Chapter 3 : Functions on $\mathbb{R}$.

3.1 : Two - sided limits.

Df 1. let $a \in \mathbb{R}$, let I be an open interval which contains a, and let f be a real function defined on I except possibly at a. Then we say that $f(x)$ converges (approaches) to L as x approaches a, and write ($\lim_{x \to a} f(x) = L$ iff $\forall \varepsilon > 0$, $\exists$ a $\delta > 0$ (which ingeneral depends on ε, f, I and a) such that

$$0 < |x - a| < \delta \longrightarrow |f(x) - L| < \varepsilon \quad *)$$ $\rightarrow a - \delta < x < a + \delta \rightarrow L - \varepsilon < f(x) < L + \varepsilon$

$\lim_{x \to 2} (x - 2) = 0 \leftarrow$ بما تشبها

RMK:

1. ε represents the maximal error allowed in the approximation $f(x)$ to L.

2. According to Df1, to show that a function has a limit, we must begin with ageneral $\varepsilon > 0$ and describe how to choose a δ which satisfies $\boxed{*}$

exp: let $f(x) = mx + b$ where $m, b \in \mathbb{R}$. prove that $\lim_{x \to a} f(x) = f(a)$, $\forall a \in \mathbb{R}$.

proof:

case1 : $m = 0 \rightarrow f(x) = b \rightarrow \underset{L}{f(a)} = b$.

$$|f(x) - L| = |b - b| = 0 < \varepsilon \text{ for all } x. \checkmark$$

Case 2 $\rightarrow$

Case 2: $m \neq 0$

given $\varepsilon > 0$, set $\boxed{\delta = \frac{\varepsilon}{|m|}}$

If $|x-a| < \delta$, then $|f(x) - f(a)| = |(mx+b) - (ma+b)|$

$= |m|\,|x-a|$

$< |m|\,\delta$ ← بدي ياها أقل من ε فبختار δ على هالأساس

$= |m| \frac{\varepsilon}{|m|}$

$= \varepsilon$ $\boxtimes$

Thus, by def 1 $\lim_{x \to a} f(x) = f(a)$

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exp 2: If $f(x) = x \sin \frac{1}{x}$, $x \neq 0$, then $\lim_{x \to 0} f(x) = 0$.

proof: $f(x) = x \sin \frac{1}{x}$, $L = 0$, $a = 0$

let $\varepsilon > 0$, set $\boxed{\delta = \varepsilon}$ $\boxed{0 < \delta \leq \varepsilon}$.

If $|x - 0| < \delta$, then $|f(x) - L| = |x \sin(\frac{1}{x}) - 0|$

$= |x|\,|\sin \frac{1}{x}|$, ≤ 1

$\leq |x|$

$< \delta$

$\leq \varepsilon$ لو كانت $\frac{\varepsilon}{2}$ أو أي اشي أقل من ε عادي

$\boxtimes$

Exp 3: If $f(x) = x^2 + x - 3$, prove that $\lim_{x \to 1} f(x) = -1$?

proof : $f(x) = x^2 + x - 3$, $a = 1$, $L = -1$

let $\varepsilon > 0$, we need to find a $\delta > 0$ s.t. $0 < |x-1| < \delta \rightarrow |(x^2+x-3) - (-1)| < \varepsilon$

i.e. $0 < |x-1| < \delta \longrightarrow |x+2||x-1| < \varepsilon$.

$\hookrightarrow |f - L| < \delta |x+2| \rightarrow \delta = \frac{\varepsilon}{|x+2|}$

وهاد غلط لانو δ ما بتعتمد على x . عشان هيك بنخد هاي الطريقة ونختار δ هيك

If $0 < \delta \leq 1$ then $|x-1| < \delta \Rightarrow 0 < x < 2$ $\quad$ $-1 < x-1 < 1 \rightarrow 0 < x < 2$

so $|x+2| \leq |x| + 2$

$< 2 + 2 = 4$

أخذت min لأنو كنت فوق مثبتة إنو δ أقل من 1 δ

set $\delta = \frac{\varepsilon}{4} \rightarrow \delta = \min\{1, \frac{\varepsilon}{4}\}$

It follows that if $|x-1| < \delta$, then $|f(x) - L| = |x-1||x+2|$

$< 4|x-1|$.

$< 4\delta$

$< 4 \cdot \frac{\varepsilon}{4} = \varepsilon$ ☒

Thus, $\lim_{x \to 1} f(x) = (-1)$ ☒

Thm 1 : If $\lim_{x\to a} f(x)$ exists, then it is unique, i.e if $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} f(x) = L_2$ then $L_1 = L_2$.

proof : suppose that $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} f(x) = L_2$.

let $\varepsilon > 0$ and $\exists$ $\delta_1, \delta_2 > 0$ s.t if $|x-a| < \delta$ then $|f(x) - L_1| < \varepsilon$ and if $|x-a| < \delta_2$ then $|f(x) - L_2| < \varepsilon$

set $\delta = \min\{\delta_1, \delta_2\}$ then

$$\begin{aligned} |L_1 - L_2| &= |L_1 - f(x) + f(x) - L_2| \\ &\leq |f(x) - L_1| + |f(x) - L_2| \\ &< \varepsilon + \varepsilon \\ &< 2\varepsilon \end{aligned}$$

i.e, $|L_1 - L_2| < 2\varepsilon$, $\forall \varepsilon > 0$.

$L_1 = L_2$ ($|a| < \varepsilon$, $\forall \varepsilon > 0$ then $a = 0$)

uniquence. ☒

The next result shows that even when a function f is defined at a, $\lim_{x\to a} f(x)$, ingeneral independent of the velue of $f(a)$.

Lemma : let $a \in \mathbb{R}$, let I be an open interval which contains a, and let f, g be real functions defined $\forall x \in I$ expect possibly at a. If $f(x) = g(x)$, $\forall x \in I \setminus \{a\}$ and $\lim_{x\to a} f(x) = L$ then $\lim_{x\to a} g(x)$ exists and $\lim_{x\to a} g(x) = \lim_{x\to a} f(x)$.

$$\frac{x^2 - 4}{x - 2} = x + 2 \quad \forall x \in \mathbb{R} \setminus \{2\}$$

$\lim_{x\to 2} g(x) =$ exist $= 4$ $\qquad$ $\lim_{x\to 2} f(x) = 4$

exp: prove that $\lim_{x\to 1} g(x)$ exists, if $g(x) = \frac{x^3+x^2-x-1}{x^2-1} = \frac{x^2(x+1)-(x+1)}{x^2-1}$

$= \frac{(x+1)(x^2-1)}{x^2-1}$

$= x+1, \; x \neq \pm 1$

set $f(x) = x+1$, observe $g(x) = f(x)$, $x \neq \pm 1$

By the last lemma, $\lim_{x\to 1} g(x)$ exists.

And $\lim_{x\to 1} g(x) = \lim_{x\to 1} f(x) = 2$ ☒

Thm 2: sequential characterization of limits.

let $a \in \mathbb{R}$, let I be an open interval contains a, and let f be a real function defined $\forall x \in I$ except possibly at a. Then $\lim_{x\to a} f(x) = L$ iff $f(x_n) \to L$ as $n \to \infty$ for every sequence $x_n \in I \setminus \{a\}$ which converges to a as $n \to \infty$.

$x_n \to a$ then $f(x_n) \to L$.

proof:

$\Rightarrow$ suppose that $\lim_{x\to a} f(x) = L$, Then given $\varepsilon > 0$, $\exists$ a $\delta > 0$ s.t

$0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon$ --- i

let $x_n \in I \setminus \{a\}$ s.t $x_n \to a$ as $n \to \infty$.

then $\exists$ an $N \in \mathbb{N}$ s.t $n \geq N \to |x_n - a| < \delta$

Since $x_n \neq a$, it follows from i that $|f(x_n) - L| < \varepsilon$, $\forall n \geq N$

Therefor, $f(x_n) \to L$ as $n \to \infty$

$\to$

Conti.

Proof: $\Leftarrow$ conversly, suppose that $f(x_n) \to L$ as $n \to \infty$ for every sequence $x_n \in I \setminus \{a\}$ which converges to a (i.e, $x_n \to a$).

suppose that $\lim_{x \to a} f(x) \neq L$, then there is an $\varepsilon > 0$ (say ε_0) such that the implication "$0 < |x-a| < \delta \to |f(x)-L| < \varepsilon$" does not hold for any $\delta > 0$.

Thus, for each $\delta = \frac{1}{n}$, $n \in \mathbb{N}$, $\exists$ a point $x_n \in I$:

$0 < |x_n - a| < \frac{1}{n}$ and $|f(x_n) - L| \geq \varepsilon_0$

$a - \frac{1}{n} < x_n < a + \frac{1}{n}$, $a - \frac{1}{n} \to a$ as $n \to \infty$, $a + \frac{1}{n} \to a$ as $n \to \infty$

Now the first condition and the squeeze thm,

$x_n \neq a$, $x_n \to a$ as $n \to \infty$ so by hypothesis

$f(x_n) \to L$ as $n \to \infty$

Inparticular, $|f(x_n) - L| < \varepsilon_0$ for large n.

Which contradicts the second condition

$\lim_{x \to a} f(x) = L$ ⊠

RMK: To show that the limit of a function f does not exist as $x \to a$ using thm 2, we need to find two sequences converges to a (say $x_n \to a$ and $y_n \to a$) whose images under f have different limits.

i.e, $f(x_n) \to L_1$ and $f(y_n) \to L_2$ where $L_1 \neq L_2$

exp 5. prove that $f(x) = \begin{cases} \sin \frac{1}{x} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$ has no limit as $x \to 0$.

let $x_n = \frac{2}{(4n+1)\pi}$, $y_n = \frac{2}{(4n+3)\pi}$, $n \in \mathbb{N}$ (بنختار 2-seq. تقرب لـ 0)

$x_n \to 0$, $y_n \to 0$ (as $n \to \infty$)

$\to f(x_n) = \sin\left(\frac{(4n+1)\pi}{2}\right) = \sin\left(\frac{\pi}{2} + 2n\pi\right) = \sin\left(\frac{\pi}{2}\right) = 1$

$\therefore f(x_n) = 1 \quad \forall n \in \mathbb{N}$

$\to f(y_n) = \sin\left(\frac{(4n+3)\pi}{2}\right) = \sin\left(\frac{3\pi}{2} + 2n\pi\right) = \sin\left(\frac{3\pi}{2}\right) = -1$

$\therefore f(y_n) = -1 \quad \forall n \in \mathbb{N}$

for all $n \in \mathbb{N}$, $f(x_n) \to 1$ as $n \to \infty$

and $f(y_n) \to -1$ as $n \to \infty$

Thus, $\lim_{x \to 0} f(x)$ does Not exist by Thm 2. ∎

لازم نجيب 2-sequ. يكون converges to 0 بحيث انه الصورة للأولى ≠ الصورة الثانية

← بقدر أوخذ $x_n = \frac{1}{2n\pi} \to 0$, $y_n = \frac{1}{2n\pi + \frac{\pi}{2}} \to 0$

$f(x_n) = \sin(2n\pi) = 0$

$f(y_n) = \sin\left(2n + \frac{\pi}{2}\right) = 1$

Thm 3: suppose that $a \in \mathbb{R}$, that I is an open interval which contains a and that f, g are real functions defined $\forall x \in I$ except possibly at a. If $f(x)$ and $g(x)$ converges as $x \to a$ (i.e. $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist). then so do $(f+g)(x) = f(x) + g(x)$, $(fg)(x) = f(x)g(x)$, $(\alpha f)(x) = \alpha f(x)$ and $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ (when $\lim_{x\to a} g(x) \neq 0$). In fact,

i. $\lim_{x\to a} (f(x) + g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$.

ii. $\lim_{x\to a} (\alpha f)(x) = \alpha \lim_{x\to a} f(x)$.

iii. $\lim_{x\to a} (fg)(x) = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right)$.

iv. $\lim_{x\to a} \left(\frac{f}{g}\right)(x) = \frac{\lim f(x)}{\lim_{x\to a} g(x)}$, ... $g(x) \neq 0$.

Proof:

(i) let $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$

If $x_n \in I\setminus\{a\}$ s.t $x_n \to a$ as $n \to \infty$, By Thm 2

$f(x) \to L$ and $g(x_n) \to M$ as $n \to \infty$

By (Thm ch2), $(f+g)(x_n) = f(x_n) + g(x_n) \to L + M$ as $n \to \infty$

By Thm 2, $\lim_{x\to a} (f+g)(x) = L + M$

$\hookrightarrow = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$ ☒

$\rightsquigarrow$

طريقة ثانية → OR, By (ε–δ) def'n :

Def 1 لا! Thm 2 و!

$\lim_{x \to a} f(x) = L$ means given $\varepsilon > 0$, $\exists\, \delta_1 > 0$ s.t $|x-a| < \delta_1 \rightarrow |f(x) - L| < \frac{\varepsilon}{2}$

$\lim_{x \to a} g(x) = M$ means $\exists\, \delta_2 > 0$ s.t $|x-a| < \delta_2 \rightarrow |g(x) - M| < \frac{\varepsilon}{2}$

Set $\delta = \min\{\delta_1, \delta_2\}$

$$|x-a| < \delta \Rightarrow |(f+g)(x) - (L+M)|$$
$$\leq |f(x) - L| + |g(x) - M|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$
$$< \varepsilon$$

ii.

Case 1 : $\alpha = 0$ trivial $\alpha f(x) = 0 \rightarrow 0$ as $x \to \infty$

Case 2 : $\alpha \neq 0$ (show $\lim_{x \to a} \alpha f(x) = \alpha L$).

since $\lim_{x \to a} f(x) = L$, given $\varepsilon > 0$, $\exists$ a $\delta > 0$ s.t

$$|x-a| < \delta \rightarrow |f(x) - L| < \frac{\varepsilon}{|\alpha|}$$

Now, $|x-a| < \delta \Rightarrow |\alpha f(x) - \alpha L|$
$$= |\alpha|\,|f(x) - L|$$
$$< |\alpha| \frac{\varepsilon}{|\alpha|} = \varepsilon$$

$\therefore \lim_{x \to a} \alpha f(x) = \alpha L = \alpha \lim_{x \to a} f(x)$.

Thm 4: squeeze Theorem for functions.

suppose that $a \in \mathbb{R}$, that I is an open interval which contains a, and that f, g, h are real functions defined $\forall x \in I$ except possibly at a.

i. If $g(x) \leq h(x) \leq f(x)$ for all $x \in I \setminus \{a\}$ and $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = L$ then $\lim_{x \to a} h(x)$ exists and $\lim_{x \to a} h(x) = L$

ii. If $|g(x)| \leq M$ for all $x \in I \setminus \{a\}$ (i.e, g is bdd) and $\lim_{x \to a} f(x) = 0$ then $\lim_{x \to a} f(x) g(x) = 0$

Thm 5: comparison Theorem for functions.

suppose that $a \in \mathbb{R}$, that I is an open interval which contains a, and that f, g are real functions defined $\forall x \in I$ except possibly at a. If f and g have limits as $x \to a$ and $f(x) \leq g(x)$ $\forall x \in I \setminus \{a\}$, then $\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$

RMK: We shall refer to thm 5 as "taking the limit of an inequality".

RMK: The limit Thms (Thm 3, 4, 5) allow us to prove that limits exist without using $(\varepsilon - \delta)$ definition.

exp: prove that $\lim_{x \to 1} \frac{x-1}{3x+1} = 0$

$\lim_{x \to 1} x-1 = 0$ and $\lim_{x \to 1} 3x+1 = 4$

Hence, by Thm 3 (iv): $\lim_{x \to 1} \frac{x-1}{3x+1} = \frac{\lim_{x \to 1} x-1}{\lim_{x \to 1} 3x+1} = \frac{0}{4} = 0$

→ Think by (ε-δ) def'n ?

given $\varepsilon > 0$ and set $\delta = \min\{1, 7\varepsilon\}$

If $0 < |x-1| < \delta$ then $\left| \frac{x-1}{3x+1} - 0 \right|$

$0 < \delta \leq 1$

$-1 < x-1 < 1$ (+1)

$(0 < x < 2)$ ×3

$(0 < 3x < 6)$ +1

$1 < 3x+1 < 7$

$$= \frac{|x-1|}{|3x+1|}$$

$$< \frac{\delta}{7}$$

$$< \frac{7\varepsilon}{7}$$

$$< \varepsilon$$