2.2 Properties of Determinants

Lemma 2.2.1 *Let A be an $n \times n$ matrix. If A_{jk} denotes the cofactor of a_{jk} for $k = 1, \ldots, n$, then*

$$a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots + a_{in}A_{jn} = \begin{cases} \det(A) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \tag{1}$$

Proof If $i = j$, (1) is just the cofactor expansion of $\det(A)$ along the ith row of A. To prove (1) in the case $i \neq j$, let A^* be the matrix obtained by replacing the jth row of A by the ith row of A:

$$A^* = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \quad j\text{th row}$$

Since two rows of A^* are the same, its determinant must be zero. It follows from the cofactor expansion of $\det(A^*)$ along the jth row that

$$\begin{aligned} 0 = \det(A^*) &= a_{i1}A^*_{j1} + a_{i2}A^*_{j2} + \cdots + a_{in}A^*_{jn} \\ &= a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots + a_{in}A_{jn} \end{aligned}$$

Let us now consider the effects of each of the three row operations on the value of the determinant.

Row Operation I

Two rows of A are interchanged.

If A is a 2×2 matrix and

$$E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

then

$$\det(EA) = \begin{vmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{vmatrix} = a_{21}a_{12} - a_{22}a_{11} = -\det(A)$$

In general, if A is an $n \times n$ matrix and E_{ij} is the $n \times n$ elementary matrix formed by interchanging the ith and jth rows of I, then

$$\det(E_{ij}A) = -\det(A)$$

Row Operation II

A row of A is multiplied by a nonzero scalar.

Let E denote the elementary matrix of type II formed from I by multiplying the ith row by the nonzero scalar α. If $\det(EA)$ is expanded by cofactors along the ith row, then

$$\begin{aligned}\det(EA) &= \alpha a_{i1}A_{i1} + \alpha a_{i2}A_{i2} + \cdots + \alpha a_{in}A_{in} \\ &= \alpha(a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}) \\ &= \alpha \det(A)\end{aligned}$$

Row Operation III

A multiple of one row is added to another row.

Let E be the elementary matrix of type III formed from I by adding c times the ith row to the jth row. Since E is triangular and its diagonal elements are all 1, it follows that $\det(E) = 1$. We will show that

$$\det(EA) = \det(A) = \det(E)\det(A)$$

If $\det(EA)$ is expanded by cofactors along the jth row, it follows from Lemma 2.2.1 that

$$\begin{aligned}\det(EA) &= (a_{j1} + ca_{i1})A_{j1} + (a_{j2} + ca_{i2})A_{j2} + \cdots + (a_{jn} + ca_{in})A_{jn} \\ &= (a_{j1}A_{j1} + \cdots + a_{jn}A_{jn}) + c(a_{i1}A_{j1} + \cdots + a_{in}A_{jn}) \\ &= \det(A)\end{aligned}$$

Determinants and Elementary Row

In summation, if E is an elementary matrix, then

$$\det(EA) = \det(E)\det(A)$$

where

$$\det(E) = \begin{cases} -1 & \text{if } E \text{ is of type I} \\ \alpha \neq 0 & \text{if } E \text{ is of type II} \\ 1 & \text{if } E \text{ is of type III} \end{cases}$$

Thus, the effects that row or column operations have on the value of the determinant can be summarized as follows:

- **I.** Interchanging two rows (or columns) of a matrix changes the sign of the determinant.
- **II.** Multiplying a single row or column of a matrix by a scalar has the effect of multiplying the value of the determinant by that scalar.
- **III.** Adding a multiple of one row (or column) to another does not change the value of the determinant.

Matrices with Proportional Rows or Columns

Theorem.

If A is a square matrix with <u>two proportional rows</u> or <u>two proportional columns</u>, then $|A| = 0$.

Example. Evaluate the determinants

a) $\begin{vmatrix} 1 & 0 & -3 & 0 \\ -1 & 3 & 3 & 2 \\ 2 & 1 & -6 & 1 \\ 1 & 4 & -3 & -4 \end{vmatrix}$

b) $\begin{vmatrix} 2 & 1 & -2 & 1 \\ 7 & 3 & 3 & 2 \\ 2 & 1 & -2 & 1 \\ 0 & 4 & -3 & -5 \end{vmatrix}$

Evaluating Determinants by Row (or Column) Reduction

- Elementary row (or column) operations can be used to develop a method for evaluating determinants that involves substantially less computation than cofactor expansion.
- The idea of the method is to reduce the given matrix to triangular form by elementary row (or column) operations, then compute the determinant of the triangular matrix (an easy computation), and then relate that determinant to that of the original matrix.

Remark.

Even with today's fastest computers, it would take millions of years to calculate a 25×25 determinant by cofactor expansion, **so methods based on row reduction are often used for large determinants**. For determinants of small size (such as those in our textbook), cofactor expansion is often a reasonable choice.

Example. Evaluate the determinant

$$|A| = \begin{vmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 1 & 6 & 1 \end{vmatrix}$$

$$\begin{bmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 1 & 6 & 1 \end{bmatrix} \xrightarrow{R_{13}} \begin{bmatrix} 1 & 6 & 1 \\ 3 & -6 & 9 \\ 0 & 1 & 5 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 3R_1} \begin{bmatrix} 1 & 6 & 1 \\ 0 & -24 & 6 \\ 0 & 1 & 5 \end{bmatrix}$$

$$\xrightarrow{R_{23}} \begin{bmatrix} 1 & 6 & 1 \\ 0 & 1 & 5 \\ 0 & -24 & 6 \end{bmatrix} \xrightarrow{R_3 \to R_3 + 24R_2} \begin{bmatrix} 1 & 6 & 1 \\ 0 & 1 & 5 \\ 0 & 0 & 126 \end{bmatrix}$$

$$|A| = (-1)(-1)(1)(1)(126) = 126$$

Example. Evaluate the determinant

$$|A| = \begin{vmatrix} 1 & 2 & 3 & -1 \\ 3 & 1 & 4 & 2 \\ 1 & -1 & -2 & 1 \\ 2 & 2 & 1 & 1 \end{vmatrix}$$

$R_2 \to R_2 - 3R_1$ $\quad$ $R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 3 & 1 & 4 & 2 \\ 1 & -1 & -2 & 1 \\ 2 & 2 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & -5 & -5 & 5 \\ 1 & -1 & -2 & 1 \\ 2 & 2 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & -5 & -5 & 5 \\ 0 & -3 & -5 & 2 \\ 2 & 2 & 1 & 1 \end{bmatrix}$$

$R_4 \to R_4 - 2R_1$

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & -5 & -5 & 5 \\ 0 & -3 & -5 & 2 \\ 0 & -2 & -5 & 3 \end{bmatrix} \xrightarrow{-\frac{1}{5}R_2} \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & -3 & -5 & 2 \\ 0 & -2 & -5 & 3 \end{bmatrix} \xrightarrow{R_3 \to R_3 + 3R_2} \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -2 & -1 \\ 0 & -2 & -5 & 3 \end{bmatrix}$$

$R_4 \to R_4 + 2R_2$

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & -3 & 1 \end{bmatrix} \xrightarrow{-\frac{1}{2}R_3} \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & -3 & 1 \end{bmatrix} \xrightarrow{R_4 \to R_4 + 3R_3} \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{5}{2} \end{bmatrix}$$

$$|A| = (-5)(-2)(1)(1)(1)\left(\frac{5}{2}\right) = 25$$

Example. Given that

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -6,$$

evaluate the determinants

a) $\begin{vmatrix} 3a & 3b & 3c \\ -d & -e & -f \\ 4g & 4h & 4i \end{vmatrix}$

b) $\begin{vmatrix} a & b & c \\ 2d & 2e & 2f \\ g+3a & h+3b & i+3c \end{vmatrix}$

Theorem 2.2.2 *An $n \times n$ matrix A is singular if and only if*

$$\det(A) = 0$$

Proof The matrix A can be reduced to row echelon form with a finite number of row operations. Thus,

$$U = E_k E_{k-1} \cdots E_1 A$$

where U is in row echelon form and the E_i's are all elementary matrices. It follows that

$$\begin{aligned}\det(U) &= \det(E_k E_{k-1} \cdots E_1 A) \\ &= \det(E_k)\det(E_{k-1}) \cdots \det(E_1)\det(A)\end{aligned}$$

Since the determinants of the E_i's are all nonzero, it follows that $\det(A) = 0$ if and only if $\det(U) = 0$. If A is singular, then U has a row consisting entirely of zeros, and hence $\det(U) = 0$. If A is nonsingular, then U is triangular with 1's along the diagonal and hence $\det(U) = 1$.

EXERCISES

18. Let A and B be $n \times n$ matrices and let $C = AB$. Prove that if B is singular then C must be singular.
Hint: Use Theorem 1.5.2.

B is singular $\Rightarrow \exists\, y \neq 0$ s.t. $By = 0$

$\Rightarrow \exists\, y \neq 0$ s.t. $Cy = 0$

$\Rightarrow$ C is singular.

Theorem 2.2.3 *If A and B are $n \times n$ matrices, then*

$$\det(AB) = \det(A)\det(B)$$

Proof If B is singular, it follows from Theorem 1.5.2 that AB is also singular (see Exercise 14 of Section 1.5), and therefore,

$$\det(AB) = 0 = \det(A)\det(B)$$

If B is nonsingular, B can be written as a product of elementary matrices. We have already seen that the result holds for elementary matrices. Thus

$$\begin{aligned}\det(AB) &= \det(AE_kE_{k-1}\cdots E_1)\\ &= \det(A)\det(E_k)\det(E_{k-1})\cdots\det(E_1)\\ &= \det(A)\det(E_kE_{k-1}\cdots E_1)\\ &= \det(A)\det(B)\end{aligned}$$ ■

EXERCISES

5. Let A be an $n \times n$ matrix and α a scalar. Show that

$$\det(\alpha A) = \alpha^n \det(A)$$

6. Let A be a nonsingular matrix. Show that

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

7. Let A and B be 3×3 matrices with $\det(A) = 4$ and $\det(B) = 5$. Find the value of

(a) $\det(AB)$
(b) $\det(3A)$
(c) $\det(2AB)$
(d) $\det(A^{-1}B)$

14. Let A and B be $n \times n$ matrices. Prove that the product AB is nonsingular if and only if A and B are both nonsingular.

16. A matrix A is said to be *skew symmetric* if $A^T = -A$. For example,

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

is skew symmetric, since

$$A^T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -A$$

If A is an $n \times n$ skew-symmetric matrix and n is odd, show that A must be singular.