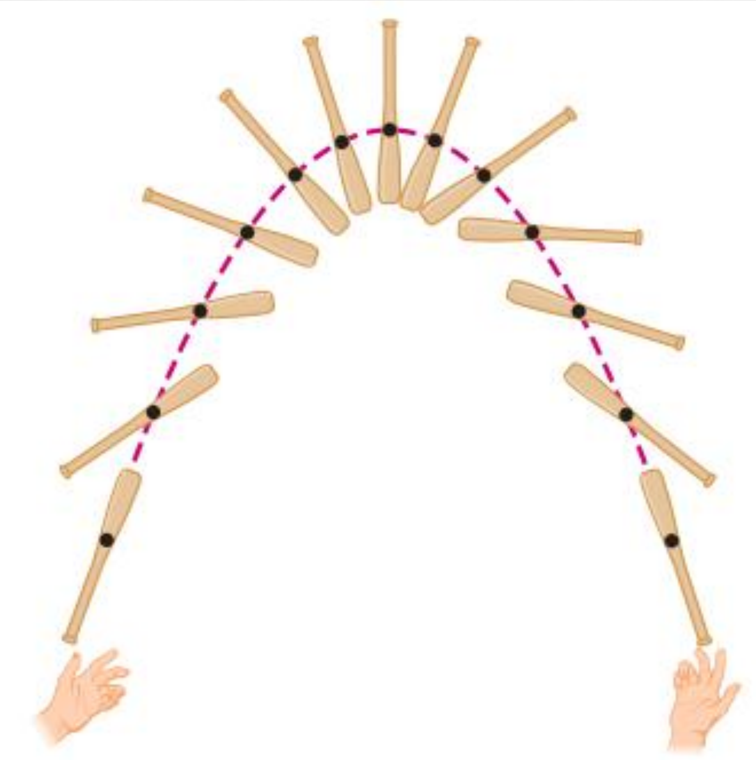
Chapter 9

Center of Mass and Linear Momentum

9-1 Center of Mass

- The motion of rotating objects can be complicated (imagine flipping a baseball bat into the air)
- But there is a special point on the object for which the motion is simple
- The center of mass of the bat traces out a parabola, just as a tossed ball does
- All other points rotate around this point



- The **center of mass (com)** of a system of particles:

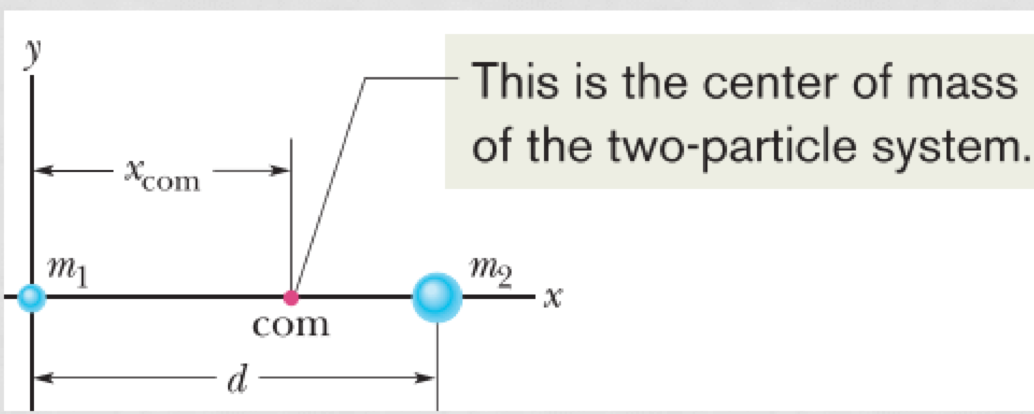
> The center of mass of a system of particles is the point that moves as though (1) all of the system's mass were concentrated there and (2) all external forces were applied there.

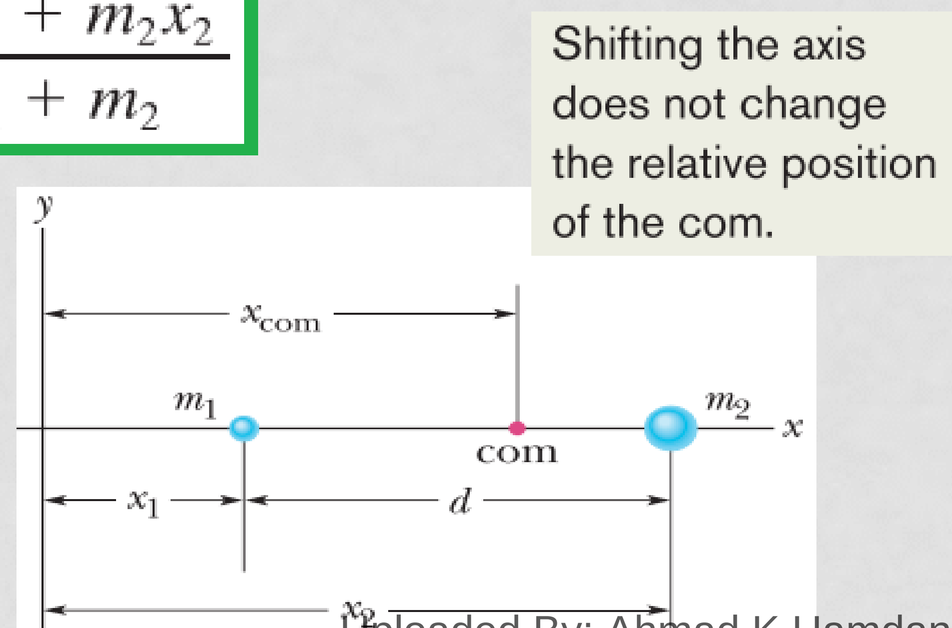
- For two particles separated by a distance *d*, where the origin is chosen at the position of particle 1:

$$x_{com} = \frac{m_2}{m_1 + m_2} d$$

- For two particles, for an arbitrary choice of origin:

$$x_{com} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$





- The center of mass is in the same location regardless of the coordinate system used.
- It is a property of the particles, not the coordinates.

For many particles, we can generalize the equation, where $M = m_1 + m_2 + . . . + m_n$:

$$x_{com} = \frac{m_1x_1 + m_2x_2 + m_3x_3 + \cdots + m_nx_n}{M} = \frac{1}{M}\sum_{i=1}^{n} m_i x_i$$

- In three dimensions, we find the center of mass along each axis separately:

$$x_{com} = \frac{1}{M}\sum_{i=1}^{n} m_i x_i \qquad y_{com} = \frac{1}{M}\sum_{i=1}^{n} m_i y_i \qquad z_{com} = \frac{1}{M}\sum_{i=1}^{n} m_i z_i$$

- More concisely, we can write in terms of vectors:

$$\vec{r}_{com} = \frac{1}{M}\sum_{i=1}^{n} m_i \vec{r}_i$$

$$\vec{r}_{com} = x_{com}\hat{\text{i}} + y_{com}\hat{\text{j}} + z_{com}\hat{\text{k}}$$

- For solid bodies, we take the limit of an infinite sum of infinitely small particles → Integration!
- Coordinate-by-coordinate, we write:

$$x_{com} = \frac{1}{M}\int x\,dm \qquad y_{com} = \frac{1}{M}\int y\,dm \qquad z_{com} = \frac{1}{M}\int z\,dm$$

- Here *M* is the mass of the object

- We limit ourselves to objects of uniform density, ρ, for the sake of simplicity

$$\rho = \frac{dm}{dV} = \frac{M}{V}$$

- The center of mass simplifies:

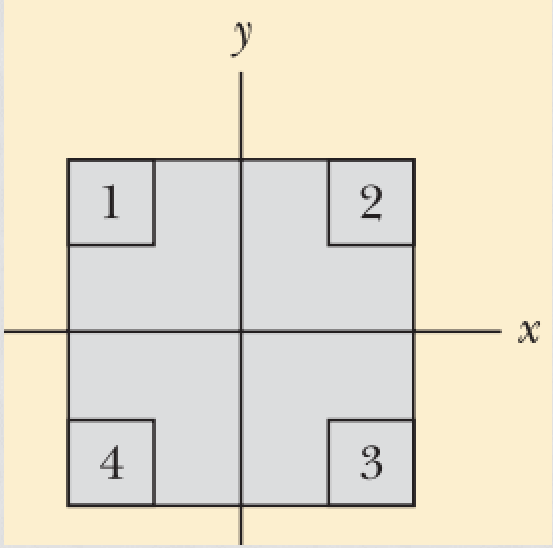
$$x_{\text{com}} = \frac{1}{V}\int x\,dV \qquad y_{\text{com}} = \frac{1}{V}\int y\,dV \qquad z_{\text{com}} = \frac{1}{V}\int z\,dV$$

- You can bypass one or more of these integrals if the object has symmetry
- The com lies at a point of symmetry (if there is one)
- It lies on the line or plane of symmetry (if there is one)
- It need not be on the object

  (consider a doughnut)

Checkpoint 1

The figure shows a uniform square plate from which four identical squares at the corners will be removed. (a) Where is the center of mass of the plate originally? Where is it after the removal of (b) square 1; (c) squares 1 and 2; (d) squares 1 and 3; (e) squares 1, 2, and 3; (f) all four squares? Answer in terms of quadrants, axes, or points (without calculation, of course).



Answer: (a) at the origin

(b) in Q4, along y=-x

(c) along the -y axis

(d) at the origin

(e) in Q3, along y=x

(f) at the origin

Example: Com of plate with missing piece:

Plate P is a metal plate of radius $2R$, with a circular hole of radius R.

- Find the com of each individual disk
- Treating the cutout as having negative mass
- On the diagram, *com*$_C$ is the center of mass for Plate *P* and Disk *S* combined
- *com*$_P$ is the center of mass for the composite plate with Disk *S* removed

Plate	Center of Mass	Location of com	Mass
P	com_P	$x_P = ?$	m_P
S	com_S	$x_S = -R$	m_S
C	com_C	$x_C = 0$	$m_C = m_S + m_P$

$$x_{S+P} = \frac{m_S x_S + m_P x_P}{m_S + m_P}.$$

$$x_{S+P} = x_C = 0.$$

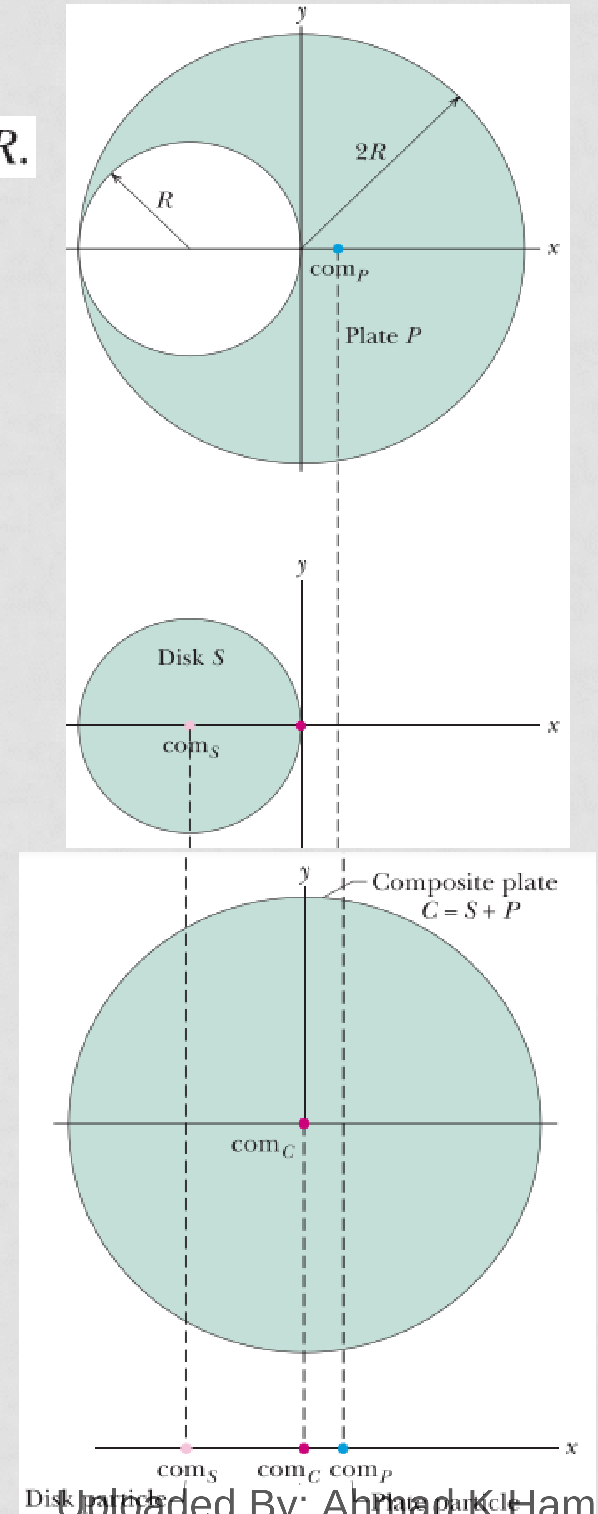
$$x_P = -x_S \frac{m_S}{m_P}.$$

$$\frac{m_S}{m_P} = \frac{\text{area}_S}{\text{area}_P} = \frac{\text{area}_S}{\text{area}_C - \text{area}_S} = \frac{\pi R^2}{\pi(2R)^2 - \pi R^2} = \frac{1}{3}$$

$$\text{mass} = \text{density} \times \text{volume} = \text{density} \times \text{thickness} \times \text{area}.$$

$$\frac{m_S}{m_P} = \frac{\text{density}_S}{\text{density}_P} \times \frac{\text{thickness}_S}{\text{thickness}_P} \times \frac{\text{area}_S}{\text{area}_P}$$

$$x_P = \tfrac{1}{3}R$$



••3 Figure 9-36 shows a slab with dimensions $d_1 = 11.0$ cm, $d_2 = 2.80$ cm, and $d_3 = 13.0$ cm. Half the slab consists of aluminum (density = 2.70 g/cm^3) and half consists of iron (density = 7.85 g/cm^3). What are (a) the x coordinate, (b) the y coordinate, and (c) the z coordinate of the slab's center of mass?

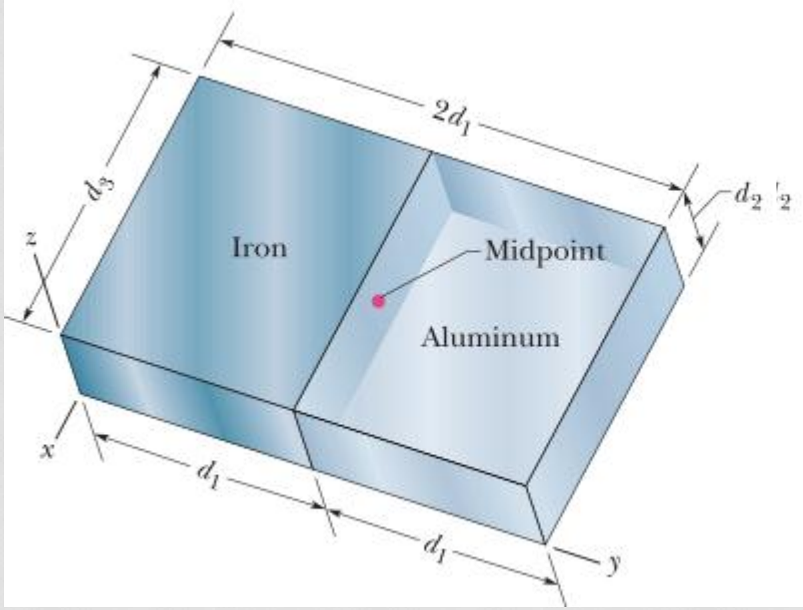
Iron com $\left(-\frac{d_3}{2}, \frac{d_1}{2}, \frac{d_2}{2}\right)$

Aluminum com $\left(-\frac{d_3}{2}, d_1 + \frac{d_1}{2}, \frac{d_2}{2}\right)$

$$x_{com} = -\frac{d_3}{2} = \frac{-13.0\text{ cm}}{2} = -6.5\text{ cm}$$

$$z_{com} = \frac{d_2}{2} = \frac{2.8\text{ cm}}{2} = 1.4\text{ cm}$$

$$y_{com} = \frac{m_{AL}\, y_{com,AL} + m_{Fe}\, y_{com,Fe}}{m_{AL} + m_{Fe}}$$



use $\rho = m/V$; $V_{AL} = V_{Fe}$

$$y_{com} = \frac{\rho_{AL} V_{AL}\, y_{com,AL} + \rho_{Fe} V_{Fe}\, y_{com,Fe}}{\rho_{AL} V_{AL} + \rho_{Fe} V_{Fe}}$$

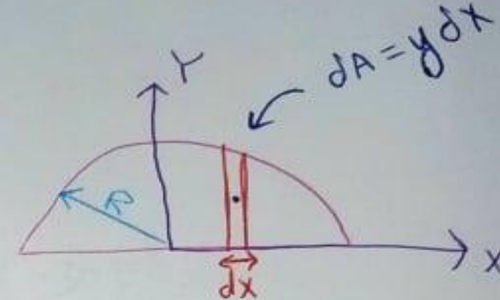
$$y_{com} = \frac{\rho_{AL}\, y_{com,AL} + \rho_{Fe}\, y_{com,Fe}}{\rho_{AL} + \rho_{Fe}}$$

$$y_{com,Fe} = \frac{d_1}{2} = \frac{11.0\text{ cm}}{2} = 5.5\text{ cm}$$

$$y_{com,AL} = d_1 + \frac{d_1}{2} = 16.5\text{ cm}$$

$$y_{com} = 8.3\text{ cm}$$

* A semi circle center of muss



$x^2 + y^2 = R^2$

⇒ $X_{com} = \text{Zero}$

⇒ $y_{com} = ??$

⇒ $y_{com} = \frac{1}{A} \int y \, dA$

$$y_{com} = \frac{1}{\frac{\pi R^2}{2}} = \int \frac{y}{2} (y \, dx) = \frac{1}{\pi R^2} \int y^2 \, dx$$

$$y_{com} = \frac{2}{\pi R^2} \int_0^R (R^2 - x^2) \, dx$$

$$y_{com} = \frac{2}{\pi R^2} \left[R^2 x - \frac{x^3}{3} \right]_0^R$$

$$= \frac{2}{\pi R^2} \left[R^3 - \frac{R^3}{3} \right] = \frac{2}{\pi R^2} \frac{2}{3} R^3$$

$$\boxed{y_{com} = \frac{4R}{3\pi}}$$

9-2 Newton's Second Law for a System of Particles

- Motion of a system's center of mass:

$$\vec{F}_{net} = M\vec{a}_{com} \quad \text{(system of particles)}$$

$$F_{net,x} = Ma_{com,x} \qquad F_{net,y} = Ma_{com,y} \qquad F_{net,z} = Ma_{com,z}$$

- Reminders:

1. F_{net} is the sum of all *external* forces
2. M is the total, constant, mass of the **closed** system
3. a_{com} is the *center of mass* acceleration

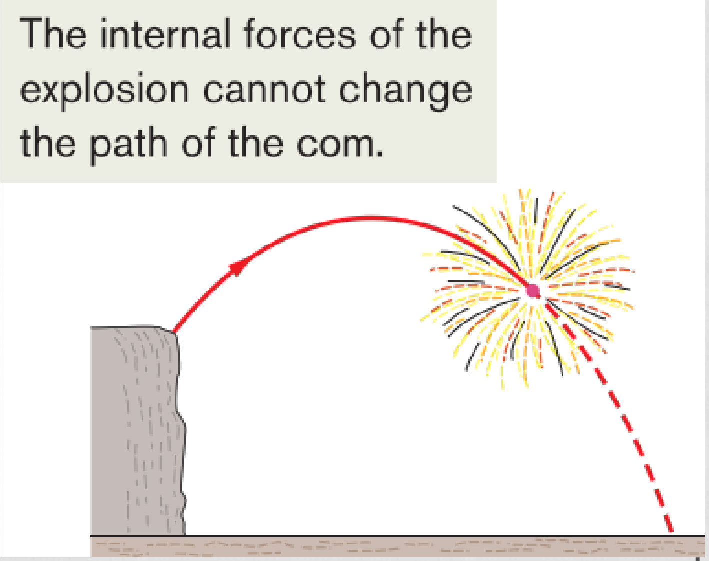
Examples: Using the center of mass motion equation:

- Billiard collision: forces are only internal, $\vec{F}_{net} = 0 \rightarrow \vec{a}_{com} = 0$

(system com, which was moving forward before the collision, must continue to move forward after the collision, with the same speed and in the same direction)

- Baseball bat: $\vec{a}_{com} = -\vec{g}$, so com follows gravitational trajectory

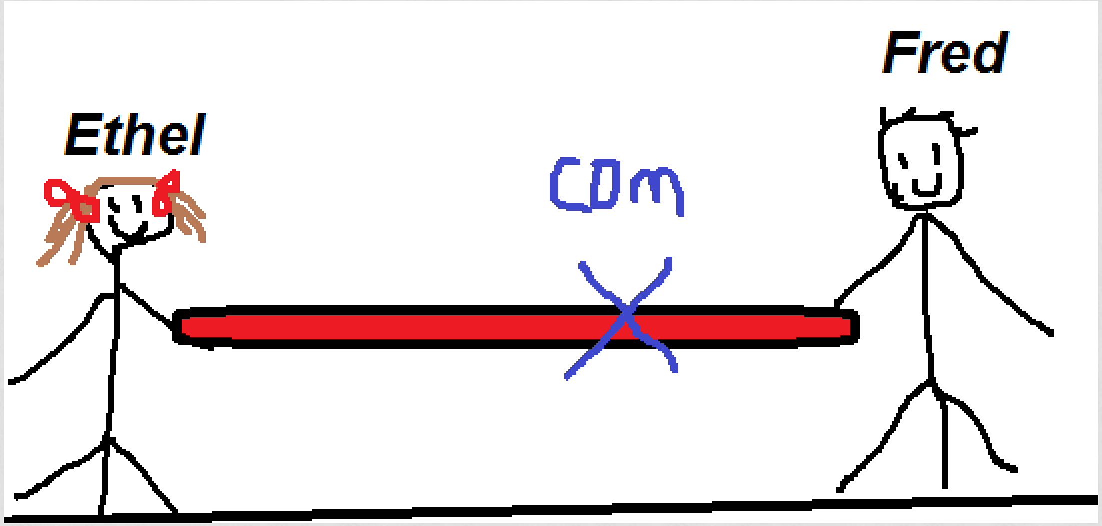
- Exploding rocket: explosion forces are internal, so only the gravitational force acts on the system, and the com follows a gravitational trajectory as long as air resistance can be ignored for the fragments.



Checkpoint 2

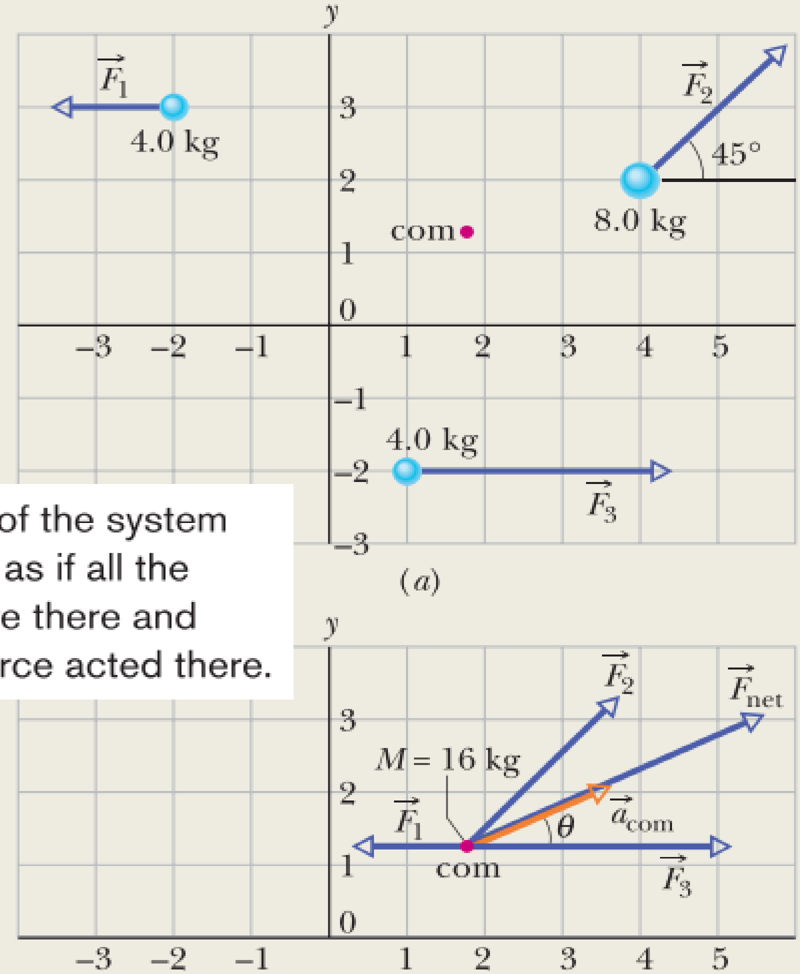
Two skaters on frictionless ice hold opposite ends of a pole of negligible mass. An axis runs along it, with the origin at the center of mass of the two-skater system. One skater, Fred, weighs twice as much as the other skater, Ethel. Where do the skaters meet if (a) Fred pulls hand over hand along the pole so as to draw himself to Ethel, (b) Ethel pulls hand over hand to draw herself to Fred, and (c) both skaters pull hand over hand?

Answer: The system consists of Fred, Ethel and the pole. All forces are internal. Therefore the com will remain in the same place. Since the origin is the com, they will meet at the origin in all three cases! (Of course the origin where the com is located is closer to Fred than to Ethel.)



Sample Problem 9.03 Motion of the com of three particles

The three particles in Fig. 9-7*a* are initially at rest. Each experiences an *external* force due to bodies outside the three-particle system. The directions are indicated, and the magnitudes are $F_1 = 6.0$ N, $F_2 = 12$ N, and $F_3 = 14$ N. What is the acceleration of the center of mass of the system, and in what direction does it move?



$$\vec{F}_{net} = M\vec{a}_{com}$$

$$\vec{F}_1 + \vec{F}_2 + \vec{F}_3 = M\vec{a}_{com}$$

$$\vec{a}_{com} = \frac{\vec{F}_1 + \vec{F}_2 + \vec{F}_3}{M}$$

The com of the system will move as if all the mass were there and the net force acted there.

$$a_{com,x} = \frac{F_{1x} + F_{2x} + F_{3x}}{M} = \frac{-6.0\text{ N} + (12\text{ N})\cos 45° + 14\text{ N}}{16\text{ kg}} = 1.03\text{ m/s}^2$$

$$a_{com,y} = \frac{F_{1y} + F_{2y} + F_{3y}}{M} = \frac{0 + (12\text{ N})\sin 45° + 0}{16\text{ kg}} = 0.530\text{ m/s}^2$$

$$a_{com} = \sqrt{(a_{com,x})^2 + (a_{com,y})^2} = 1.16\text{ m/s}^2 \approx 1.2\text{ m/s}^2$$

$$\theta = \tan^{-1}\frac{a_{com,y}}{a_{com,x}} = 27°$$

$$x_{com} = \frac{(8kg)(4m) + (4kg)(-2m) + (4\ kg)(1m)}{(8+4+4)\ kg} = +1.75\ m$$

$$y_{com} = \frac{(8kg)(2m) + (4kg)(3m) + (4\ kg)(-2m)}{(8+4+4)\ kg} = +1.25\ m$$

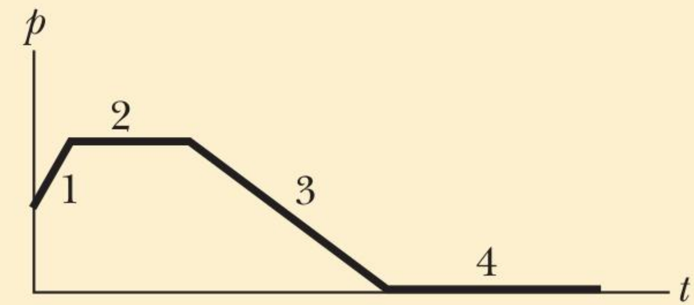
9-3 Linear Momentum

- The **linear momentum** is defined as: $$\vec{p} = m\vec{v}$$
- The momentum:
  - Vector quantity
  - Points in the same direction as the velocity
  - Can only be changed by a net external force
- We can write Newton's second law thus: $$\vec{F}_{net} = \frac{d\vec{p}}{dt}$$

The time rate of change of the momentum of a particle is equal to the net force acting on the particle and is in the direction of that force.

Checkpoint 3

The figure gives the magnitude p of the linear momentum versus time t for a particle moving along an axis. A force directed along the axis acts on the particle. (a) Rank the four regions indicated according to the magnitude of the force, greatest first. (b) In which region is the particle slowing?



Answer: (a) 1, 3, 2 & 4 (b) region 3

- The linear momentum of a system of particles:

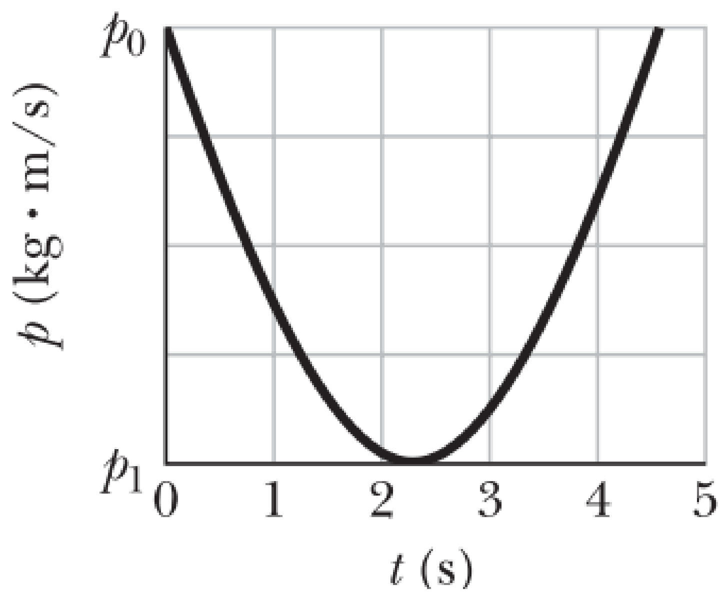
$$\vec{P} = \vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \cdots + \vec{p}_n$$
$$= m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + \cdots + m_n\vec{v}_n$$

$$\vec{P} = M\vec{v}_{com} \quad \text{(linear momentum, system of particles)}$$

$$\vec{F}_{net} = \frac{d\vec{P}}{dt} \quad \text{(system of particles)}$$

The net external force on a system changes system linear momentum!

••20 GO At time $t = 0$, a ball is struck at ground level and sent over level ground. The momentum p versus t during the flight is given by Fig. 9-46 (with $p_0 = 6.0\ kg \cdot m/s$ and $p_1 = 4.0\ kg \cdot m/s$). At what initial angle is the ball launched? (*Hint*: Find a solution that does not require you to read the time of the low point of the plot.)



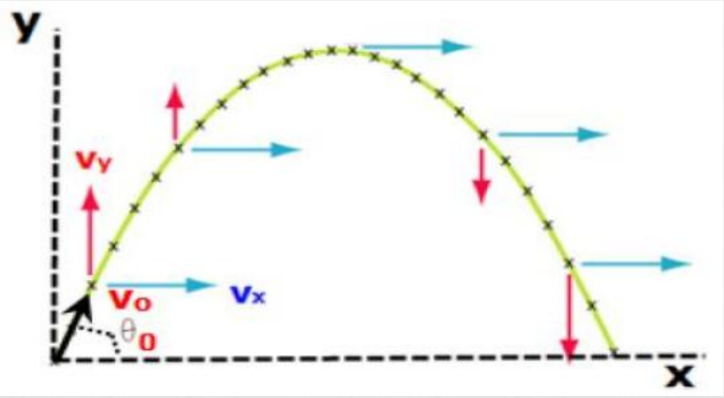
The initial component of momentum $p_0 = 6.0\ kg \cdot m/s$

$\vec{p} = m\vec{v}$

The horizontal component of momentum $p_x = 4.0\ kg \cdot m/s$

$$p_x = p_0 \cos\theta_0$$

$$\cos\theta_0 = \frac{p_x}{p_0} = \frac{4.0\ kg \cdot m/s}{6.0\ kg \cdot m/s} \to\to \theta_0 = 48^\circ$$



9-4 Collision and Impulse

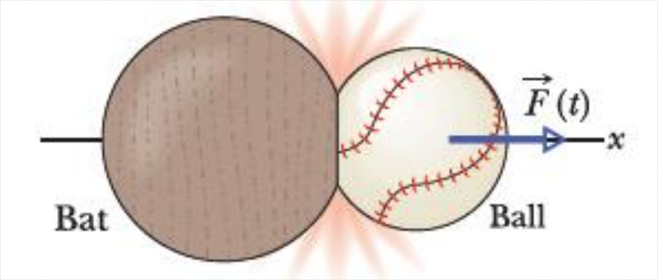
- In a collision, momentum of a particle can change
- We define the **Impulse *J*** acting during a collision:

$$\vec{J} = \int_{t_i}^{t_f} \vec{F}(t)\, dt$$

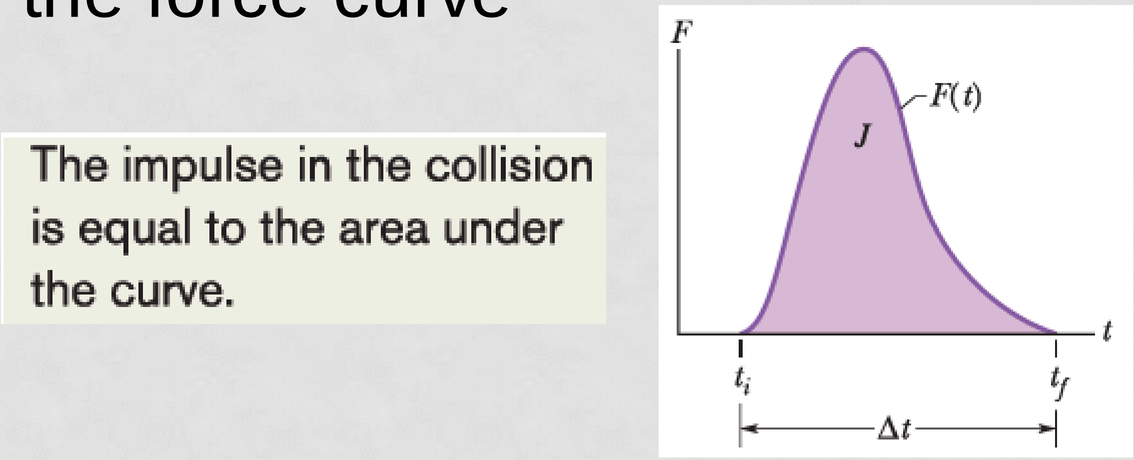
- This means that the applied impulse is equal to the change in momentum of the object during the collision:

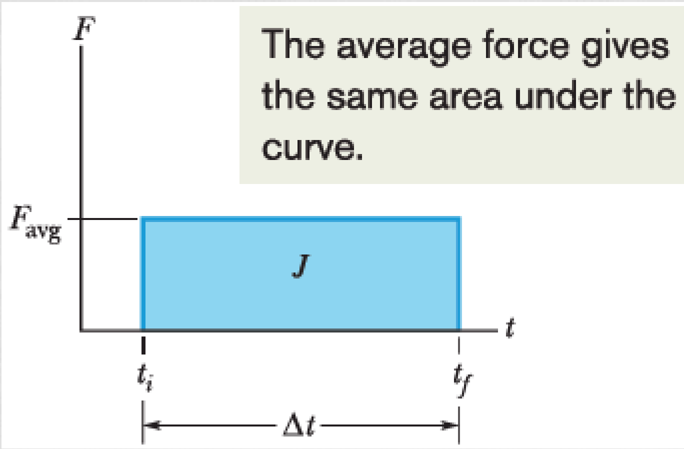
$$\Delta\vec{p} = \vec{J} \quad \text{(linear momentum–impulse theorem)}$$

- This equation can be rewritten component-by-component, like other vector equations



- Given F_{avg} and duration: $J = F_{\text{avg}}\,\Delta t$
- We are integrating: we only need to know the area under the force curve





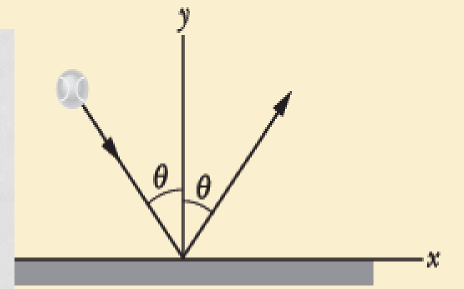
Checkpoint 4

A paratrooper whose chute fails to open lands in snow; he is hurt slightly. Had he landed on bare ground, the stopping time would have been 10 times shorter and the collision lethal. Does the presence of the snow increase, decrease, or leave unchanged the values of (a) the paratrooper's change in momentum, (b) the impulse stopping the paratrooper, and (c) the force stopping the paratrooper?

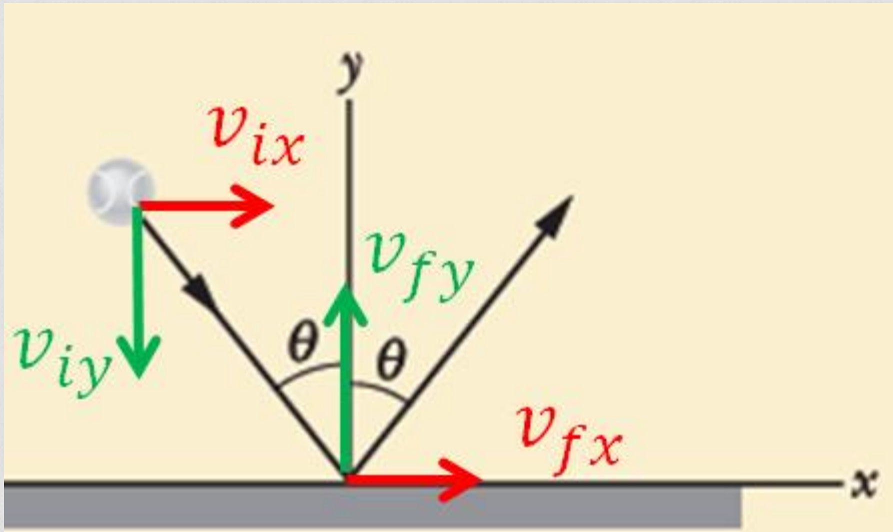
Answer: (a) unchanged (b) unchanged (c) decreased

Checkpoint 5

The figure shows an overhead view of a ball bouncing from a vertical wall without any change in its speed. Consider the change $\Delta\vec{p}$ in the ball's linear momentum. (a) Is Δp_x positive, negative, or zero? (b) Is Δp_y positive, negative, or zero? (c) What is the direction of $\Delta\vec{p}$?



Linear momentum is a vector quantity (magnitude and direction.



$$P_{ix} = P_{fx} \rightarrow \Delta P_x = 0$$

$$P_{iy} = -P_{fy} \rightarrow \Delta P_y = 2mv\cos\theta$$

$$v_{ix} = v_{fx} = v\sin\theta$$

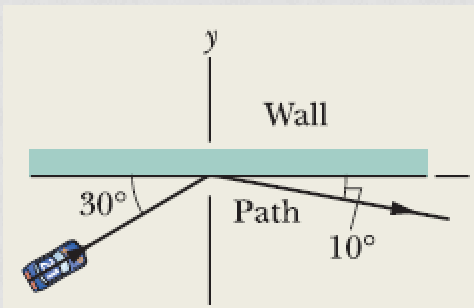
$$v_{iy} = -v_{fy} = -v\cos\theta$$

Answer:

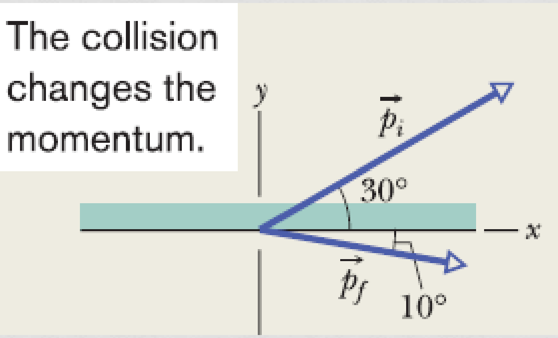
(a) zero (b) positive (c) along the positive *y*-axis (normal force)

Sample Problem 9.04 Two-dimensional impulse, race car–wall collision

Race car–wall collision. Figure 9-11*a* is an overhead view of the path taken by a race car driver as his car collides with the racetrack wall. Just before the collision, he is traveling at speed $v_i = 70$ m/s along a straight line at 30° from the wall. Just after the collision, he is traveling at speed $v_f = 50$ m/s along a straight line at 10° from the wall. His mass m is 80 kg.



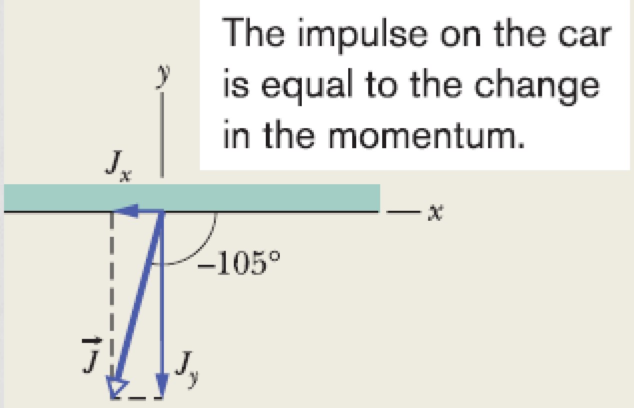
(a) What is the impulse $\vec{J}$ on the driver due to the collision?



$$\vec{J} = \vec{p}_f - \vec{p}_i = m\vec{v}_f - m\vec{v}_i = m(\vec{v}_f - \vec{v}_i)$$

$$J_x = m(v_{fx} - v_{ix})$$
$$= (80\text{ kg})[(50\text{ m/s})\cos(-10°) - (70\text{ m/s})\cos 30°] = -910\text{ kg}\cdot\text{m/s}.$$

$$J_y = m(v_{fy} - v_{iy})$$
$$= (80\text{ kg})[(50\text{ m/s})\sin(-10°) - (70\text{ m/s})\sin 30°] \approx -3500\text{ kg}\cdot\text{m/s}.$$



$$\vec{J} = (-910\hat{i} - 3500\hat{j})\text{ kg}\cdot\text{m/s}$$

$$\theta = \tan^{-1}\frac{J_y}{J_x}$$

$$\theta = 75.4°$$

$$J = \sqrt{J_x^2 + J_y^2} = 3616\text{ kg}\cdot\text{m/s} \approx 3600\text{ kg}\cdot\text{m/s}.$$

$$\theta = -105°$$

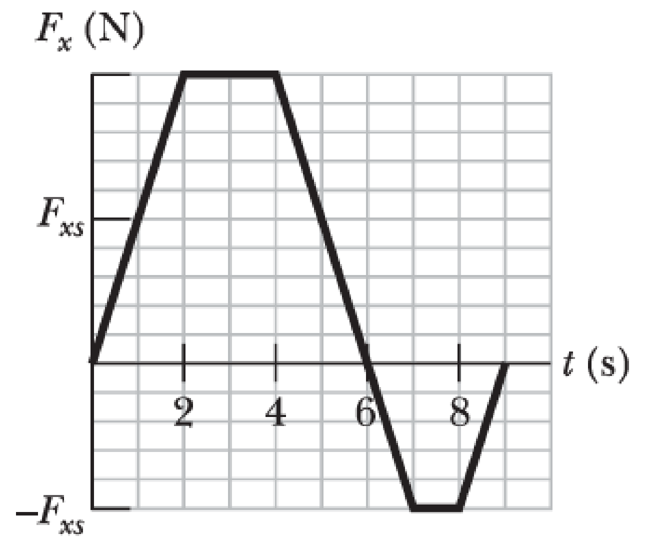
(b) The collision lasts for 14 ms. What is the magnitude of the average force on the driver during the collision?

$$F_{avg} = \frac{J}{\Delta t} = \frac{3616\text{ kg}\cdot\text{m/s}}{0.014\text{ s}}$$
$$= 2.583 \times 10^5\text{ N} \approx 2.6 \times 10^5\text{ N}.$$

Driver's average acceleration:

$$a_{avg} = \frac{F_{avg}}{m} = 3.22 \times 10^3\ m/s^2 = 329\ g$$ (Fatal!)

••32 A 5.0 kg toy car can move along an x axis; Fig. 9-50 gives F_x of the force acting on the car, which begins at rest at time $t = 0$. The scale on the F_x axis is set by $F_{xs} = 5.0$ N. In unit-vector notation, what is $\vec{p}$ at (a) $t = 4.0$ s and (b) $t = 7.0$ s, and (c) what is $\vec{v}$ at $t = 9.0$ s?



$$\Delta\vec{p} = \vec{J} \quad \text{(linear momentum–impulse theorem)}$$

$$\vec{J} = \int_{t_i}^{t_f} \vec{F}(t)\, dt$$

Impulse equals the area under the force versus time curve

(a) $\Delta\vec{P}(0\text{ s to }4\text{ s}) = \frac{1}{2}(2\text{ s})(10\text{N}) + (2\text{ s})(10\text{ N}) = (30\, kg\, m/s)\hat{\imath}$

$$\vec{P}(t = 4\ s) = (30\ kg\ m/s)\hat{\imath}$$

(b) $\Delta\vec{P}(0\text{ s to }7\text{ s}) = \frac{1}{2}((6+2)\text{ s})(10\text{N}) - \frac{1}{2}(1\text{ s})(5\text{N}) = (37.5\, kg\, m/s)\hat{\imath}$

$$\vec{P}(t = 7\ s) = (37.5\ kg\ m/s)\hat{\imath}$$

(c) $\vec{P}(t = 9\ s) = (30\ kg\ m/s)\hat{\imath} = \text{m}\vec{v}$

$$\vec{v} = (6\ m/s)\hat{\imath}$$

9-5 Conservation of Linear Momentum

- For an impulse of zero we find:

$$\vec{P}_i = \vec{P}_f \quad \text{(closed, isolated system)}$$

$$\vec{P} = \text{constant} \quad \text{(closed, isolated system)}$$

- **law of conservation of linear momentum:**

If no net external force acts on a system of particles, the total linear momentum $\vec{P}$ of the system cannot change.

Depending on the forces acting on a system, Linear momentum might be conserved in one or two directions but not in all directions. So you must check the components of the net external force separately to know if you should apply this or not

If the component of the net *external* force on a closed system is zero along an axis, then the component of the linear momentum of the system along that axis cannot change.

- Internal forces can change momenta of parts of the system, but cannot change the linear momentum of the entire system
- Do not confuse momentum and energy

Checkpoint 6

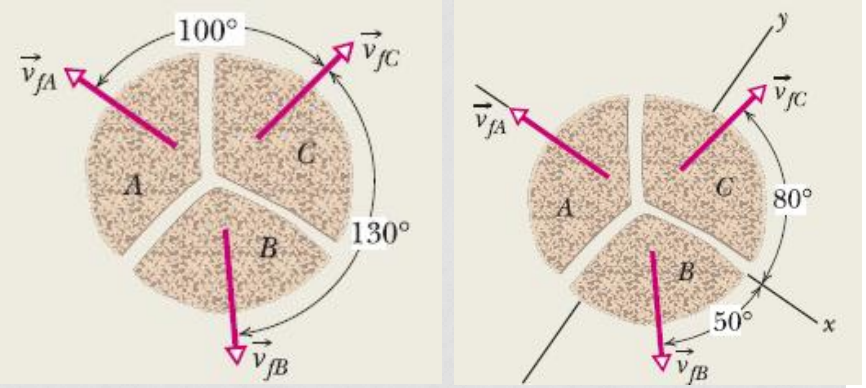
An initially stationary device lying on a frictionless floor explodes into two pieces, which then slide across the floor, one of them in the positive x direction. (a) What is the sum of the momenta of the two pieces after the explosion? (b) Can the second piece move at an angle to the x axis? (c) What is the direction of the momentum of the second piece?

Answer: (a) zero (b) no (c) the negative x- direction

Sample Problem 9.06 Two-dimensional explosion, momentum, coconut

Two-dimensional explosion: A firecracker placed inside a coconut of mass M, initially at rest on a frictionless floor, blows the coconut into three pieces that slide across the floor. An overhead view is shown in Fig. 9-13*a*. Piece C, with mass $0.30M$, has final speed $v_{fC} = 5.0$ m/s.

(a) What is the speed of piece B, with mass $0.20M$?



Closed system (The explosion forces are internal and no external force acts on the system →→ Conservation of linear momentum!

$\vec{P}_i = \vec{P}_f \rightarrow P_{ix} = P_{fx}$ and $P_{iy} = P_{fy}$

The coconut is initially at rest: $P_{ix} = P_{iy} = 0$

$$p_{fA,y} = 0,$$
$$p_{fB,y} = -0.20Mv_{fB,y} = -0.20Mv_{fB}\sin 50°$$
$$p_{fC,y} = 0.30Mv_{fC,y} = 0.30Mv_{fC}\sin 80°$$

$$p_{fA,x} = -0.50Mv_{fA}$$
$$p_{fB,x} = 0.20Mv_{fB,x} = 0.20Mv_{fB}\cos 50°$$
$$p_{fC,x} = 0.30Mv_{fC,x} = 0.30Mv_{fC}\cos 80°$$

$$P_{iy} = P_{fy} = p_{fA,y} + p_{fB,y} + p_{fC,y}$$

$$0 = 0 - 0.20Mv_{fB}\sin 50° + (0.30M)(5.0 \text{ m/s})\sin 80°$$

$$v_{fB} = 9.64 \text{ m/s} \approx 9.6 \text{ m/s}.$$

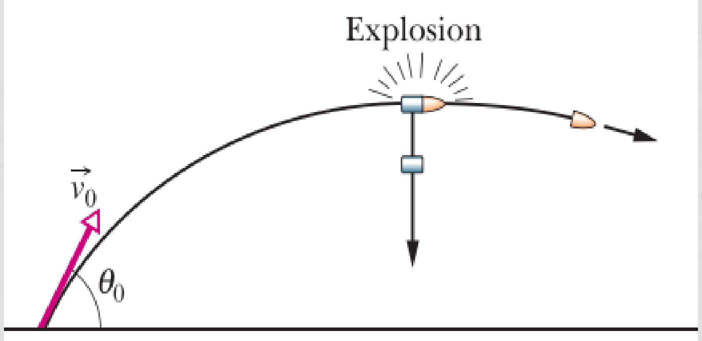
(b) What is the speed of piece A?

$$P_{ix} = P_{fx} = p_{fA,x} + p_{fB,x} + p_{fC,x}$$

$$0 = -0.50Mv_{fA} + 0.20M(9.64 \text{ m/s})\cos 50° + 0.30M(5.0 \text{ m/s})\cos 80°$$

$$v_{fA} = 3.0 \text{ m/s}.$$

••13 SSM A shell is shot with an initial velocity $\vec{v}_0$ of 20 m/s, at an angle of $\theta_0 = 60°$ with the horizontal. At the top of the trajectory, the shell explodes into two fragments of equal mass (Fig. 9-42). One fragment, whose speed immediately after the explosion is zero, falls vertically. How far from the gun does the other fragment land, assuming that the terrain is level and that air drag is negligible?



The coordinates of the trajectory highest point where the shell explodes into 2 fragments:

$$t = \frac{v_{0y}}{g} = \frac{v_0 sin\theta}{g}$$

$$x = v_{0x}t = \frac{{v_0}^2 \cos\theta\, sin\theta}{g} = \frac{(20\ m/s)^2 \cos 60^\circ \sin 60^\circ}{9.8\ m/s^2} = 17.7\ \text{m}$$

$$y = v_{0y}t - \frac{1}{2}gt^2 = \frac{{v_0}^2 sin^2\theta}{g} - \frac{1}{2}g\left(\frac{v_0 sin\theta}{g}\right)^2 = \frac{{v_0}^2 sin^2\theta}{2\,g} = \frac{1}{2}\frac{(20\text{m}/s)^2 sin^2 60^\circ}{9.8\,m/s^2} = 15.3\ \text{m}$$

The horizontal component of the linear momentum is conserved (No horizontal forces act):

$$P_{ix} = P_{fx}$$

$$M\, v_0 \cos\theta = 0 + \frac{M}{2} v_x \rightarrow\rightarrow v_x = 2\, v_0 \cos\theta = 2(20\ m/s)\cos 60^\circ = 20\ m/s$$

Fragement 2 from explosion point to landing point:

$$\Delta y = v_{0y}\text{t} - \frac{1}{2}gt^2 \rightarrow\rightarrow \Delta y = 0 - \frac{1}{2}gt^2 \rightarrow\rightarrow t = \sqrt{\frac{2\,\Delta y}{g}} = \sqrt{\frac{2\,(15.3\,m)}{(9.8\,m/s^2)}} = 1.77s$$

$$\Delta x = v_{0x}\text{t} = (20\ m/s)(1.77s) = 35.3\ \text{m}$$

$$\Delta x = x - x_0 = 35.3\ \text{m} \rightarrow\rightarrow x = x_0 + 35.3\ \text{m} = 17.7\text{m} + 35.3\ \text{m} = 53\ \text{m}$$

9-6 Momentum and Kinetic Energy in Collisions

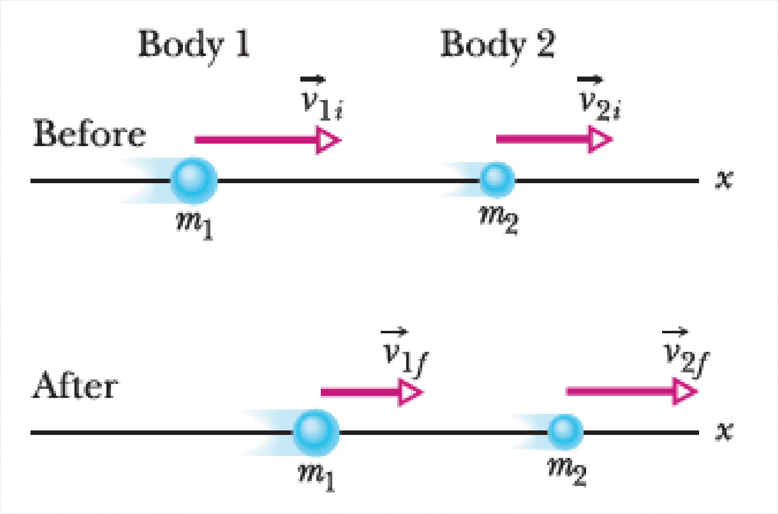
$$\vec{p}_{1i} + \vec{p}_{2i} = \vec{p}_{1f} + \vec{p}_{2f} \qquad \text{(conservation of linear momentum)}$$

- Types of collisions:

1. **Elastic collisions**:
   - Total kinetic energy is unchanged (conserved)
   - A useful approximation for common situations
   - In real collisions, some energy is always transferred
2. **Inelastic collisions**: some energy is transferred
3. **Completely inelastic collisions**:
   - The objects stick together
   - Greatest loss of kinetic energy

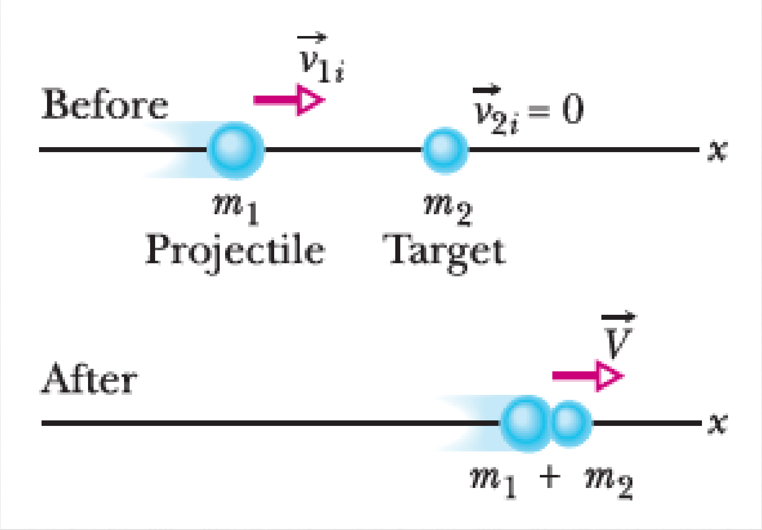
- Inelastic collision in one dimension:

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$



- Completely inelastic collision, for target at rest:

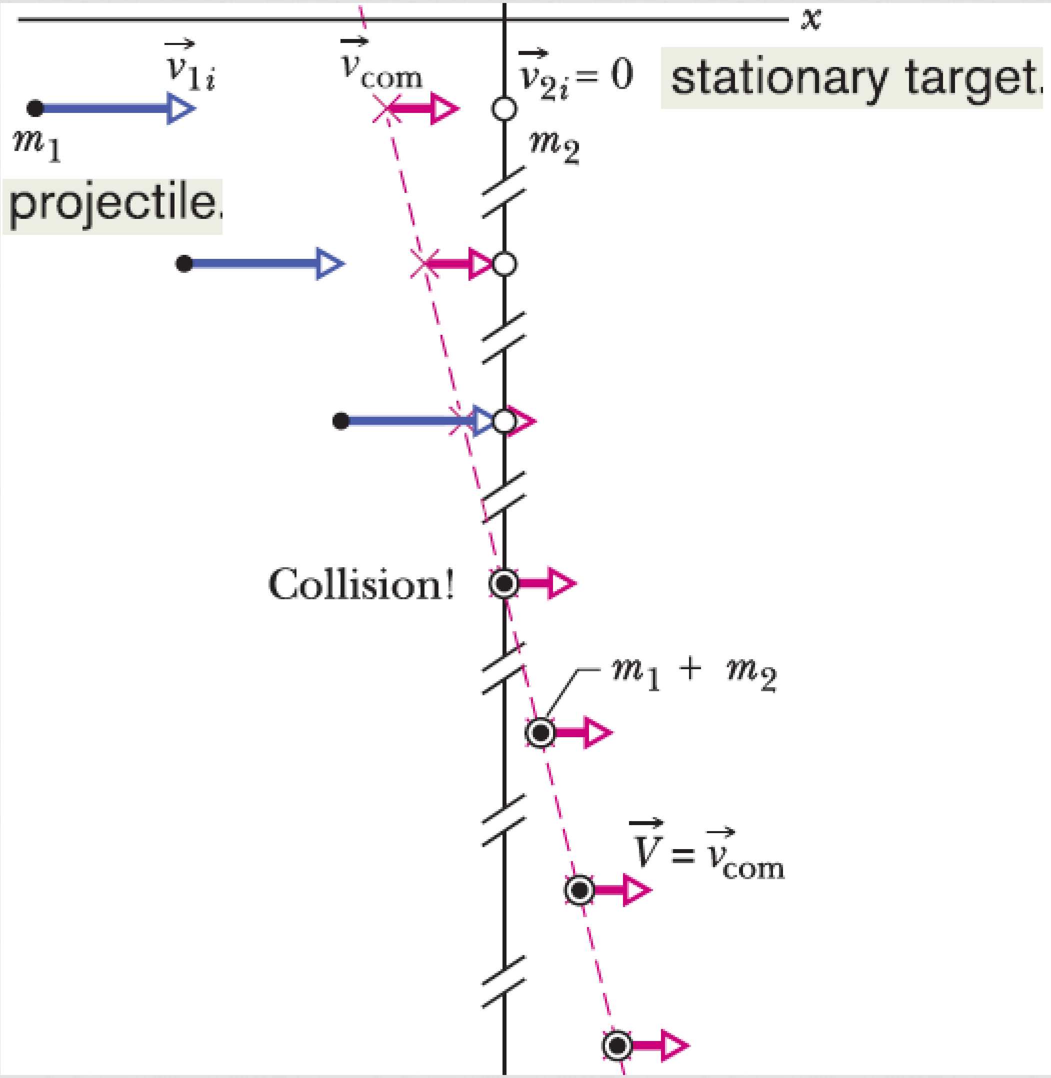
$$m_1 v_{1i} = (m_1 + m_2)V$$



- The center of mass velocity remains unchanged:

Completely Inelastic Collision

$$\vec{v}_{com} = \frac{\vec{P}}{m_1 + m_2} = \frac{\vec{p}_{1i} + \vec{p}_{2i}}{m_1 + m_2}$$



The com moves at the same velocity even after the bodies stick together

Unaffected by collisions/internal forces

Checkpoint 7

Body 1 and body 2 are in a completely inelastic one-dimensional collision. What is their final momentum if their initial momenta are, respectively, (a) 10 kg · m/s and 0; (b) 10 kg · m/s and 4 kg · m/s; (c) 10 kg · m/s and −4 kg · m/s?

Answer:

(a) 10 kg m/s

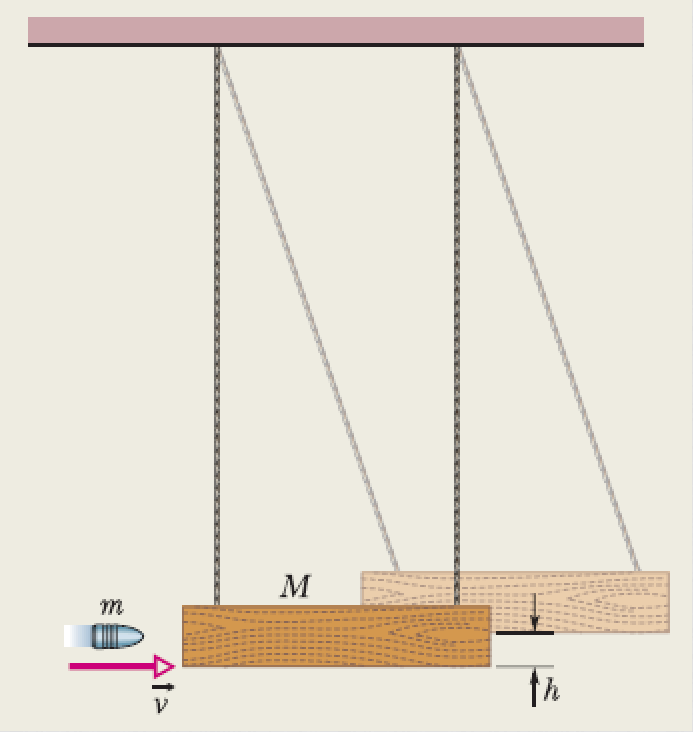
(b) 14 kg m/s

(c) 6 kg m/s

$$\vec{P}_i = \vec{P}_f$$

Sample Problem 9.07 Conservation of momentum, ballistic pendulum

The *ballistic pendulum* was used to measure the speeds of bullets before electronic timing devices were developed. The version shown in Fig. 9-17 consists of a large block of wood of mass $M = 5.4$ kg, hanging from two long cords. A bullet of mass $m = 9.5$ g is fired into the block, coming quickly to rest. The *block + bullet* then swing upward, their center of mass rising a vertical distance $h = 6.3$ cm before the pendulum comes momentarily to rest at the end of its arc. What is the speed of the bullet just prior to the collision?



Completely Inelastic collision in one-dimension:

$$\begin{pmatrix}\text{total momentum}\\ \text{before the collision}\end{pmatrix} = \begin{pmatrix}\text{total momentum}\\ \text{after the collision}\end{pmatrix}$$

$$V = \frac{m}{m+M}v$$

$$\begin{pmatrix}\text{mechanical energy}\\ \text{at bottom}\end{pmatrix} = \begin{pmatrix}\text{mechanical energy}\\ \text{at top}\end{pmatrix}$$

$$\tfrac{1}{2}(m+M)V^2 = (m+M)gh$$

$$v = \frac{m+M}{m}\sqrt{2gh}$$

$$= \left(\frac{0.0095\text{ kg} + 5.4\text{ kg}}{0.0095\text{ kg}}\right)\sqrt{(2)(9.8\text{ m/s}^2)(0.063\text{ m})} = 630\text{ m/s}$$

••52 In Fig. 9-59, a 10 g bullet moving directly upward at 1000 m/s strikes and passes through the center of mass of a 5.0 kg block initially at rest. The bullet emerges from the block moving directly upward at 400 m/s. To what maximum height does the block then rise above its initial position?

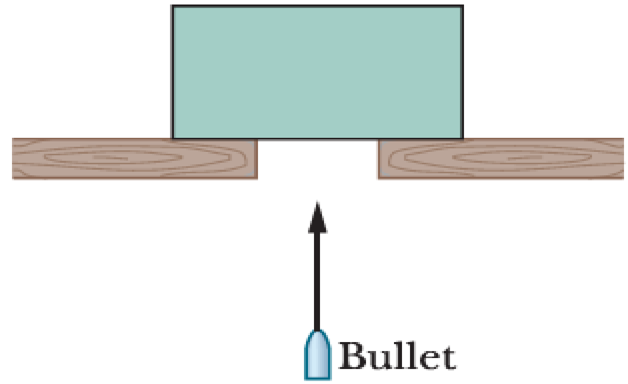
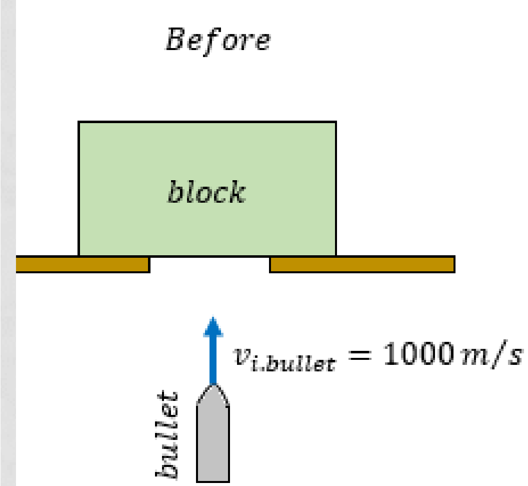
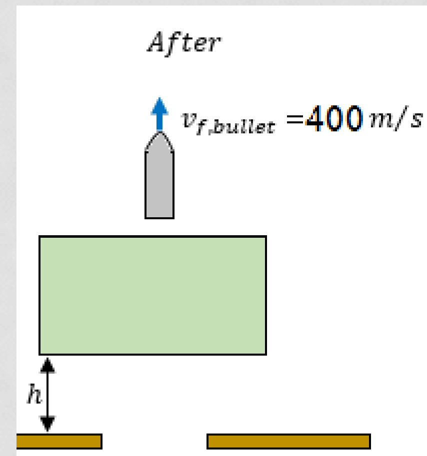


Figure 9-59 Problem 52.





The collision between the bullet that block is inelastic, thus the linear momentum of the bullet-block system is conserved.

$$p_i = p_f$$

$$mv_{i,bullet} + Mv_{i,block} = mv_{f,bullet} + Mv_{f,block}$$

$$(0.01\ kg)(1000 m/s) + (5kg)(0) = (0.01\ kg)(400\ m/s) + (5\ kg)v_{f,block}$$

$$v_{f,block} = 1.2\ m/s$$

The mechanical energy of the block is conserved (neglect air resistance), thus

$$\frac{1}{2}Mv_{f,block}^2 = Mgh$$

$$h = \frac{v_{f,block}^2}{2g} = \frac{(1.2\ m/s)^2}{2(9.8\ m/s^2)} = 0.073m = 7.3\ cm$$

9-7 Elastic Collisions in One Dimension

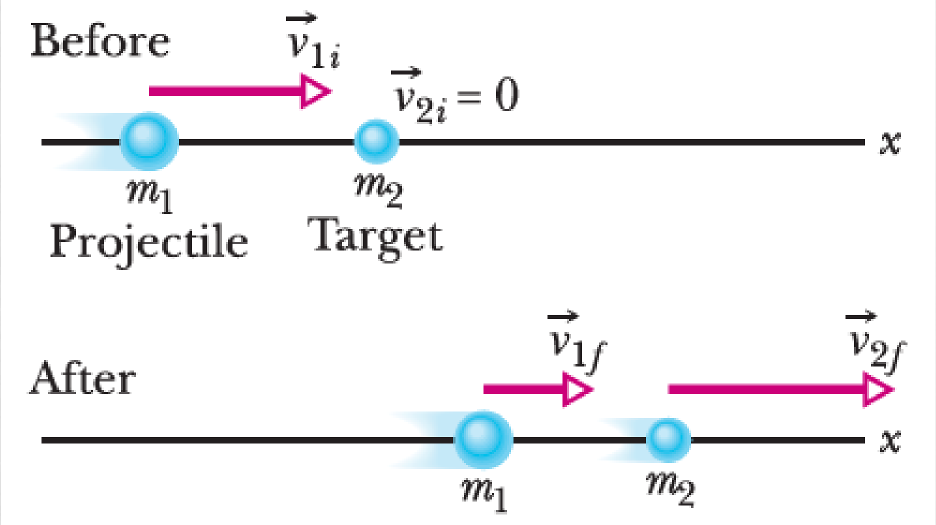
- Linear momentum is conserved
- Total kinetic energy is conserved in elastic collisions

In an elastic collision, the kinetic energy of each colliding body may change, but the total kinetic energy of the system does not change.

- For a stationary target, conservation laws give:

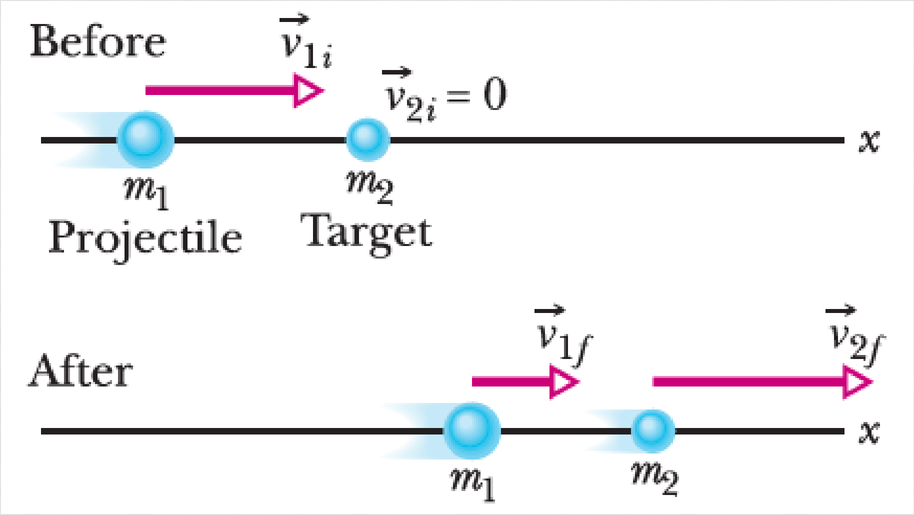
$$m_1 v_{1i} = m_1 v_{1f} + m_2 v_{2f} \quad \text{(linear momentum)}$$

$$\tfrac{1}{2} m_1 v_{1i}^2 = \tfrac{1}{2} m_1 v_{1f}^2 + \tfrac{1}{2} m_2 v_{2f}^2 \quad \text{(kinetic energy)}$$



$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i}$$

$$v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i}$$



- Results:
  - Equal masses: $v_{1f} = 0,\ \ v_{2f} = v_{1i}$, the first object stops
  - Massive target, $m_2 >> m_1$: $v_{1f} = -v_{1i}$, the first object just bounces back, speed mostly unchanged
  - Massive projectile: $v_{1f} = v_{1i}\ ,\ \ v_{2f} = 2v_{1i}$, the first object keeps going, the target flies forward at about twice its speed

Checkpoint 8

What is the final linear momentum of the target in Fig. 9-18 if the initial linear momentum of the projectile is 6 kg · m/s and the final linear momentum of the projectile is (a) 2 kg · m/s and (b) −2 kg · m/s? (c) What is the final kinetic energy of the target if the initial and final kinetic energies of the projectile are, respectively, 5 J and 2 J?

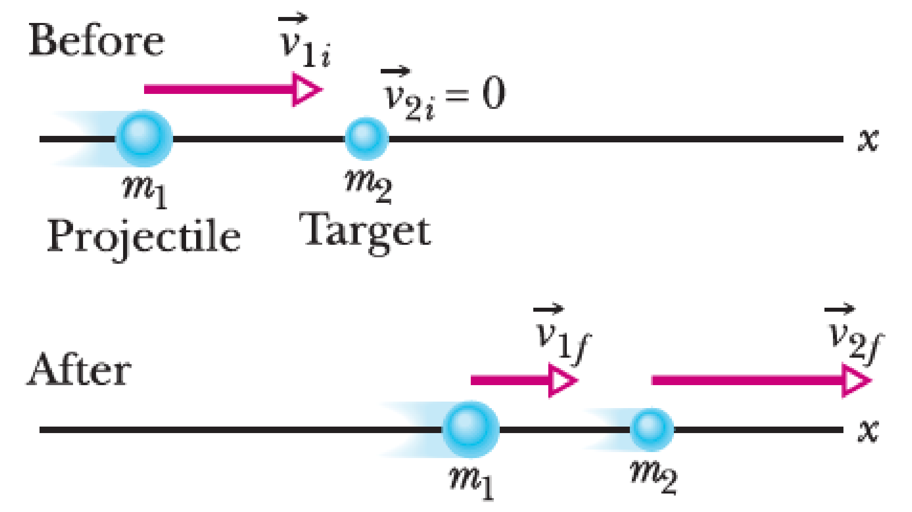
Answer: (a) 4 kg m/s

(b) 8 kg m/s

(c) 3 J

$$\vec{P}_i = \vec{P}_f$$
$$\vec{P}_{1i} = \vec{P}_{1f} + \vec{P}_{2f}$$

$$K_i = K_f$$
$$K_{1i} = K_{1f} + K_{2f}$$



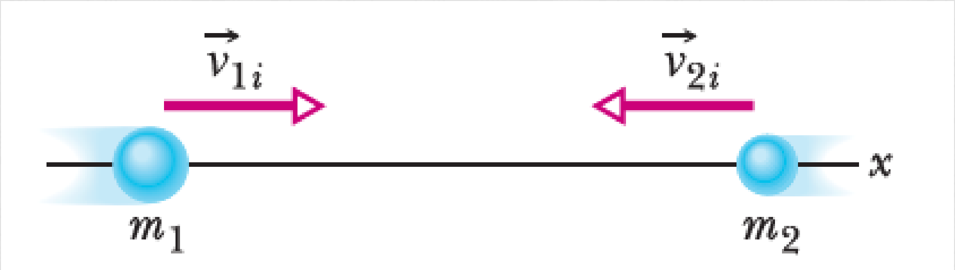
Elastic Collisions in One Dimension with moving target:

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

$$\frac{1}{2}m_1 v_{1i}^2 + \frac{1}{2}m_2 v_{2i}^2 = \frac{1}{2}m_1 v_{1f}^2 + \frac{1}{2}m_2 v_{2f}^2$$

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} + \frac{2m_2}{m_1 + m_2} v_{2i}$$

$$v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i} + \frac{m_2 - m_1}{m_1 + m_2} v_{2i}.$$



Remember:

The linear momentum and the velocity are vector quantities!

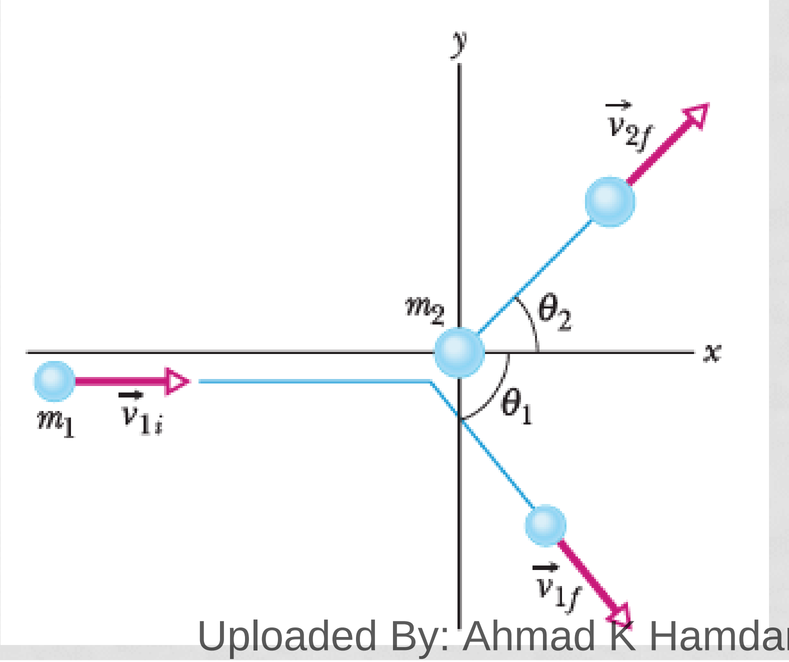
9-8 Collisions in Two Dimensions

- Apply the conservation of momentum along each axis
- Apply conservation of energy for elastic collisions

Example Elastic collision (A projectile with a stationary target):

- $P_{ix} = P_{fx}$: $m_1 v_{1i} = m_1 v_{1f} \cos\theta_1 + m_2 v_{2f} \cos\theta_2$
- $P_{iy} = P_{fy}$: $0 = -m_1 v_{1f} \sin\theta_1 + m_2 v_{2f} \sin\theta_2$
- Kinetic Energy:

$$\frac{1}{2} m_1 v_{1i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2$$



Checkpoint 9

In Fig. 9-21, suppose that the projectile has an initial momentum of 6 kg · m/s, a final x component of momentum of 4 kg · m/s, and a final y component of momentum of −3 kg · m/s. For the target, what then are (a) the final x component of momentum and (b) the final y component of momentum?

$$P_{1ix} = P_{1fx} + P_{2fx}$$

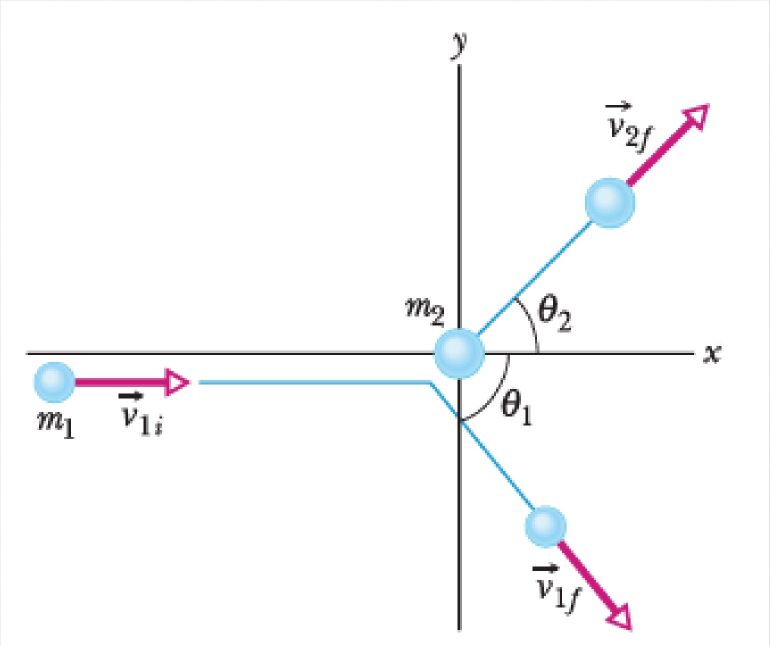
$$6\ kg \cdot m/s = 4\ kg \cdot m/s + P_{2fx}$$

$$P_{2fx} = 2\, kg \cdot m/s$$

$$0 = P_{1fy} + P_{2fy}$$

$$0 = -3\, kg \cdot m/s + P_{2fy}$$

$$P_{2fy} = 3\, kg \cdot m/s$$



•61 **SSM** A cart with mass 340 g moving on a frictionless linear air track at an initial speed of 1.2 m/s undergoes an elastic collision with an initially stationary cart of unknown mass. After the collision, the first cart continues in its original direction at 0.66 m/s. (a) What is the mass of the second cart? (b) What is its speed after impact? (c) What is the speed of the two-cart center of mass?

Elastic collision:

1. Conservation of momentum $P_i = P_f$

$$P_{1i} = P_{1f} + P_{2f}$$

$$m_1 v_{1i} = m_1 v_{1f} + m_2 v_{2f}$$

$$(0.34\ kg)(1.2\ m/s) = (0.34\ kg)(0.66\ m/s) + m_2 v_{2f}$$

$$m_2 v_{2f} = 0.184\ kg \cdot m/s\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

2. Conservation of kinetic energy:

$$\tfrac{1}{2} m_1 v_{1i}^2 = \tfrac{1}{2} m_1 v_{1f}^2 + \tfrac{1}{2} m_2 v_{2f}^2 \quad \text{(kinetic energy)}$$

$$\frac{1}{2}(0.34\ kg)(1.2\ m/s)^2 = \frac{1}{2}(0.34\ kg)(0.66\ m/s)^2 + \frac{1}{2} m_2 v_{2f}^2$$

$$m_2 v_{2f}^2 = 0.34\text{N.m}\ \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$v_{2f} = 1.85\ m/s$$

$$m_2 v_{2f} = 0.184\ kg \cdot m/s \rightarrow m_2 = \frac{0.184\ kg\cdot m/s}{1.85\ m/s} = 0.099\ \text{kg}$$

(c) The center of mass speed:

$$\vec{v}_{com} = \frac{\vec{P}}{m_1 + m_2}$$

$$v_{com} = \frac{P_i}{m_1 + m_2} = \frac{P_f}{m_1 + m_2}$$

$$v_{com} = \frac{P_i}{m_1 + m_2} = \frac{(0.34\ kg)(1.2\ m/s)}{(0.34\ kg) + (0.099\ kg)} = 0.93\ \text{m/s}$$

$$v_{com} = \frac{P_f}{m_1 + m_2} = \frac{(0.34\ kg)(0.66\ m/s) + (0.099\ kg)(1.85\ m/s)}{(0.34\ kg) + (0.099\ kg)} = 0.93\ \text{m/s}$$

9-9 Systems with Varying Mass: A Rocket

- Rocket and exhaust products form an isolated system
- Conserve momentum:

$$P_i = P_f$$

$$Mv = -dM\,U + (M + dM)(v + dv)$$

- Using relative speed:

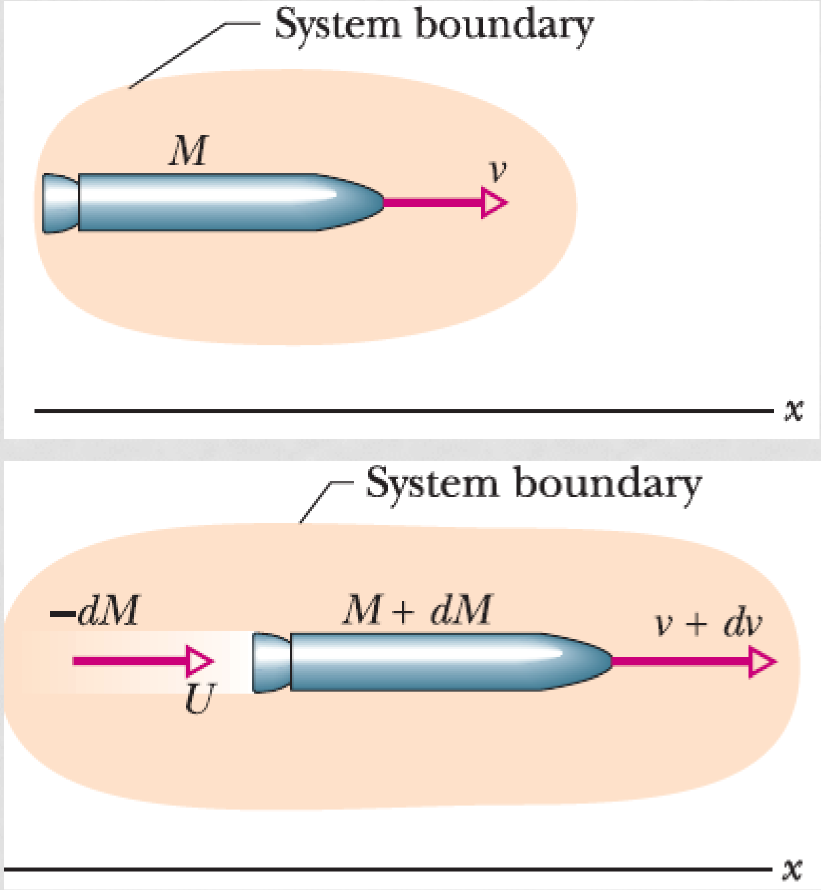
$$\begin{pmatrix}\text{velocity of rocket}\\ \text{relative to frame}\end{pmatrix} = \begin{pmatrix}\text{velocity of rocket}\\ \text{relative to products}\end{pmatrix} + \begin{pmatrix}\text{velocity of products}\\ \text{relative to frame}\end{pmatrix}.$$

$$(v + dv) = v_{\text{rel}} + U$$

$$U = v + dv - v_{\text{rel}}$$

$$-dM\,v_{\text{rel}} = M\,dv$$

$$-\frac{dM}{dt}v_{\text{rel}} = M\frac{dv}{dt}$$



The ejection of mass from the rocket's rear increases the rocket's speed.

- The first rocket equation: $$Rv_{\text{rel}} = Ma$$

  - R is the mass rate of fuel consumption
  - The left side of the equation is **Thrust**, T
  - Thrust unit is Newton

- Derive the velocity change for a given consumption of fuel as the second rocket equation:

$$v_f - v_i = v_{\text{rel}} \ln \frac{M_i}{M_f} \quad \text{(second rocket equation)}$$

Sample Problem 9.09 Rocket engine, thrust, acceleration

A rocket whose initial mass M_i is 850 kg consumes fuel at the rate $R = 2.3$ kg/s. The speed v_{rel} of the exhaust gases relative to the rocket engine is 2800 m/s. What thrust does the rocket engine provide?

$$T = Rv_{rel} = (2.3 \text{ kg/s})(2800 \text{ m/s}) = 6440 \text{ N} \approx 6400 \text{ N}.$$

(b) What is the initial acceleration of the rocket?

$$a = \frac{T}{M_i} = \frac{6440 \text{ N}}{850 \text{ kg}} = 7.6 \text{ m/s}^2$$

To be launched from Earth's surface, a rocket must have an initial acceleration greater than $g = 9.8 \text{ m/s}^2$. That is, it must be greater than the gravitational acceleration at the surface. Put another way, the thrust T of the rocket engine must exceed the initial gravitational force on the rocket, which here has the magnitude $M_i g$, which gives us

$$(850 \text{ kg})(9.8 \text{ m/s}^2) = 8330 \text{ N}$$

Because the acceleration or thrust requirement is not met (here $T = 6400$ N), our rocket could not be launched from Earth's surface by itself; it would require another, more powerful rocket.

•79 SSM ILW A rocket that is in deep space and initially at rest relative to an inertial reference frame has a mass of 2.55×10^5 kg, of which 1.81×10^5 kg is fuel. The rocket engine is then fired for 250 s while fuel is consumed at the rate of 480 kg/s. The speed of the exhaust products relative to the rocket is 3.27 km/s. (a) What is the rocket's thrust? After the 250 s firing, what are (b) the mass and (c) the speed of the rocket?

$$v_{rev} = 3.27\, km/s = 3.27 \times 10^3 m/s$$

(a) Thrust; $T = Rv_{rev} = (480 kg/s)(3.27 \times 10^3 m/s) = 1.6$ MN

(b) After 250s, The consumed fuel mass

$$m_{consumed\ fuel} = R\, t = (480 kg/s)(250\, s) = 1.2 \times 10^5 kg$$

Rocket mass after 250 s: $M_f = M_i - m_{consumed\ fuel}$

$$M_f = (2.55 \times 10^5 kg) - (1.2 \times 10^5 kg) = 1.35 \times 10^5 kg$$

(c) $v_f = v_i + v_{rev} \ln \frac{M_i}{M_f} = 0 + (3.27 \times 10^3 m/s) \ln\left(\frac{2.55 \times 10^5 kg}{1.35 \times 10^5 kg}\right) = 2.1\ km/s$

96 A rocket is moving away from the solar system at a speed of 6.0×10^3 m/s. It fires its engine, which ejects exhaust with a speed of 3.0×10^3 m/s relative to the rocket. The mass of the rocket at this time is 4.0×10^4 kg, and its acceleration is 2.0 m/s^2. (a) What is the thrust of the engine? (b) At what rate, in kilograms per second, is exhaust ejected during the firing?

(a) Thrust; $T = Rv_{rev} = Ma = (4.0 \times 10^4 kg)(2.0\, m/s^2) = 8.0 \times 10^4 N$

(b) $v_{rev} = 3.0 \times 10^3 m/s$

$$R = \frac{T}{v_{rev}} = \frac{8.0 \times 10^4 N}{3.0 \times 10^3 m/s} = 26.7\ kg/s$$

102 In Fig. 9-79, an 80 kg man is on a ladder hanging from a balloon that has a total mass of 320 kg (including the basket passenger). The balloon is initially stationary relative to the ground. If the man on the ladde begins to climb at 2.5 m/s relative to the ladder, (a) in what direction and (b) at what speed does the balloon move? (c) If the man then stops climbing, what is the speed of the balloon?

Man-balloon system is an isolated one; system com does not move ($v_{com} = 0$)

(a)The balloon will move downward with a certain speed v_{bg} relative to the ground.

(b) The speed of man relative to the ground is $v_{mg} = v_{mb} - v_{bg}$

v_{mg} is the speed of man relative to the ground.

v_{mb} is the speed of man relative to the balloon.

v_{bg} is the speed of balloon relative to the ground.

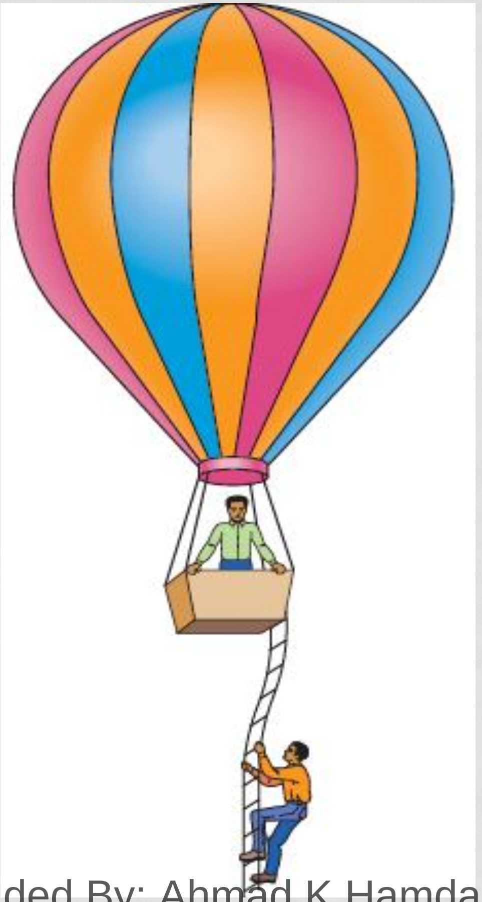
M is the mass of balloon= $320kg$

m is the mass of man = $80\ kg$

$$v_{com} = \frac{mv_{mg} - Mv_{bg}}{m+M} = \frac{m(v_{mb} - v_{bg}) - Mv_{bg}}{m+M} = 0$$

$$mv_{mb} - mv_{bg} - Mv_{bg} = 0$$

$$v_{bg} = \frac{mv_{mb}}{m+M} = \frac{(80\ kg)(2.5\,m/s)}{(80\ kg) + (320\ kg)} = 0.5\,m/s$$



(c) The balloon will again be stationary! $v_{com} = 0$