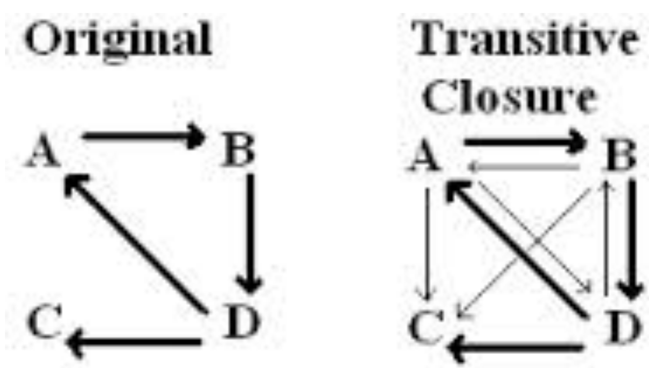
The Transitive Closure of a Relation

The **smallest** transitive relation that contains the relation.

• Definition

Let A be a set and R a relation on A. The **transitive closure** of R is the relation R^t on A that satisfies the following three properties:

1. R^t is transitive.
2. $R \subseteq R^t$.
3. If S is any other transitive relation that contains R, then $R^t \subseteq S$.



Exercise

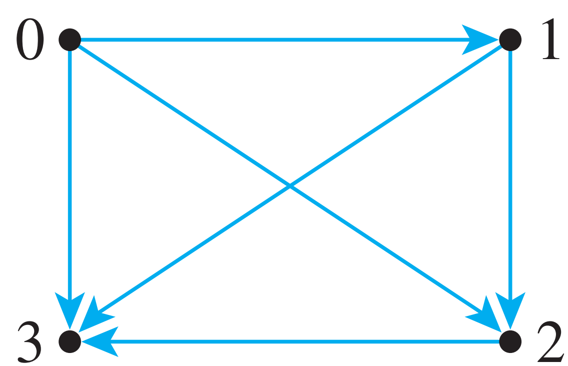
Let $A = \{0, 1, 2, 3\}$ and consider the relation R defined on A as:

$$R = \{(0, 1), (1, 2), (2, 3)\}.$$

Find the transitive closure of R.

$R^t = \{(0, 1), (0, 2), (0, 3), (1, 2), (1, 3), (2, 3)\}.$

R



$\mathbf{R^t}$

Relations

8.1. Introduction to Relations

8.2 Properties of Relations

8.3 Equivalence Relations

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Relations

8.3 Equivalence Relations

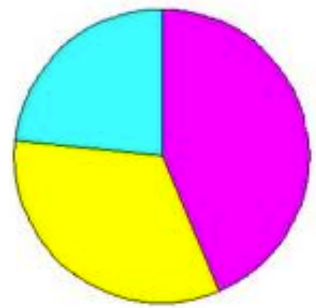
In this lecture:

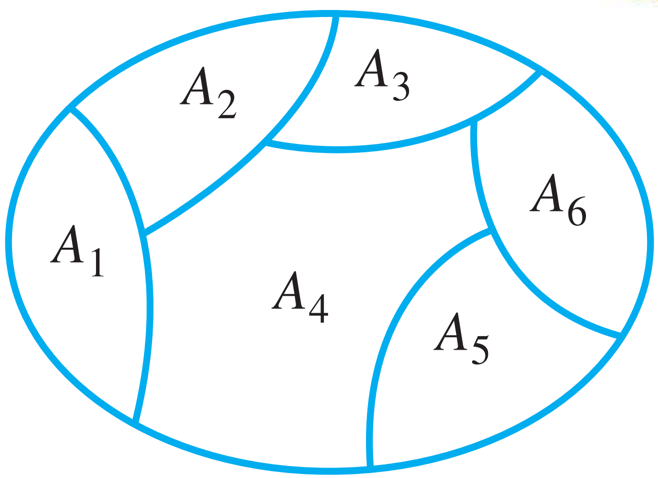
- ❑ Part 1: **Partitioned Sets**
- ❑ Part 2: **Equivalence Classes**
- ❑ Part 3: **Equivalence Relation**

Partitioned Sets

Sets can be partitioned into disjoint sets

A **partition** of a set *A* is a finite or infinite collection of nonempty, mutually disjoint subsets whose union is *A*.





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Total (جامع)

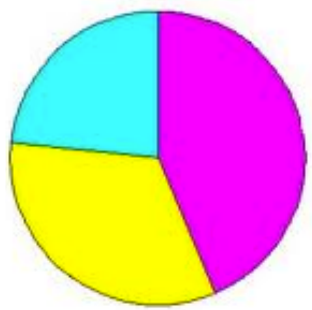
$$A_i \cup A_2 \cup \ldots \cup A_6 = A$$

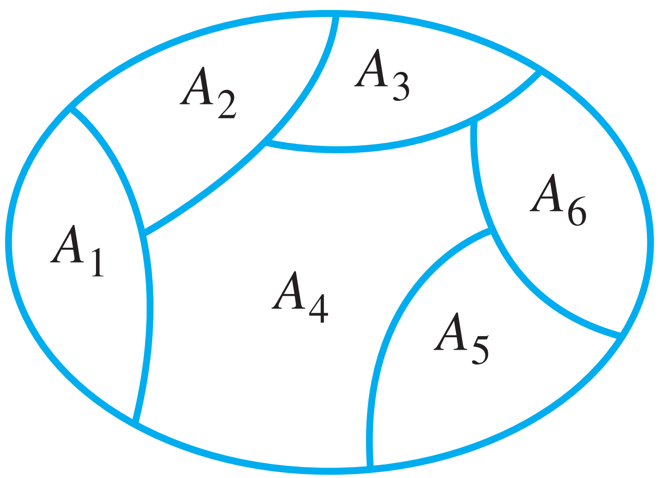
Disjoint (مانع)

$A_i \cap A_j = \phi$, whenever $i \neq j$

Partitioned Sets

Sets can be partitioned into disjoint sets





Total (جامع)

$$A_i \cup A_2 \cup \ldots \cup \; A_6 = A$$

Disjoint (مانع)

$$A_i \cap A_j = \phi, \text{ whenever } i \neq j$$

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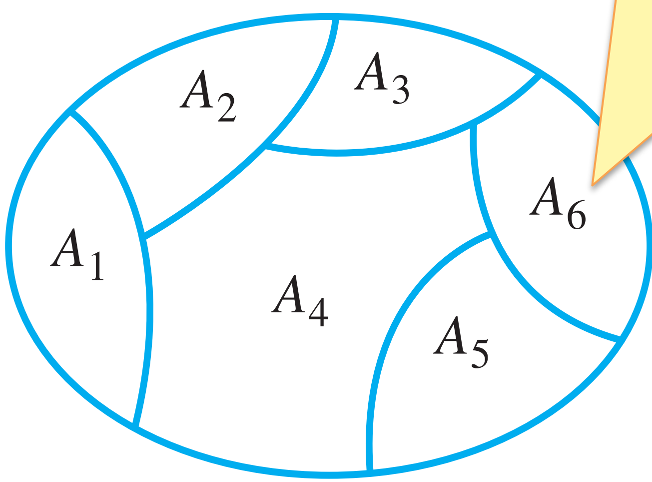
Relations Induced by a Partition

A **relation induced by a partition**, is a relation between two element in the same partition.



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Total (جامع)

$$A_i \cup A_2 \cup \ldots \cup \; A_6 = A$$

Disjoint (مانع)

$$A_i \cap A_j = \phi, \text{ whenever } i \neq j$$

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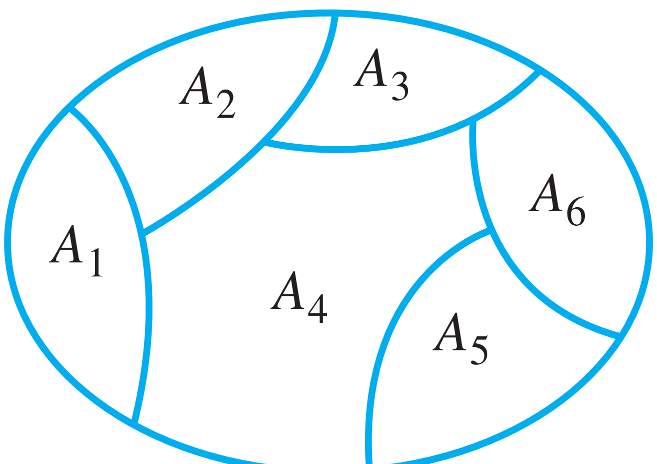
Relations Induced by a Partition

• Definition

Given a partition of a set A, the **relation induced by the partition,** R, is defined on A as follows: For all $x, y \in A$,

$$x \; R \; y \quad \Leftrightarrow \quad \text{there is a subset } A_i \text{ of the partition such that both } x \text{ and } y \text{ are in } A_i.$$



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Total (جامع)

$$A_i \cup A_2 \cup \ldots \cup A_6 = A$$

Disjoint (مانع)

$$A_i \cap A_j = \phi, \text{ whenever } i \neq j$$

Example

Let $A = \{0, 1, 2, 3, 4\}$ and consider the following partition of A:

$$\{0, 3, 4\}, \{1\}, \{2\}.$$

Find the relation R induced by this partition.

Since $\{0, 3, 4\}$ is a subset of the partition,

$0\ R\ 3$ because both 0 and 3 are in $\{0, 3, 4\}$,
$3\ R\ 0$ because both 3 and 0 are in $\{0, 3, 4\}$,
$0\ R\ 4$ because both 0 and 4 are in $\{0, 3, 4\}$,
$4\ R\ 0$ because both 4 and 0 are in $\{0, 3, 4\}$,
$3\ R\ 4$ because both 3 and 4 are in $\{0, 3, 4\}$, and
$4\ R\ 3$ because both 4 and 3 are in $\{0, 3, 4\}$.

Also, $0\ R\ 0$ because both 0 and 0 are in $\{0, 3, 4\}$
$3\ R\ 3$ because both 3 and 3 are in $\{0, 3, 4\}$, and
$4\ R\ 4$ because both 4 and 4 are in $\{0, 3, 4\}$.

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Example

Let $A = \{0, 1, 2, 3, 4\}$ and consider the following partition of A:

$$\{0, 3, 4\}, \{1\}, \{2\}.$$

Find the relation R induced by this partition.

Since {1} is a subset of the partition,

$1 \; R \; 1$ because both 1 and 1 are in {1},

and since {2} is a subset of the partition,

$2 \; R \; 2$ because both 2 and 2 are in {2}.

Hence

$$R = \{(0,0),(0,3),(0,4),(1,1),(2,2),(3,0),$$
$$(3,3),(3,4),(4,0),(4,3),(4,4)\}.$$

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Relations Induced by a Partition

Theorem 8.3.1

Let A be a set with a partition and let R be the relation induced by the partition. Then R is reflexive, symmetric, and transitive.

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Relations

8.3 Equivalence Relations

In this lecture:

- Part 1: **Partitioned Sets**
- Part 2: **Equivalence Classes**
- Part 3: **Equivalence Relation**

Equivalence Relation

علاقة تكافؤ

Definition

Let A be a set and R a relation on A. R is an **equivalence relation** if, and only if, R is reflexive, symmetric, and transitive.

➔ The relation induced by a partition is an equivalence relation

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Example

Let X be the set of all nonempty subsets of $\{1, 2, 3\}$. Then

$$X = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

Define a relation R on X as follows: For all A and B in X,

$$A\ R\ B \Leftrightarrow \text{the least element of } A \text{ equals the least element of } B.$$

Prove that R is an equivalence relation on X.

R *is reflexive*: Suppose A is a nonempty subset of $\{1, 2, 3\}$. *[We must show that A* **R** *A.]* It is true to say that the least element of A equals the least element of A. Thus, by definition of R, A **R** A.

R *is symmetric*: Suppose A and B are nonempty subsets of $\{1, 2, 3\}$ and A **R** B. *[We must show that B* **R** *A.]* Since A **R** B, the least element of A equals the least element of B. But this implies that the least element of B equals the least element of A, and so, by definition of **R**, B **R** A.

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Example

Let X be the set of all nonempty subsets of $\{1, 2, 3\}$. Then

$$X = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

Define a relation R on X as follows: For all A and B in X,

$$A\ R\ B \Leftrightarrow \text{the least element of } A \text{ equals the least element of } B.$$

Prove that R is an equivalence relation on X.

R *is transitive*: Suppose A, B, and C are nonempty subsets of $\{1, 2, 3\}$, A **R** B, and $B\ R\ C$. *[We must show that A* **R** *C.]* Since A **R** B, the least element of A equals the least element of B and since B **R** C, the least element of B equals the least element of C. Thus the least element of A equals the least element of C, and so, by definition of **R**, A **R** C.

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Example

Let S be the set of all digital circuits with a fixed number n of inputs. Define a relation E on S as follows: For all circuits $C1$ and $C2$ in S,

$$C_1 \mathbf{E} C_2 \Leftrightarrow C1 \text{ has the same input/output table as } C2.$$

E ***is reflexive*****:** Suppose C is a digital logic circuit in S. *[We must show that C **E** C.]* Certainly C has the same input/output table as itself. Thus, by definition of **E**, C **E** C

E ***is symmetric*****:** Suppose C_1 and C_2 are digital logic circuits in S such that C_1 **E** C_2. By definition of **E**, since C_1 **E** C_2, then C_1 has the same input/output table as C_2. It follows that C_2 has the same input/output table as C_1. Hence, by definition of **E**, C_2 **E** C_1

E ***is transitive*****:** Suppose C_1, C_2, and C_3 are digital logic circuits in S such that C_1 **E** C_2 and C_2 **E** C_3. By definition of **E**, since C_1 **E** C_2 and C_2 **E** C_3, then C_1 has the same input/output table as C_2 and C_2 has the same input/output table as C_3. It follows that C_1 has the same input/output table as C_3. Hence, by definition of **E**, C_1 **E** C_3

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Example

Let L be the set of all allowable identifiers in a certain computer language, and define a relation R on L as follows: For all strings s and t in L,

$s\ R\ t \Leftrightarrow$ the first eight characters of s equal the first eight characters of t.

***R* is reflexive**: Let $s \in L$. Clearly s has the same first eight characters as itself. Thus, by definition of R, $s\ R\ s$.

***R* is symmetric**: Let s and t be in L and suppose that $s\ R\ t$. By definition of R, since $s\ R\ t$, the first eight characters of s equal the first eight characters of t. But then the first eight characters of t equal the first eight characters of s. And so, by definition of R, $t\ R\ s$

***R* is transitive**: Let s, t, and u be in L and suppose that $s\ R\ t$ and $t\ R\ u$. By definition of R, since $s\ R\ t$ and $t\ R\ u$, the first eight characters of s equal the first eight characters of t, and the first eight characters of t equal the first eight characters of u. Hence the first eight characters of s equal the first eight characters of u. Thus, by definition of R, $s\ R\ u$

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Relations

8.3 Equivalence Relations

In this lecture:

- Part 1: **Partitioned Sets**
- Part 2: **Equivalence Classes**
- Part 3: **Equivalence Relation**

Equivalence Class

- **Definition**

Suppose A is a set and R is an equivalence relation on A. For each element a in A, the **equivalence class of *a***, denoted **[*a*]** and called the **class of *a*** for short, is the set of all elements x in A such that x is related to a by R.

In symbols:

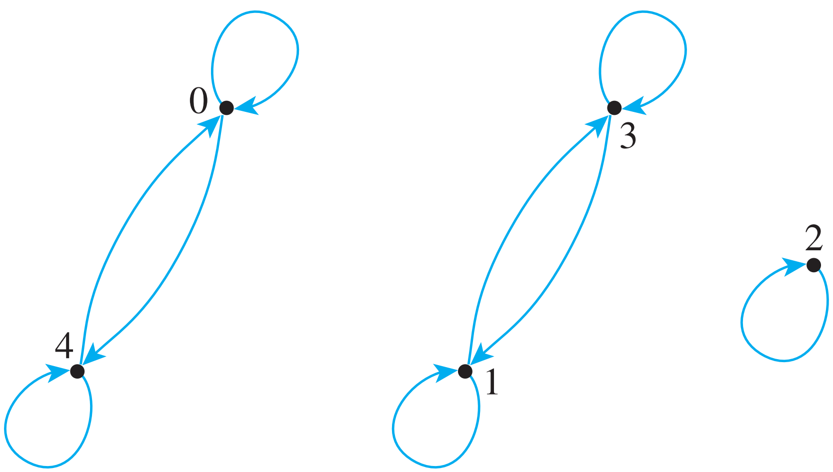
$$[a] = \{x \in A \mid x \; R \; a\}$$

for all $x \in A$, $\quad x \in [a] \quad \Leftrightarrow \quad x \; R \; a.$

Example

Let $A = \{0,1,2,3,4\}$ and define a relation R on A as :
$R = \{(0, 0), (0, 4), (1, 1), (1, 3), (2, 2), (3, 1), (3, 3), (4, 0), (4, 4)\}$.

Find the distinct equivalence classes of R.



$[0] = \{x \in A \mid x \; R \; 0\} = \{0, 4\}$

$[1] = \{x \in A \mid x \; R \; 1\} = \{1, 3\}$

$[2] = \{x \in A \mid x \; R \; 2\} = \{2\}$

$[3] = \{x \in A \mid x \; R \; 3\} = \{1, 3\}$

$[4] = \{x \in A \mid x \; R \; 4\} = \{0, 4\}$

$[0] = [4]$ and $[1] = [3]$. Thus the *distinct* equivalence classes of the relation are $\{0, 4\}$, $\{1, 3\}$, and $\{2\}$.

Equivalence Class

Lemma 8.3.2

Suppose A is a set, R is an equivalence relation on A, and a and b are elements of A. If $a \; R \; b$, then $[a] = [b]$.

Lemma 8.3.3

If A is a set, R is an equivalence relation on A, and a and b are elements of A, then

$$\text{either} \quad [a] \cap [b] = \emptyset \quad \text{or} \quad [a] = [b].$$

• Definition

Suppose R is an equivalence relation on a set A and S is an equivalence class of R. A **representative** of the class S is any element a such that $[a] = S$.

Congruence Modulo 3

Let R be the relation of congruence modulo 3 on the set $\mathbf{Z}$ of all integers. That is, for all integers m and n,

$$m\ R\ n \Leftrightarrow 3|(m-n) \Leftrightarrow m \equiv n \pmod 3.$$

Describe the distinct equivalence classes of R.

For each integer a,

$$\begin{aligned}[a] &= \{x \in \mathbf{Z} \mid x\ R\ a\} \\ &= \{x \in \mathbf{Z} \mid 3 \mid (x-a)\} \\ &= \{x \in \mathbf{Z} \mid x - a = 3k, \text{ for some integer } k\}.\end{aligned}$$

Therefore

$$[a] = \{x \in \mathbf{Z} \mid x = 3k + a, \text{ for some integer } k\}.$$

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Congruence Modulo 3

Let R be the relation of congruence modulo 3 on the set $\mathbf{Z}$ of all integers. That is, for all integers m and n,

$$mRn \Leftrightarrow 3|(m-n) \Leftrightarrow m \equiv n (\text{mod} 3).$$

In particular:

$$\begin{aligned}
[0] &= \{x \in \mathbf{Z} \mid x = 3k + 0, \text{ for some integer } k\} \\
&= \{x \in \mathbf{Z} \mid x = 3k, \text{ for some integer } k\} \\
&= \{\ldots -9, -6, -3, 0, 3, 6, 9, \ldots\}, \\
[1] &= \{x \in \mathbf{Z} \mid x = 3k + 1, \text{ for some integer } k\} \\
&= \{\ldots -8, -5, -2, 1, 4, 7, 10, \ldots\}, \\
[2] &= \{x \in \mathbf{Z} \mid x = 3k + 2, \text{ for some integer } k\} \\
&= \{\ldots -7, -4, -1, 2, 5, 8, 11, \ldots\}.
\end{aligned}$$

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Congruence Modulo 3

Let R be the relation of congruence modulo 3 on the set **Z** of all integers. That is, for all integers m and n,

$$mRn \Leftrightarrow 3|(m-n) \Leftrightarrow m \equiv n (\text{mod} 3).$$

Now since 3 R 0, then by Lemma 8.3.2,

$$[3] = [0].$$

More generally, by the same reasoning,

$$[0] = [3] = [-3] = [6] = [-6] = \ldots, \text{ and so on.}$$

Similarly,

$$[1] = [4] = [-2] = [7] = [-5] = \ldots, \text{ and so on.}$$

And

$$[2] = [5] = [-1] = [8] = [-4] = \ldots, \text{ and so on.}$$

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Congruence Modulo 3

Let R be the relation of congruence modulo 3 on the set **Z** of all integers. That is, for all integers m and n,

$$mRn \Leftrightarrow 3|(m-n) \Leftrightarrow m \equiv n \pmod{3}.$$

Notice that every integer is in class [0], [1], or [2]. Hence the distinct equivalence classes are

$$\{x \in \mathbf{Z} \mid x = 3k, \text{ for some integer } k\},$$

$$\{x \in \mathbf{Z} \mid x = 3k + 1, \text{ for some integer } k\},$$

$$\{x \in \mathbf{Z} \mid x = 3k + 2, \text{ for some integer } k\}.$$

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Congruence Modulo 3

Determine which of the following congruences are true and which are false.
a. $12 \equiv 7 (\text{mod}5)$ b. $6 \equiv -8 (\text{mod}4)$ c. $3 \equiv 3 (\text{mod}7)$

a. True. $12 - 7 = 5 = 5 \cdot 1$. Hence $5 \mid (12 - 7)$, and so $12 \equiv 7 \pmod 5$.

b. False. $6 - (-8) = 14$, and $4 \nmid 14$ because $14 \neq 4 \cdot k$ for any integer k. Consequently, $6 \not\equiv -8 \pmod 4$.

c. True. $3 - 3 = 0 = 7 \cdot 0$. Hence $7 \mid (3 - 3)$, and so $3 \equiv 3 \pmod 7$. ■

Exercise

Let A be the set of all ordered pairs of integers for which the second element of the pair is nonzero. Symbolically,

$$A = \mathbf{Z} \times (\mathbf{Z} - \{0\}).$$

Define a relation R on A as follows: For all $(a, b), (c, d) \in A$,

$$(a,b) R (c,d) \Leftrightarrow ad = bc.$$

Describe the distinct equivalence classes of R

For example, the class (1,2):

$$[(1, 2)] = \{(1, 2), (-1, -2), (2, 4), (-2, -4), (3, 6), (-3, -6), \ldots\}$$

since $\frac{1}{2} = \frac{-1}{-2} = \frac{2}{4} = \frac{-2}{-4} = \frac{3}{6} = \frac{-3}{-6}$ and so forth.