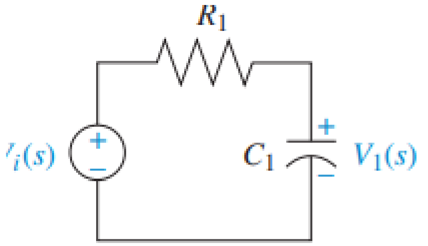
Control Systems Representation

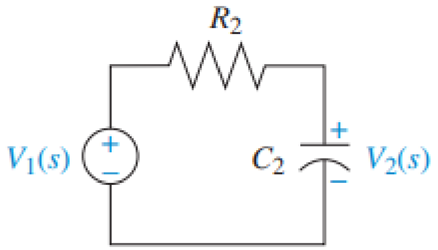
Block and Signal-flow Diagrams

Block Diagrams and Signal-flow Diagrams:

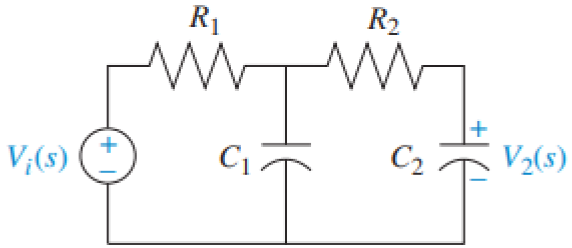
- A control system is composed of several subsystems that interact and exchange signals and employ signal combination through sum nodes and distribution through derivation points.
- A simple representation that describes the subsystems interaction and signal flow become necessary to analyze and study.
- Block diagrams and Signal-flow diagrams are among the most used universal languages in control systems.
- An analogy relation exists between Block diagrams and signal flow diagrams. That is between the input and output vocabularies of the these languages and their grammar.
- Whenever, these representation are used, it is inherently assumed that the chain rule is satisfied, that is the connection of two subsystems does not affect the validity of their mathematical models. That is each system maintains its transfer relation.
- Specific subsystems interconnection (cascade, parallel, and feedback) and other rules related to signals combination (sum nodes) and extraction (derivation points) are used to reduce the system representation to an equivalent one that includes only the necessary components for the control systems objectives.



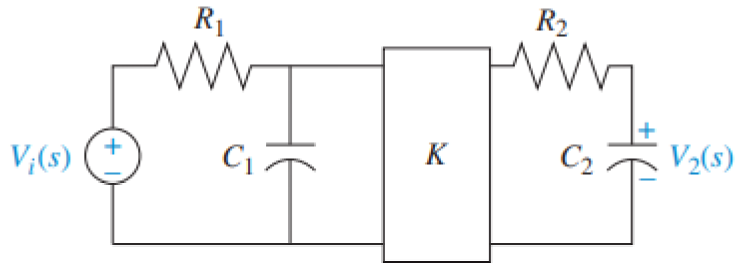
$$G_1(s) = \frac{V_1(s)}{V_i(s)}$$



$$G_2(s) = \frac{V_2(s)}{V_1(s)}$$



$$G_T(s) = \frac{V_2(s)}{V_i(s)} \neq G_2(s)G_1(s)$$



$$G_T(s) = \frac{V_2(s)}{V_i(s)} = KG_2(s)G_1(s)$$

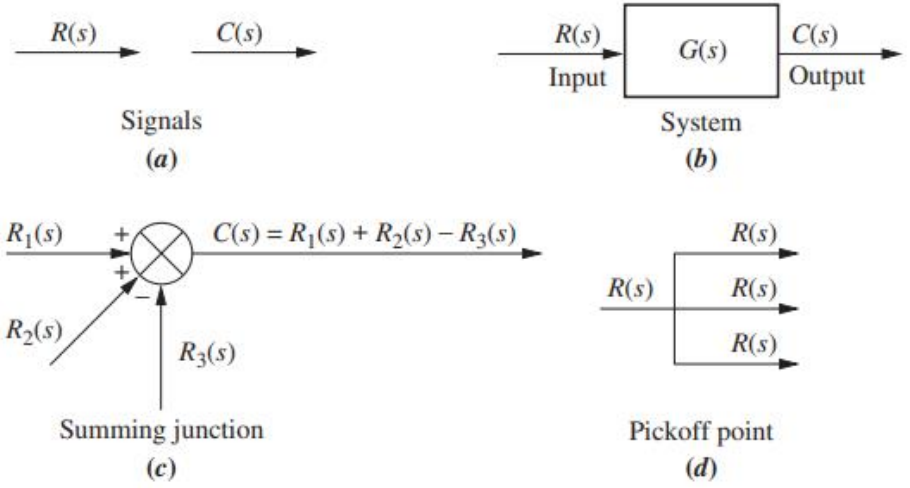
$$G_1(s) = \frac{V_1(s)}{V_i(s)} = \frac{\frac{1}{R_1C_1}}{s + \frac{1}{R_1C_1}}$$

$$G_2(s) = \frac{V_2(s)}{V_1(s)} = \frac{\frac{1}{R_2C_2}}{s + \frac{1}{R_2C_2}}$$

$$G(s) = \frac{V_2(s)}{V_i(s)} = \frac{\frac{1}{R_1C_1R_2C_2}}{s^2 + \left(\frac{1}{R_1C_1} + \frac{1}{R_2C_2} + \frac{1}{R_2C_1}\right)s + \frac{1}{R_1C_1R_2C_2}}$$

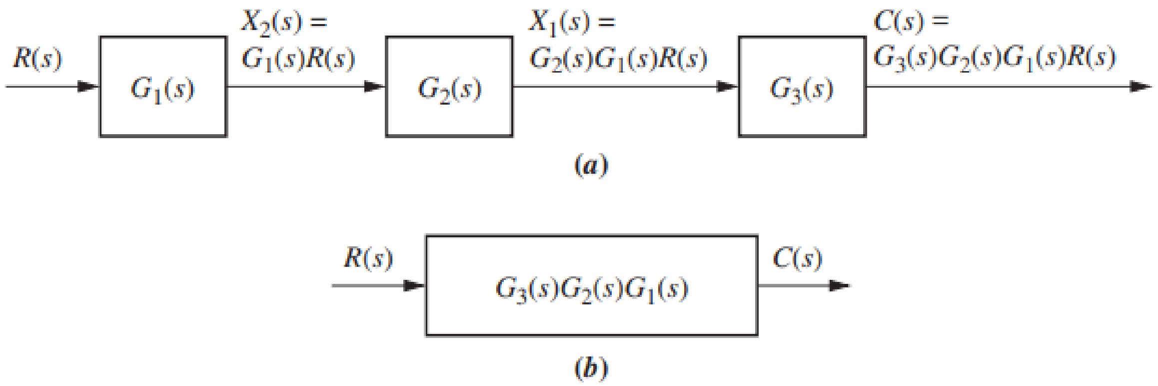
Block Diagrams:

The basic input vocabulary components of the block diagrams are shown in figure, with the signal represented by an arrow, system transfer relation (gain) represented by a block and signal combination and extraction represented by sum and derivation nodes.

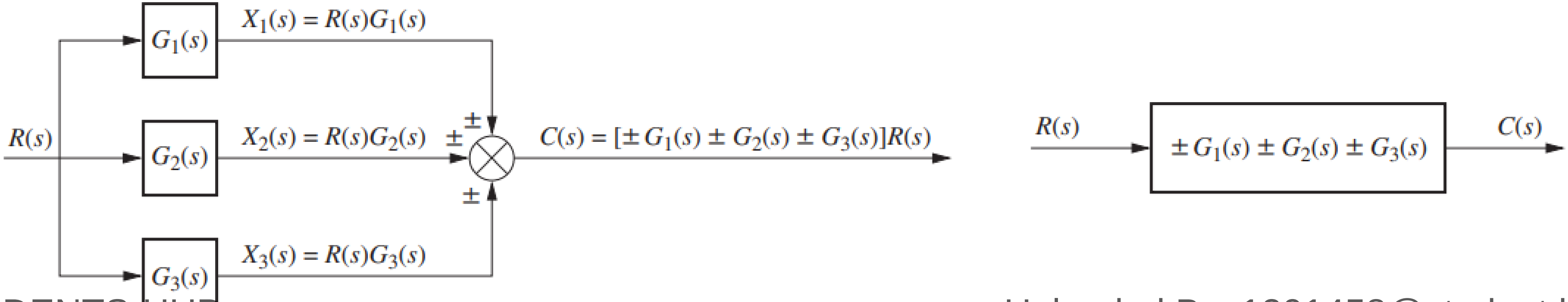


Block Algebra:

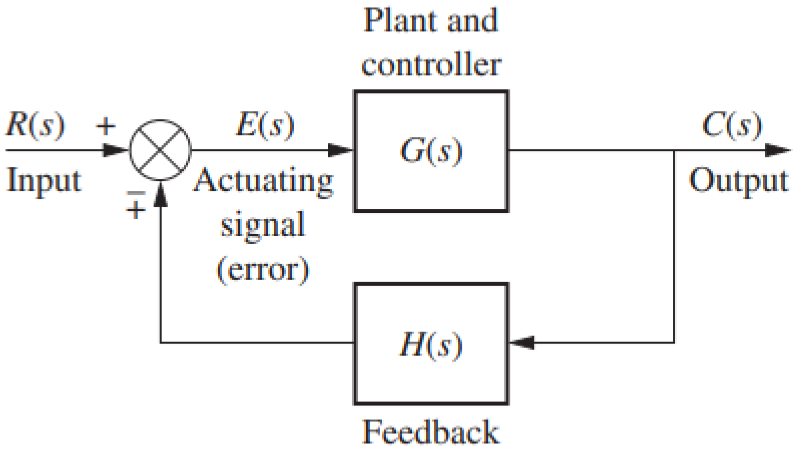
- Cascade connection



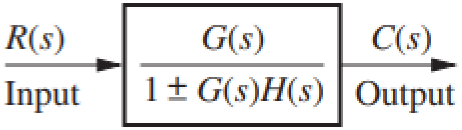
- Parallel connection



- Feedback connection



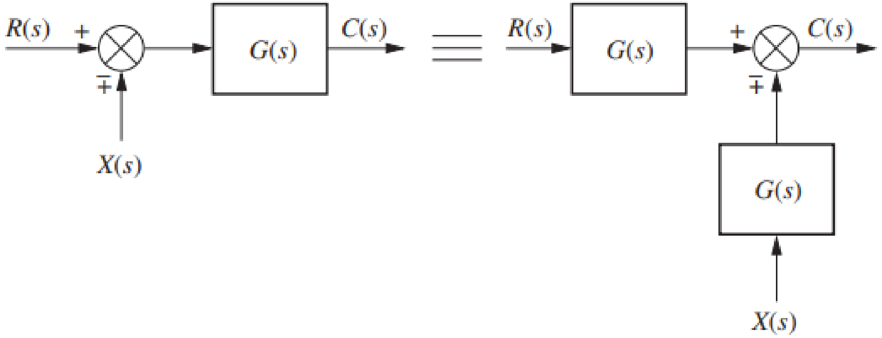
$$C(s) = G(s)E(s)$$
$$E(s) = R(s) - H(s)C(s) = R(s) - H(s)G(s)E(s) \rightarrow$$
$$E(s)\big(1 + H(s)G(s)\big) = R(s) \rightarrow E(s) = \frac{1}{1 + H(s)G(s)}R(s) \rightarrow$$
$$C(s) = \frac{G(s)}{1 + H(s)G(s)}R(s)$$



<u>Moving Blocks to Create Familiar Forms:</u>

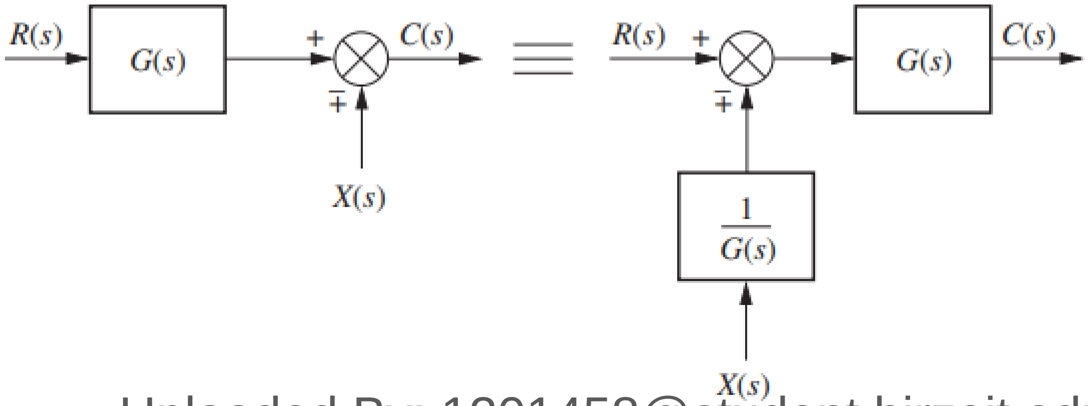
- Transfer of a sum node from the input to the output of a block:

$$C(s) = G(s)[R(s) + X(s)] = G(s)R(s) + G(s)X(s)$$

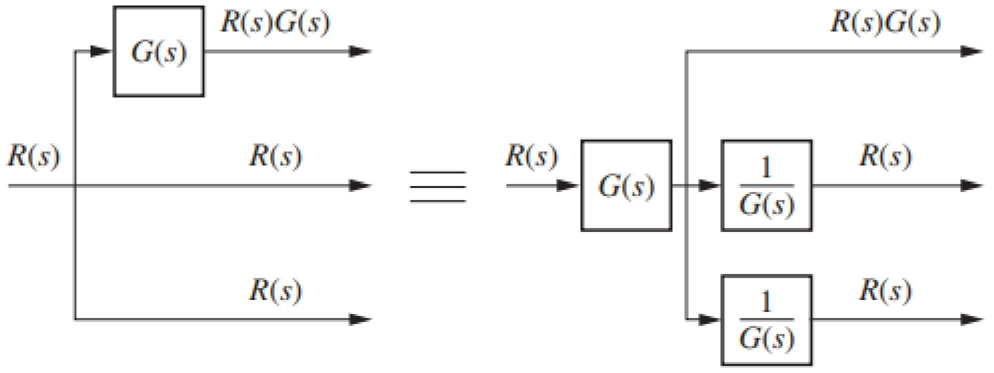


- Transfer of a sum node from the output to the input of a block:

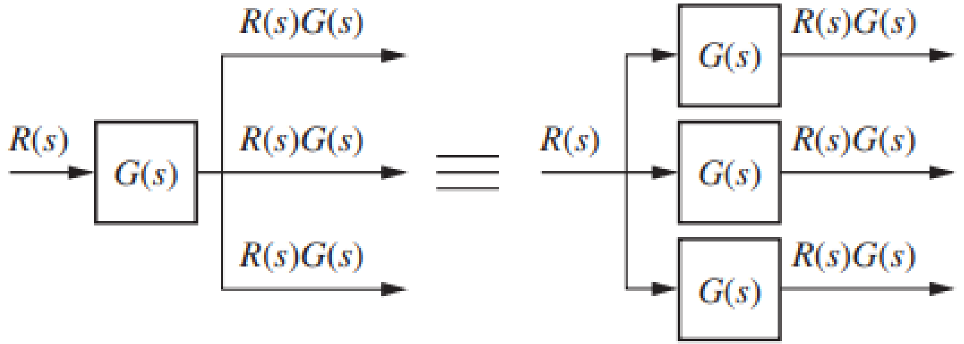
$$C(s) = G(s)R(s) + X(s) = G(s)[R(s) + \frac{1}{G(s)}X(s)]$$



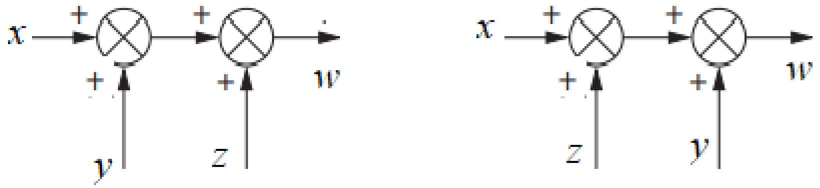
- Transfer of a derivation point from the input to the output of a block:



- Transfer of a derivation point from the output to the input of a block:



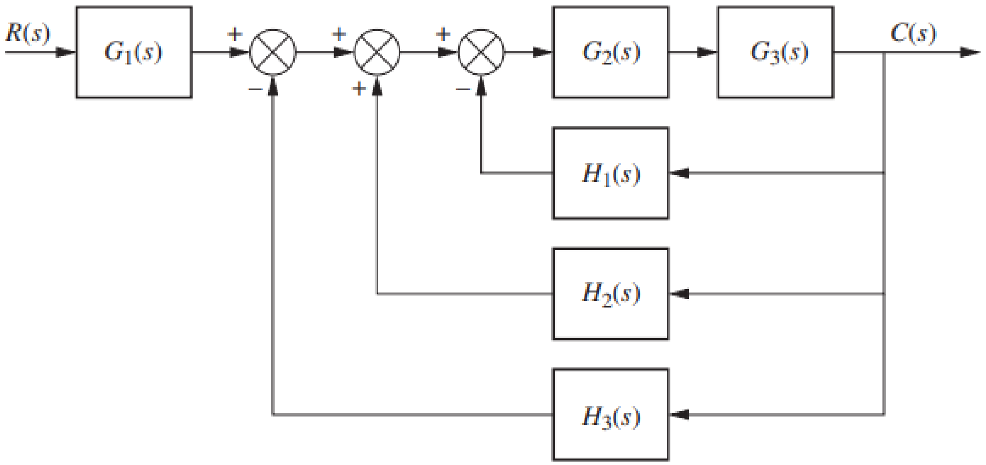
- Signal-node switching:

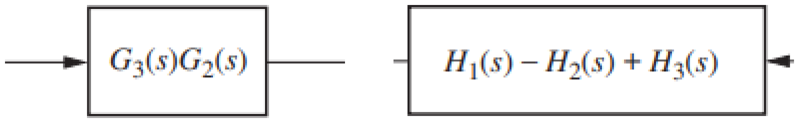


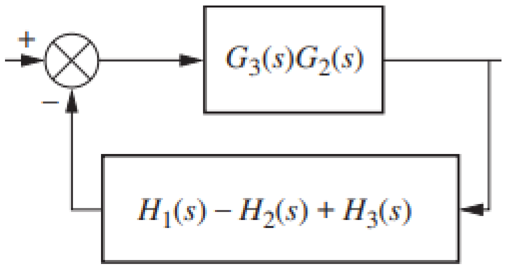
Example1: Reduce the block diagram shown in Figure to a single transfer function.

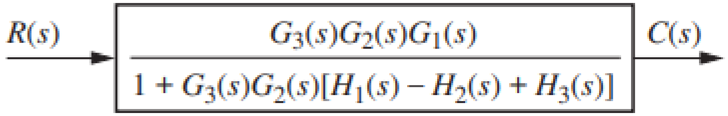
Reduction Steps:

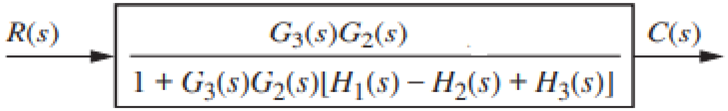
1. $Cascade(G_3, G_2) = T_1$
2. $Parallel\ (H_1, H_2, H_3) = T_2$
3. $feedback(T_1, T_2) = T_3$
4. $Cascade(T_3, G_1) = T$



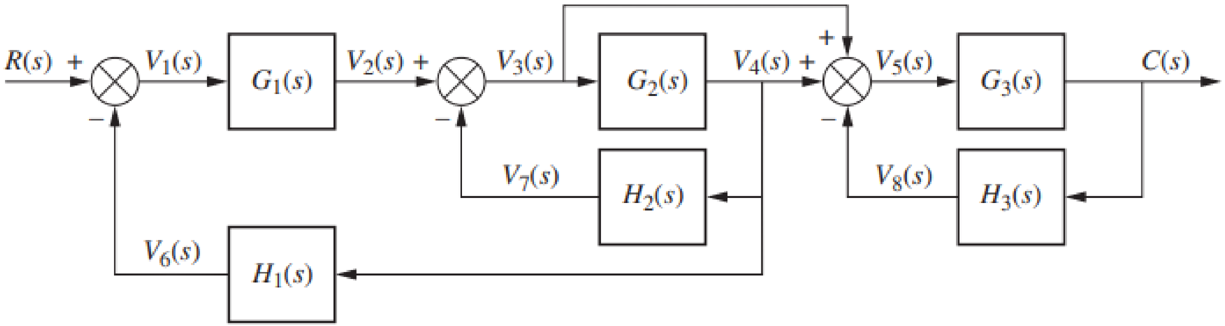


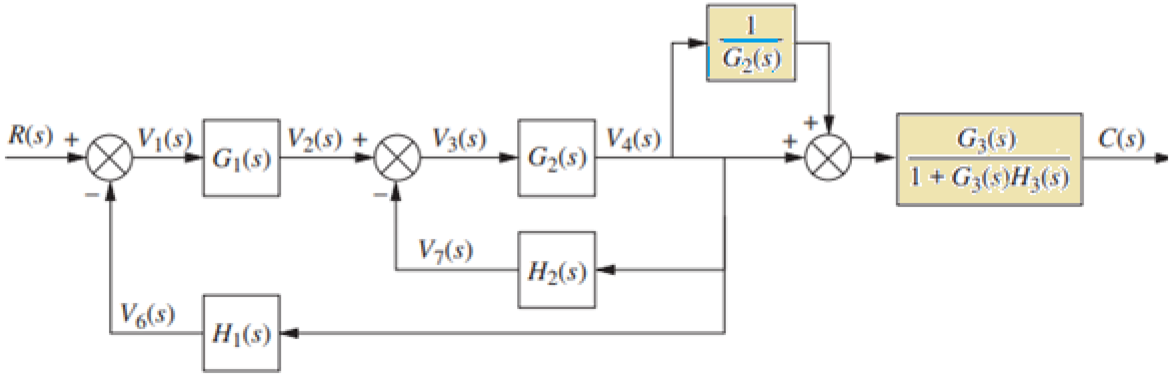






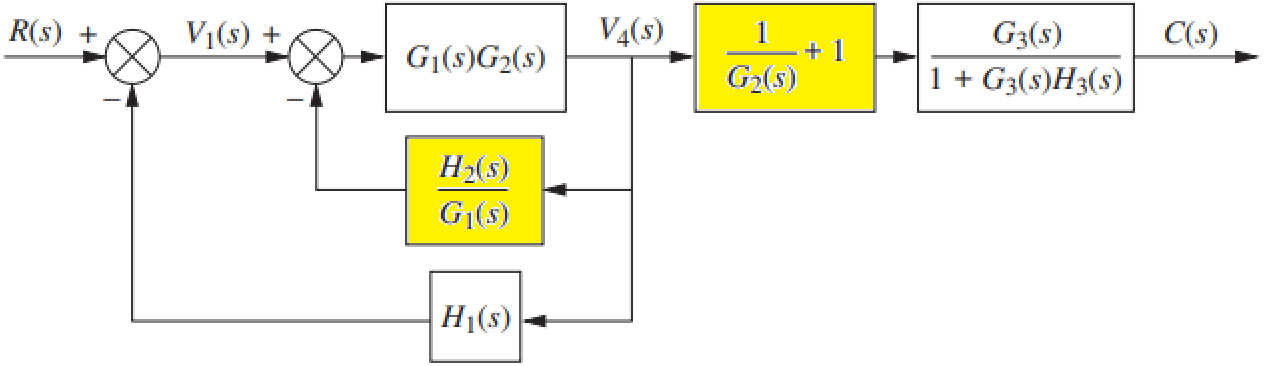
Example2: Reduce the block diagram shown in Figure to a single transfer function.

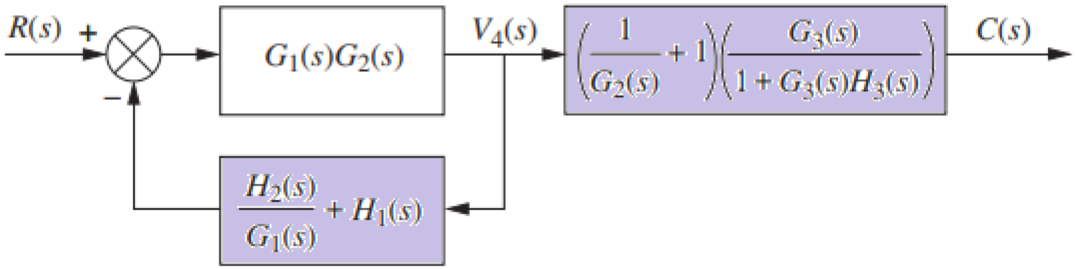


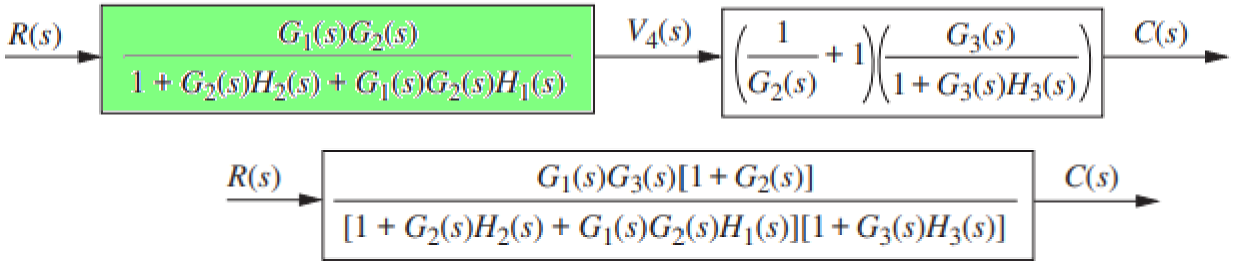


Reduction Steps:

- *Transfer the derivation point from the*
- *input of G_2 to its output and apply feedback(G_3, H_3)*
- *Transfer the sum node from the output of G_1 to its output + parallel($\frac{1}{G_2(s)}$, 1)*
- *cascade and parallel*
- *feedback and then cascade*

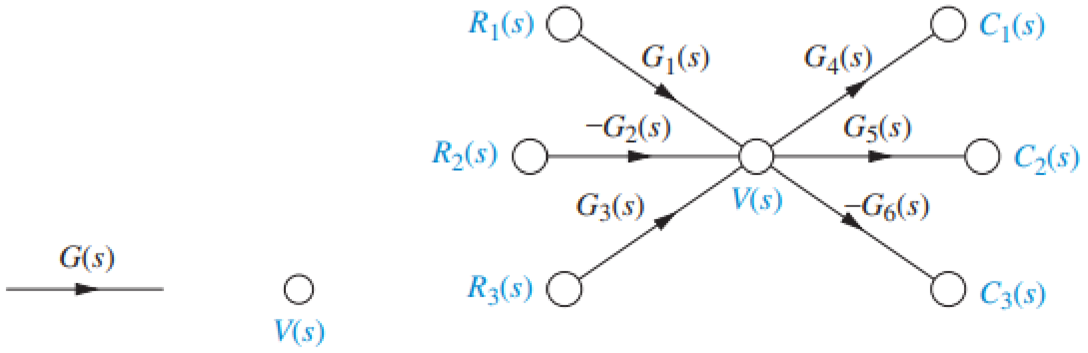


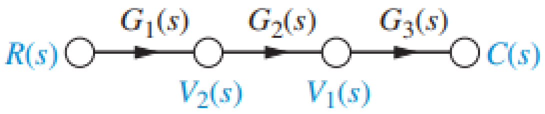


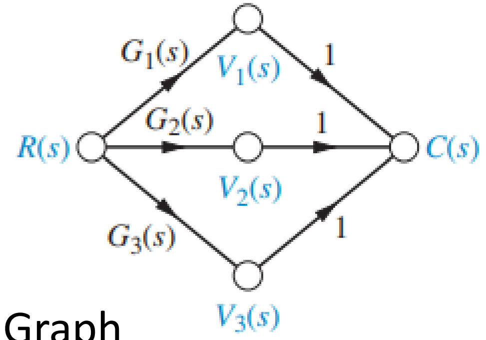


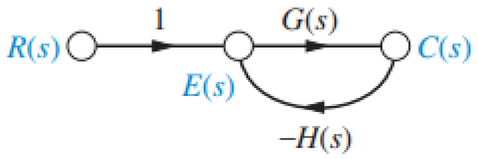
Signal-Flow diagrams:

The basic input vocabulary of the signal flow diagrams are shown in figure, with the signal represented by a node, system transfer relation (gain) represented by an arrow, and signal combination and extraction represented by rows converging in a node and rows diverging from it.

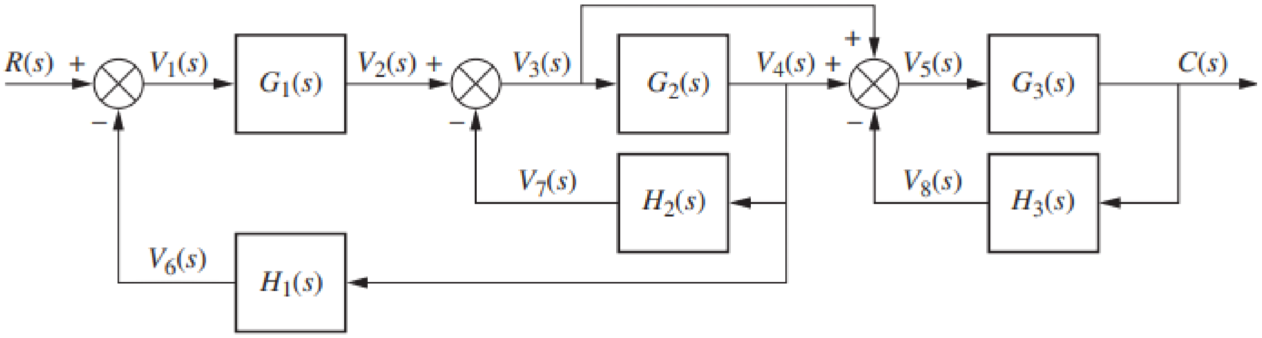


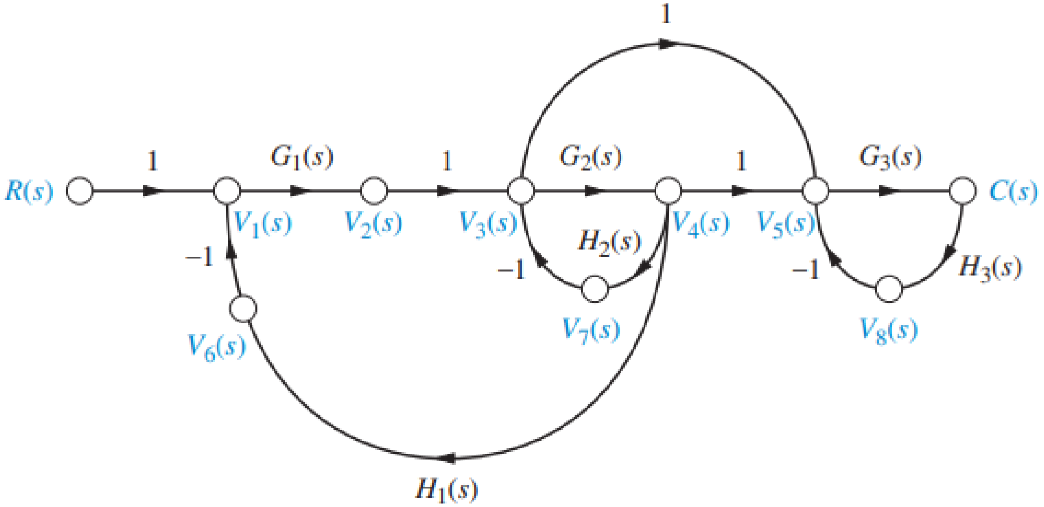






Converting a Block Diagram to a Signal-Flow Graph





Mason's Rule:

We define the following:

- Pathi,j : the sequence of branching that connects node i and node j without going through any node more than one time.
- Path-gain: the product of the gains of all the branches of the path.
- Loop: a closed path.
- Loop gain: its relative path gain.
- Nontouching loops: Loops that do not have any nodes and branches in common. Nontouching loops are inspected as two, three, four, or more at a time.
- Nontouching loops gains: the product of nontouching loops taken as two, three, four, or more at a time.
- Loops and nontouching loops with a path_{ij}: the loops and nontoucing loops that do not have any nodes or branches in common.
- Loops and nontouching loops with a path_{ij} gains: the product of the gains of the Loops and nontouching loops with the path_{ij} .

Masons Formula:

$$\boxed{G(s) = \frac{C(s)}{R(s)} = \frac{\sum_k T_k \Delta_k}{\Delta}}$$

where

k = number of forward paths

T_k = the kth forward-path gain

Δ = 1 − Σ loop gains + Σ nontouching-loop gains taken two at a time − Σ nontouching-loop gains taken three at a time + Σ nontouching-loop gains taken four at a time − . . .

Δ_k = Δ − Σ loop gain terms in Δ that touch the kth forward path. In other words, Δ_k is formed by eliminating from Δ those loop gains that touch the kth forward path.

Example 1: determine the transfer function of the following system using Masons rule.

Solution:

graph elements gains will be written directly.

Path gains:

$P_1 = G_1G_2G_3G_4G_5G_7$

$P_2 = G_1G_2G_3G_4G_6G_7$

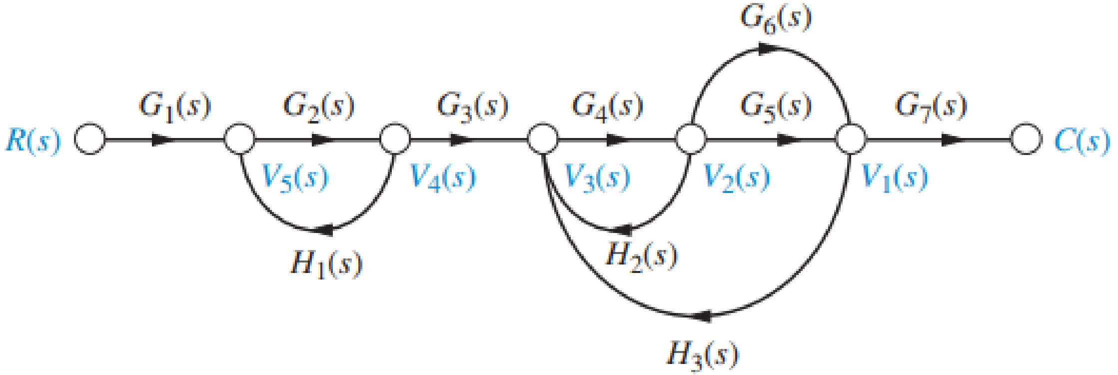
Loop gains:

$L_1 = H_1G_2$

$L_2 = H_2G_4$

$L_3 = H_3G_4G_5$

$L_4 = H_3G_4G_6$



Nontouching Loops gains (2-2):

$L_{12} = H_1H_2G_2G_4$

$L_{13} = H_1H_3G_2G_4G_5$

$L_{14} = H_1H_3G_2G_4G_6$

Nontouching Loops gains (3-3):

Do not exist

Nontouching Loops with P_1 gains:

Do not exist

Nontouching Loops with P_2 gains:

Do not exist

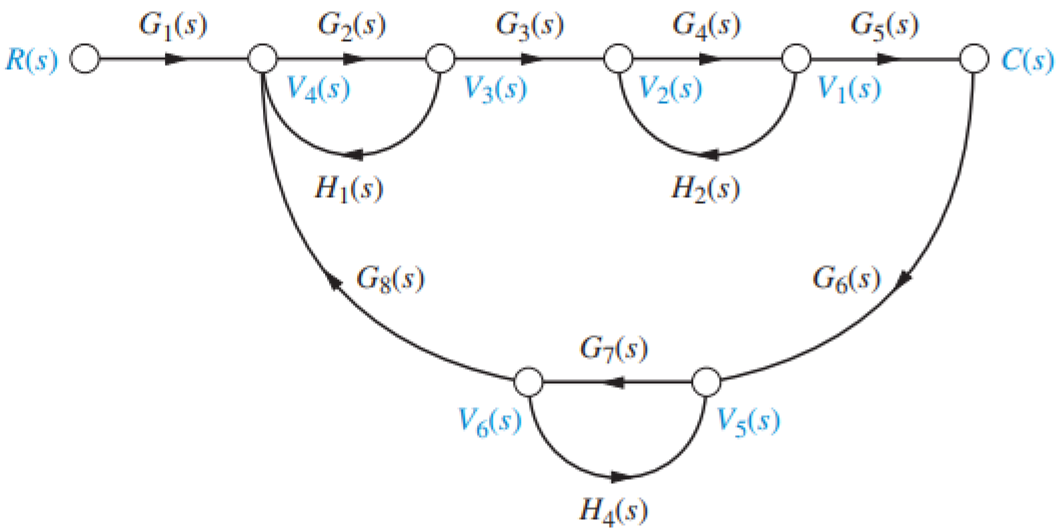
Computation:

$\Delta = 1 - (H_1G_2 + H_2G_4 + H_3G_4G_5 + H_3G_4G_6) + (H_1H_2G_2G_4 + H_1H_3G_2G_4G_5 + H_1H_3G_2G_4G_6)$

$\Delta_{P_1} = 1, \Delta_{P_2} = 1$

$$T(s) = \frac{C(s)}{R(s)} = \frac{G_1G_2G_3G_4G_7(G_5 + G_6)}{1 - (H_1G_2 + H_2G_4 + H_3G_4G_5) + (H_1H_2G_2G_4 + H_1H_3G_2G_4G_5)}$$

Example 2: determine the transfer function of the following system using Masons rule.



Solution:

graph elements gains will be written directly.

Path gains:

$P_1 = G_1G_2G_3G_4G_5$

Loop gains:

$L_1 = H_1G_2$

$L_2 = H_2G_4$

$L_3 = H_4G_7$

$L_4 = G_2G_3G_4G_5G_6G_7G_8$

Nontouching Loops gains (2-2):

$L_{12} = H_1H_2G_2G_4$

$L_{13} = H_1H_4G_2G_7$

$L_{23} = H_2H_4G_4G_7$

Nontouching Loops gains (3-3):

$L_{123} = H_1H_2H_4G_2G_4G_7$

Nontouching Loops with P_1 gains:

$L_{P_{1-3}} = H_4G_7$

Computation:

$\Delta = 1 - (H_1G_2 + H_2G_4 + H_4G_7 + G_2G_3G_4G_5G_6G_7G_8) + (H_1H_2G_2G_4 + H_1H_4G_2G_7 + H_2H_4G_4G_7) - (H_1H_2H_4G_2G_4G_7)$, $\quad \Delta_{P_1} = 1 - H_4G_7$

$$T(s) = \frac{C(s)}{R(s)} = \frac{G_1G_2G_3G_4G_5 \cdot (1 - H_4G_7)}{1 - (H_1G_2 + H_2G_4 + H_4G_7 + G_2G_3G_4G_5G_6G_7G_8) + (H_1H_2G_2G_4 + H_1H_4G_2G_7 + H_2H_4G_4G_7) - (H_1H_2H_4G_2G_4G_7)}$$

State Space Representation

Dr. Jamal Siam

State Space Representation:

- It is an internal system time-domain representation composed of a set of simultaneous first-order differential equations that describes the evolution of the internal state variables (memory elements variables or other related variables) and a second set of algebraic equations that set the relation between the input and the state.

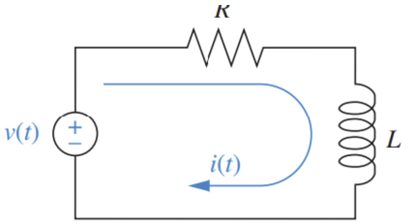
$$\dot{x} = f(x,u)$$
$$y = g(x,u)$$

- The number of independent state equations is equal to the order of the system.
- The natural selection of the independent state variables is the energy variable of the conservative elements.
- The state equation includes only state variables and input excitation.
- For a linear type invariant system of order n with m inputs and d outputs, the state equations representation is formulated as follow:

$$\begin{cases} \dot{x} = A_{n\times n}x_{n\times 1} + B_{n\times m}u_{m\times 1} \\ y_{d\times 1} = C_{d\times n}x + D_{d\times m}u \end{cases}$$

x: state vector, A: state Space matrix, B: state−input matrix, C: output−ste matrix, D:output−input Matrix

Example:

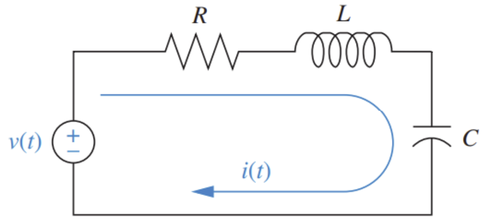


- The system is first order system, thus we need one state variable. The output is $v_R(t)$.
- Select the mesh current which is equal to the inductor current as state variable.
- The energy element equation is $v_L(t) = L\frac{di_L(t)}{dt}$ which is not a state equation because $v_L(t)$ is not a state variable an has to be eliminated.
- Applying KVL and the resistor characteristic equation we obtain $v_L(t) = v(t) - Ri(t)$
- Substituting in the energy equation, we obtain $\frac{di(t)}{dt} = \frac{1}{L}v(t) - \frac{R}{L}i(t)$ ....state equation

$v_R(t) = Ri(t)$ ......output equation

Example2

- The system of a second-order system, thus we need two independent state variables.
- The natural selection of state variables is $i_L(t)$ and $V_c(t)$.
- Assume the output variable is $V_c(t)$. The output equation becomes $\mathrm{y(t)} = V_c(t)$
- Solution:
- The energy equations are $v_L(t) = L\dfrac{di_L(t)}{dt}$ and $i_c(t) = c\dfrac{dv_c(t)}{dt}$ which are both not state equations.
- From the node equations $i_c(t) = i_L(t)$ →the first state equation: $\dfrac{dv_c(t)}{dt} = \dfrac{1}{C}i_L(t)$
- From the KVL and the resistor characteristic equation: $v_L(t) = v(t) - Ri_L(t) - v_c(t)$
- Applying in the inductor characteristic equation and ordering we obtain:

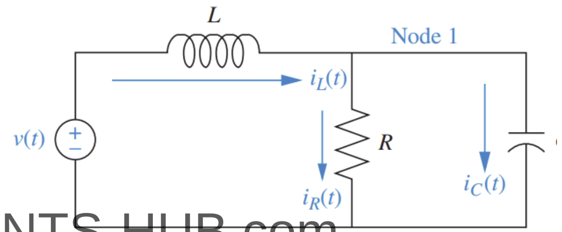


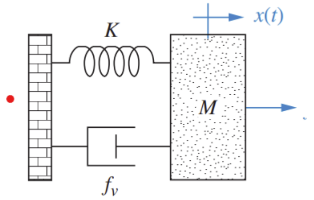
$$\frac{di_L(t)}{dt} = \frac{1}{L}v(t) - \frac{R}{L}i_L(t) - \frac{1}{L}v_c(t)$$

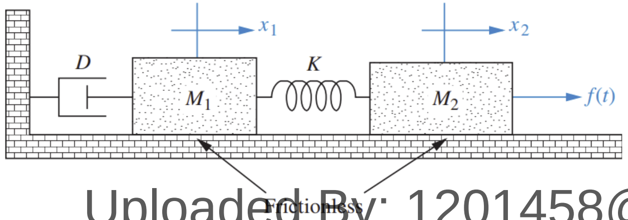
In matrix form
$$\begin{bmatrix} \dot{v}_C \\ \dot{i}_l \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} V_c(t) \\ i_L(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} v_c(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} V_c(t) \\ i_L(t) \end{bmatrix}$$

Exercise: Write the state equations of the following systems in algebraic matrix form. Outputs: $v_L(t), x, x_1 and x_2$, respectively.







Transforming Internal representation to external representation (unique form)

- State space representation → Transfer Matrix/System of differential equations.
- For a SISO system: State space representation → Transfer function/ differential equations.

Given the state and output equations

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{3.68a}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \tag{3.68b}$$

take the Laplace transform assuming zero initial conditions:[8]

$$s\mathbf{X}(s) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s) \tag{3.69a}$$

$$\mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s) \tag{3.69b}$$

Solving for $\mathbf{X}(s)$ in Eq. (3.69a),

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{B}\mathbf{U}(s) \tag{3.70}$$

or

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) \tag{3.71}$$

where $\mathbf{I}$ is the identity matrix.

Substituting Eq. (3.71) into Eq. (3.69b) yields

$$\mathbf{Y}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) + \mathbf{D}\mathbf{U}(s) = \left[\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}\right]\mathbf{U}(s) \tag{3.72}$$

We call the matrix $\left[\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}\right]$ the transfer function matrix, since it relates the output vector, $\mathbf{Y}(s)$, to the input vector, $\mathbf{U}(s)$. However, if $\mathbf{U}(s) = U(s)$ and $\mathbf{Y}(s) = Y(s)$ are scalars, we can find the transfer function, Thus,

$$\boxed{T(s) = \frac{Y(s)}{U(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}}$$

Example:

Determine the transfer function of the system defined by the following state space representation.

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\mathbf{x}$$

Solution:

$$(s\mathbf{I}-\mathbf{A}) = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 2 & s+3 \end{bmatrix} \rightarrow (s\mathbf{I}-\mathbf{A})^{-1} = \frac{\operatorname{adj}(s\mathbf{I}-\mathbf{A})}{\det(s\mathbf{I}-\mathbf{A})} = \frac{\begin{bmatrix} (s^2+3s+2) & s+3 & 1 \\ -1 & s(s+3) & s \\ -s & -(2s+1) & s^2 \end{bmatrix}}{s^3+3s^2+2s+1}$$

$$\mathbf{B} = \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} \quad \rightarrow \quad \boxed{T(s) = \frac{Y(s)}{U(s)} = \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B} + \mathbf{D}} \quad \rightarrow \quad T(s) = \frac{10(s^2+3s+2)}{s^3+3s^2+2s+1}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

$$\mathbf{D} = 0$$

Exercise: determine the transfer function of the system represented by the following state space representation

$$\dot{\mathbf{x}} = \begin{bmatrix} -4 & -1.5 \\ 4 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u(t)$$

$$y = \begin{bmatrix} 1.5 & 0.625 \end{bmatrix}\mathbf{x}$$

Converting external representation to internal representation:

differential equation/ transfer function $\rightarrow$ state space representation (not unique)

Phase-variable state space representation:

Consider the following differential equation and the following variable assignment:

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_1\frac{dy}{dt} + a_0 y = b_0 u$$

$$\begin{aligned} x_1 &= y \\ x_2 &= \frac{dy}{dt} \\ x_3 &= \frac{d^2y}{dt^2} \\ &\vdots \\ x_n &= \frac{d^{n-1}y}{dt^{n-1}} \end{aligned} \quad \rightarrow \quad \begin{aligned} \dot{x}_1 &= \frac{dy}{dt} \\ \dot{x}_2 &= \frac{d^2y}{dt^2} \\ \dot{x}_3 &= \frac{d^3y}{dt^3} \\ &\vdots \\ \dot{x}_n &= \frac{d^ny}{dt^n} \end{aligned} \quad \rightarrow \quad \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= -a_0x_1 - a_1x_2 \cdots - a_{n-1}x_n + b_0u \end{aligned}$$

In matrix form-The state matrix is called companion matrix because it includes the coefficient of the transfer equation :

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ \vdots & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ b_0 \end{bmatrix} u \qquad y = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

Example1: Consider the following transfer and

- determine the system differential equation and the phase variable representation.
- Plot the block diagram of the system

$$\frac{C(s)}{R(s)} = \frac{24}{(s^3 + 9s^2 + 26s + 24)}$$

Solution:

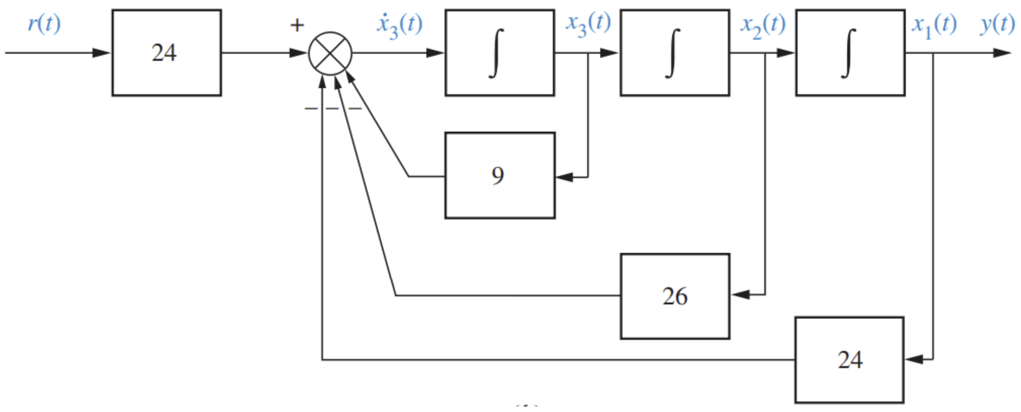
$$(s^3 + 9s^2 + 26s + 24)C(s) = 24R(s) \quad \rightarrow \quad \dddot{c} + 9\ddot{c} + 26\dot{c} + 24c = 24r$$

State variable assignment $\rightarrow$

$$\begin{aligned} x_1 &= c \\ x_2 &= \dot{c} \\ x_3 &= \ddot{c} \end{aligned} \quad \rightarrow \quad \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -24x_1 - 26x_2 - 9x_3 + 24r \\ y &= c = x_1 \end{aligned} \quad \rightarrow \quad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 24 \end{bmatrix} r$$

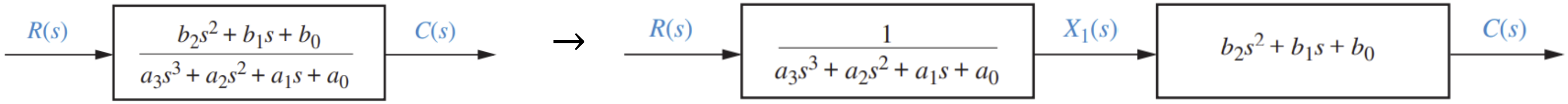
$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Block Diagram:

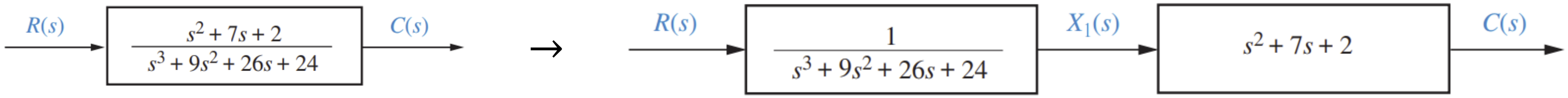


Example2: Transfer function with polynomial numerator

Determine the state space representation of the following system and plot the corresponding block diagram



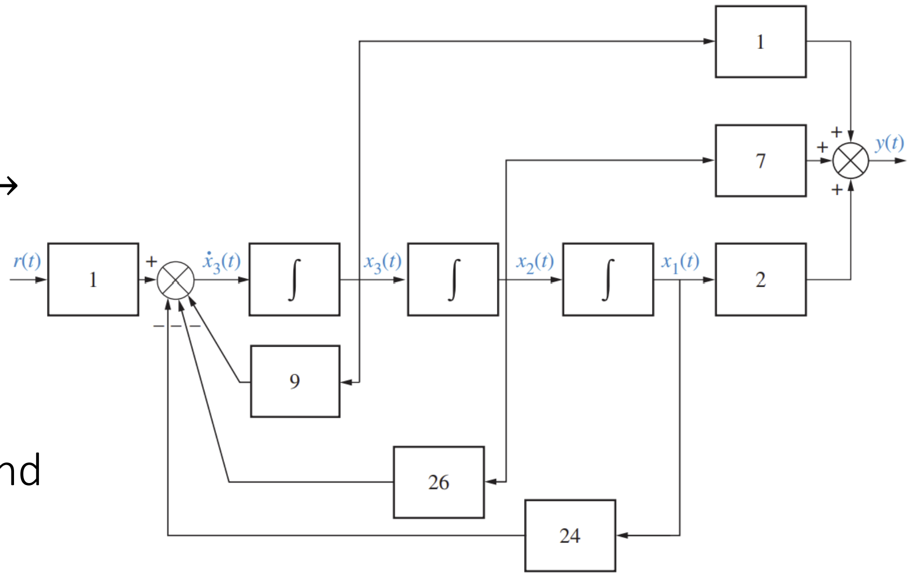
$$y(t) = b_0x_1 + b_1x_2 + b_2x_3$$



$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -24 & -26 & -9\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 1\end{bmatrix} r$$

$$C(s) = (b_2s^2 + b_1s + b_0)X_1(s) = (s^2 + 7s + 2)X_1(s) \rightarrow c = \ddot{x}_1 + 7\dot{x}_1 + 2x_1$$ using $$\begin{aligned} x_1 &= x_1 \\ \dot{x}_1 &= x_2 \\ \ddot{x}_1 &= x_3 \end{aligned}$$ →

$$y = \begin{bmatrix} b_0 & b_1 & b_2 \end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix} 2 & 7 & 1 \end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$



Exercise: Determine the state space representation of the following system and plot the representation block diagram

$$G(s) = \frac{2s+1}{s^2+7s+9}.$$

Alternative Representations in State Space:

Controller Canonical Form:(a variant of the phase variable representation with companion matrix)

$$G(s)=\frac{C(s)}{R(s)}=\frac{s^2+7s+2}{s^3+9s^2+26s+24}$$

Phase variable:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -24 & -26 & -9\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}$$

$$y=\begin{bmatrix}2 & 7 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

The controller canonical representation is obtained by changing the numbers of the variables and reordering the equations

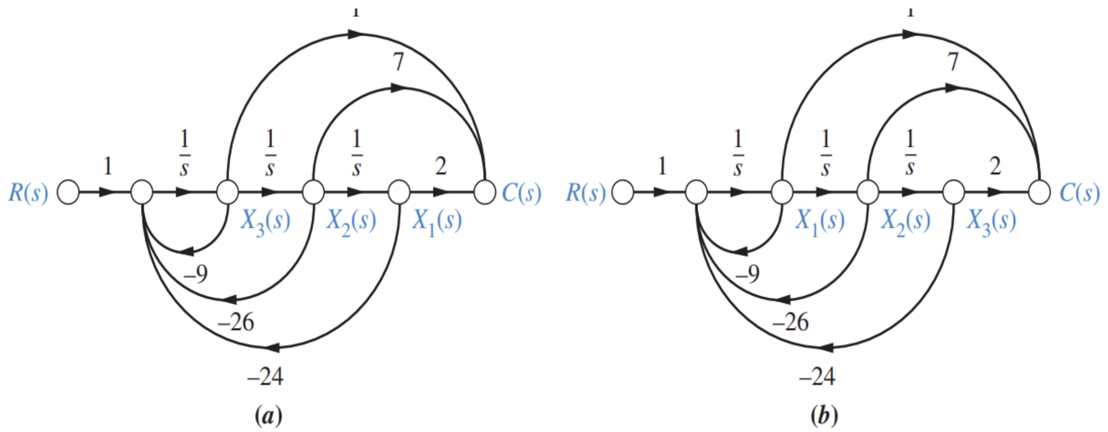
$$\begin{bmatrix}\dot{x}_3\\ \dot{x}_2\\ \dot{x}_1\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -24 & -26 & -9\end{bmatrix}\begin{bmatrix}x_3\\ x_2\\ x_1\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}r$$

$$y=\begin{bmatrix}2 & 7 & 1\end{bmatrix}\begin{bmatrix}x_3\\ x_2\\ x_1\end{bmatrix}$$

$\rightarrow$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}-9 & -26 & -24\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}r$$

$$y=\begin{bmatrix}1 & 7 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$



a. phase-variable form; **b.** controller canonical form

Observer Canonical Form:

The transfer function/differential equation are written in integral form which is then written as a sequence of integration and variables are assigned accordingly.

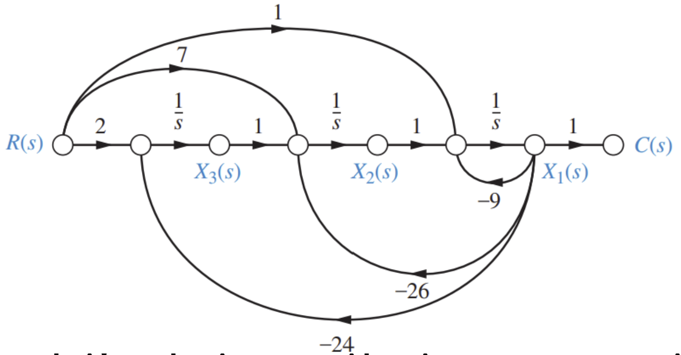
Example:

$$G(s)=\frac{C(s)}{R(s)}=\frac{s^2+7s+2}{s^3+9s^2+26s+24} \quad\rightarrow\quad \frac{C(s)}{R(s)}=\frac{\frac{1}{s}+\frac{7}{s^2}+\frac{2}{s^3}}{1+\frac{9}{s}+\frac{26}{s^2}+\frac{24}{s^3}} \quad\rightarrow\quad \left[\frac{1}{s}+\frac{7}{s^2}+\frac{2}{s^3}\right]R(s)=\left[1+\frac{9}{s}+\frac{26}{s^2}+\frac{24}{s^3}\right]C(s) \quad\rightarrow$$

$$C(s)=\frac{1}{s}[R(s)-9C(s)]+\frac{1}{s^2}[7R(s)-26C(s)]+\frac{1}{s^3}[2R(s)-24C(s)] \quad\rightarrow\quad C(s)=\frac{1}{s}\left[[R(s)-9C(s)]+\frac{1}{s}\left([7R(s)-26C(s)]+\frac{1}{s}[2R(s)-24C(s)]\right)\right]$$

$$\begin{aligned}\dot{x}_1 &= -9x_1+x_2 && +r\\ \dot{x}_2 &= -26x_1 && +x_3+7r\\ \dot{x}_3 &= -24x_1 && +2r\\ y &= c(t)=x_1\end{aligned} \quad\rightarrow\quad \dot{\mathbf{x}}=\begin{bmatrix}-9&1&0\\-26&0&1\\-24&0&0\end{bmatrix}\mathbf{x}+\begin{bmatrix}1\\7\\2\end{bmatrix}r$$

$$y=[1\quad 0\quad 0]\,\mathbf{x}$$



Controller-Observer Duality:

The controller representation of the same system is given by:

Observe the duality relation between the two representations:

$$\mathbf{A_D}=\mathbf{A}^T,\ \mathbf{B_D}=\mathbf{C}^T,\ \mathbf{C_D}=\mathbf{B}^T.$$

$$\begin{bmatrix}\dot{x}_1\\\dot{x}_2\\\dot{x}_3\end{bmatrix}=\begin{bmatrix}-9&-26&-24\\1&0&0\\0&1&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}+\begin{bmatrix}1\\0\\0\end{bmatrix}r$$

$$y=\begin{bmatrix}1&7&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

Exercise: Determine the phase-variable, controller, and observer representation of the following system represented by state space and plot the signal flow diagrams

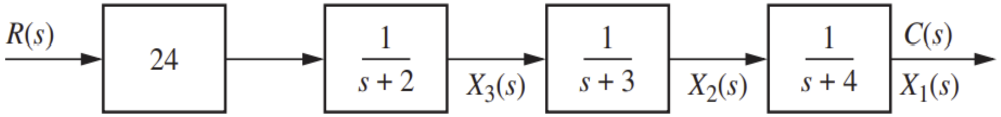
Hint: convert the state space representation to the transfer function

$$\dot{\mathbf{x}}=\begin{bmatrix}-105&-506\\1&0\end{bmatrix}\mathbf{x}+\begin{bmatrix}1\\0\end{bmatrix}r \qquad y=[100\quad 500]\,\mathbf{x}$$

Cascade representation(for transfer functions with simple roots(Triangular Matrix Form):

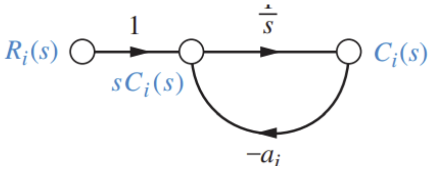
The Transfer function is written as the product of its basic-first-order terms and cascaded with the numerator term.

$$\frac{C(s)}{R(s)} = \frac{24}{(s+2)(s+3)(s+4)} \quad \rightarrow$$

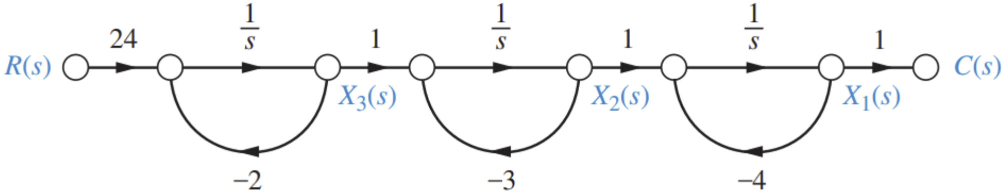


<u>Representation of the general first-order term</u>

$$\frac{C_i(s)}{R_i(s)} = \frac{1}{(s+a_i)} \rightarrow (s+a_i)C_i(s) = R_i(s) \rightarrow \frac{dc_i(t)}{dt} = -a_i c_i(t) + r_i(t) \rightarrow$$



The system can be represented using this representation as:



Writing the equations of each block we obtain:

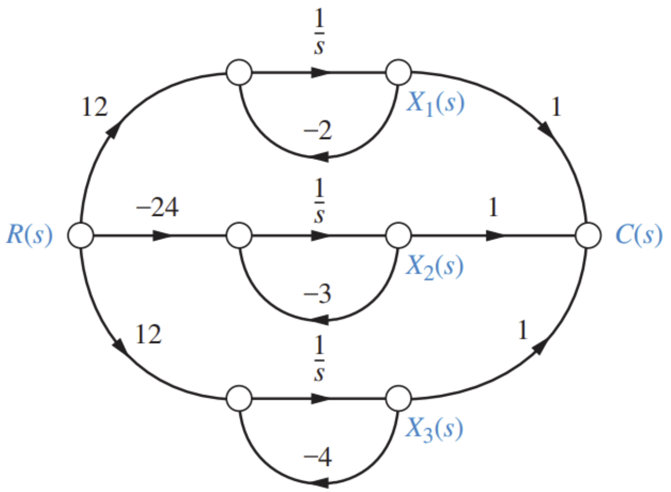
$$\begin{aligned} \dot{x}_1 &= -4x_1 + x_2 \\ \dot{x}_2 &= \quad -3x_2 + x_3 \\ \dot{x}_3 &= \quad\quad -2x_3 + 24r \end{aligned} \quad \rightarrow \quad \dot{\mathbf{x}} = \begin{bmatrix} -4 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -2 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 24 \end{bmatrix} r$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\mathbf{x}$$

Parallel representation: systems with simple roots written in the form of the partial fraction(Diagonal Matrix)

$$\frac{C(s)}{R(s)} = \frac{24}{(s+2)(s+3)(s+4)} = \frac{12}{(s+2)} - \frac{24}{(s+3)} + \frac{12}{(s+4)} \rightarrow C(s) = R(s)\frac{12}{(s+2)} - R(s)\frac{24}{(s+3)} + R(s)\frac{12}{(s+4)}$$

Using the general first-order representation we can obtain the parallel plot in the figure. Writing the equation of each block we obtain:



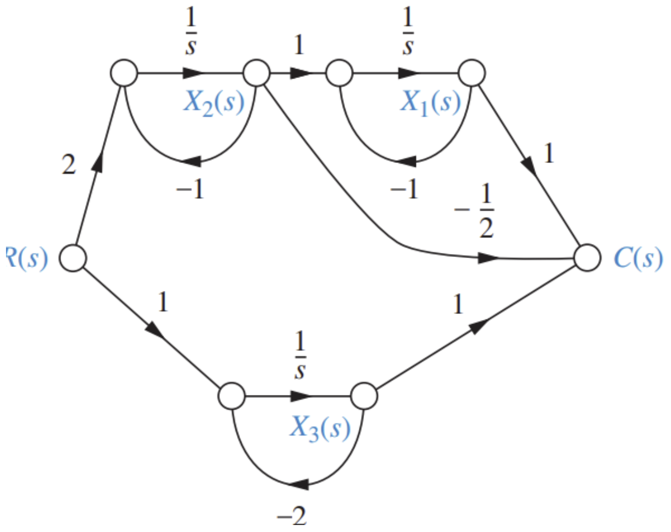
$$\begin{aligned} \dot{x}_1 &= -2x_1 \qquad\qquad\qquad +12r \\ \dot{x}_2 &= \qquad -3x_2 \qquad\quad -24r \\ \dot{x}_3 &= \qquad\qquad -4x_3 \ +12r \end{aligned} \rightarrow \dot{\mathbf{x}} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 12 \\ -24 \\ 12 \end{bmatrix} r$$

$$y = c(t) = x_1 + x_2 + x_3 \qquad y = [1 \quad 1 \quad 1]\mathbf{x}$$

Mixed Parallel-Cascade representation: partial fractions with repeated roots(Jordan Matrix):

$$\frac{C(s)}{R(s)} = \frac{(s+3)}{(s+1)^2(s+2)} \rightarrow \frac{C(s)}{R(s)} = \frac{2}{(s+1)^2} - \frac{1}{(s+1)} + \frac{1}{(s+2)}$$

Plotting using the first-order cell and reading the equation we obtain:



$$\begin{aligned} \dot{x}_1 &= -x_1 \quad + x_2 \\ \dot{x}_2 &= \qquad - x_2 \qquad + 2r \\ \dot{x}_3 &= \qquad\qquad -2x_3 + r \end{aligned} \rightarrow \dot{\mathbf{x}} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} r$$

$$y = c(t) = x_1 - \frac{1}{2}x_2 + x_3 \qquad y = \begin{bmatrix} 1 & -\frac{1}{2} & 1 \end{bmatrix}\mathbf{x}$$