Homework 1 (chapter 2) Reem Ali

Abistract 1

Q11: prove that the set of all 2x2 matrices with entries from R and determinent +1 is a group under matrix multiplication. $(GL(2,\mathbb{R}),\ det = 1,\ \cdot)$.

① Associative: let $A, B, C \in G$:

$(A.B).C = A.(B.C)$ since multiplication in matrices is associative

② Identity: $e = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in G$ since $|e| = 1$ ✓

③ Inverse: let $A \in G \Rightarrow A^{-1} \in G$ since $|A^{-1}| = \frac{1}{|A|} = 1$ ✓

So its a Group #.

Q12: For any integer $n > 2$, show that there are at least two elements in $U(n)$ that satisfy $x^2 = 1$. $U(n) = \{1, \dots, n-1\}$.

$1 \in U(n)$ $\forall n > 2$ and $1^2 = 1$ First element.

$n-1 \in U(n)$ $\forall n > 2$ and $(n-1)^2 \overset{?}{=} 1$

second element.

$n^2 - 2n + 1 \overset{?}{=} 1$

$(n-2)(n) + 1 \overset{?}{=} 1$

$1 = 1$ since $n = 0$.

Q14: let G be a group with the following property : whenever a, b and c belong to G and $ab = ca$, then $b = c$. prove that G is Abelian $\rightarrow$ iff $a * b = b * a$.

since G is a group $\Rightarrow$ G is Associative.

let $x, y \in G$

$x(yx) = (xy)x$

suppose $yx = b$, $xy = a$

$\rightarrow$ $xb = ax$

$\rightarrow$ $b = a$

$\rightarrow$ $yx = xy$ $\quad\therefore G$ is Abelian.

Q17: prove that a group G is Abelian iff $(ab)^{-1} = a^{-1}b^{-1}$ for all a and b in G.

suppose G is Abelian : $\forall a, b \in G \rightarrow a * b = b * a$.

$\boxed{\Rightarrow}$

$\rightarrow$ $ab = ba$ inverse to other side.

$\rightarrow$ $(ab)^{-1} = (ba)^{-1}$

$\rightarrow$ $a^{-1} b^{-1}$

$\boxed{\Leftarrow}$

$(ab)^{-1} = a^{-1} b^{-1}$

$((ab)^{-1})^{-1} = (a^{-1} b^{-1})^{-1}$

$(b^{-1}a^{-1})^{-1} = (a^{-1} b^{-1})^{-1}$

$ab = ba$

So G is Abelian #

Q18: Prove that in agroup, $(a^{-1})^{-1} = a$ for all a.

let $a \in G$, Then $a^{-1} \in G$ and $(a^{-1})^{-1} \in G$ « since G is agroup ».

$\rightarrow \quad a^{-1}(a^{-1})^{-1} = e$ multiply by a from aright.

$$a a^{-1} (a^{-1})^{-1} = a e$$

$$e (a^{-1})^{-1} = a$$

$$(a^{-1})^{-1} = a \quad \#.$$

Q19: For any elemnts a and b from agroup and any integer n, prove that $(a^{-1}ba)^n = a^{-1}b^n a$.

$$(a^{-1}ba)^n = a^{-1}\underline{b}\cancel{a} \cdot \cancel{a^{-1}}\underline{b}\cancel{a} \cdot \cancel{a^{-1}}\underline{b} \cdot \cancel{a} \, \cancel{\ } \cdots \cancel{\ } \cdot \cancel{a^{-1}}\underline{b}a.$$

$$= a^{-1} b^n a \quad \forall n \geq 1 \quad \#$$

Q20: If $a_1, a_2, \ldots, a_n$ belong to a group, What is the inverse of $a_1 a_2 \ldots a_n$?

We Know $(ab)^{-1} = b^{-1}a^{-1}$

so, $(a_1, a_2, \ldots a_n)^{-1} = a_n^{-1}, a_{n-1}^{-1}, a_{n-2}^{-1}, \ldots, a_1^{-1}$.

Q25: suppose the table below is agroup table, Fill in the blank entries.

	e	a	b	c	d
e	e	a	b	c	d
a	a	b	c	d	e
b	b	c	d	e	a
c	c	d	e	a	b
d	d	e	a	b	c

Q26 : prove that if $(ab)^2 = a^2 b^2$ in a group G, then $ab = ba$. Abelian.

supse that $(ab)^2 = a^2 b^2$

$$abab = aa\,bb \qquad \text{multiply by } a^{-1} \text{ from left}$$

$$a^{-1}abab = a^{-1}aa\,bb$$

$$bab = a\,bb \qquad \text{multiply by } b^{-1} \text{ from Right.}$$

$$ba\cancel{b}\cancel{b^{-1}} = abb\cancel{b^{-1}}$$

$$ba = ab$$

#

Q27: let a, b and c be elements of a group. solve the equation $axb = c$ for x .

solve $a^{-1}xa = c$ for x .

$$axb = c$$

$$a^{-1}axb = a^{-1}c \qquad \text{multiply by } a^{-1} \text{ from left.}$$

$$xb = a^{-1}c$$

$$xbb^{-1} = a^{-1}cb^{-1} \qquad \text{multiply by } b^{-1} \text{ from right.}$$

$$x = a^{-1}cb^{-1}$$

$$a^{-1}xa = c$$

$$aa^{-1}xa = ac \qquad \text{multiply by } a \text{ from left}$$

$$xa = ac$$

$$xaa^{-1} = aca^{-1} \qquad \text{multiply by } a^{-1} \text{ from right.}$$

$$\boxed{x = aca^{-1}}$$

Q35: Prove that if G is a group with the property that the square of every element is the identity, then G is Abelian.

Let $a \in G \Rightarrow a^2 = e$.

Let $b \in G \Rightarrow b^2 = e$.

$\rightsquigarrow a.b \in G$ since G is closure

$\rightsquigarrow (a.b)^2 = e$.

$\rightsquigarrow a^2 b^2 = aa\,bb = e = (ab)(ab) = (ab)^2 = e$

$\Rightarrow aabb = abab$

$a^{-1}aabb = a^{-1}abab$

$abb = bab$

$abbb^{-1} = babb^{-1}$

$ab = ba$ G is Abelian.

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Q36: Prove that the set of all 3x3 matrices with Real ientries of the form

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$$ is a group with multiplication.

* Closure: Let $A = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & \bar{a} & \bar{b} \\ 0 & 1 & \bar{c} \\ 0 & 0 & 1 \end{bmatrix} \in G$, Then

$$A.B = \begin{bmatrix} 1 & a+\bar{a} & \bar{b}+a\bar{c}+b \\ 0 & 1 & \bar{c}+c \\ 0 & 0 & 1 \end{bmatrix} \in G.$$

* Associtive: yes.

* identity $= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

* inverse of $\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -a & ac-b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix} \in G$

Q39 : let $G = \left\{ \begin{bmatrix} a & a \\ a & a \end{bmatrix} \mid a \in \mathbb{R}, a \neq 0 \right\}$, show that G is a group under matrix multiplication.

$\rightarrow$ closure : let $\begin{bmatrix} a & a \\ a & a \end{bmatrix} \in G$ and $\begin{bmatrix} b & b \\ b & b \end{bmatrix} \in G$

$$\begin{bmatrix} a & a \\ a & a \end{bmatrix} \begin{bmatrix} b & b \\ b & b \end{bmatrix} = \begin{bmatrix} 2ab & 2ab \\ 2ab & 2ab \end{bmatrix} \in G \checkmark$$

$\rightarrow$ Associtive : let $A, B, C \in G$

$(A.B).C = A.(B.C)$ since multiplication of matrix is Associative.

$\rightarrow$ Idintity : $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \in G$ since $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} a & a \\ a & a \end{bmatrix} = \begin{bmatrix} a & a \\ a & a \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} a & a \\ a & a \end{bmatrix}$.

$\rightarrow$ Inverse : $\begin{bmatrix} a & a \\ a & a \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{4a} & \frac{1}{4a} \\ \frac{1}{4a} & \frac{1}{4a} \end{bmatrix} \checkmark$

So G it's a Group.