6.2 Volume by Using Cylindrical Shell Method

To Find Volume by shell Method :-

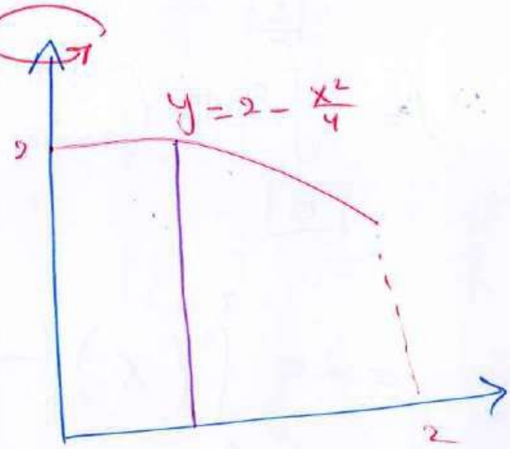
* Graph

* V → About x_axis, About Horizantal line: $V = \int_{\square}^{\square} 2\pi \left(\begin{matrix}\text{Shell}\\ \text{Radius}\end{matrix}\right)\left(\begin{matrix}\text{Shell}\\ \text{Length}\end{matrix}\right) dy$

→ About y-axis, About vertical line: $V = \int_{\square}^{\square} 2\pi \left(\begin{matrix}\text{Shell}\\ \text{Radius}\end{matrix}\right)\left(\begin{matrix}\text{Shell}\\ \text{height}\end{matrix}\right) dx$

Note. Shell Method: نرسم خط يقطع منطقة التكامل ويوازي محور الدوران

$\boxed{Q_2}$ use the shall Method to find the Volume of the solid generated by revolving the shaded region about the y-axis



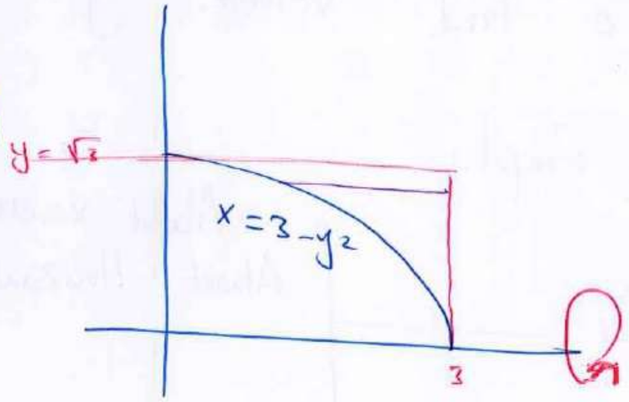
$$V = \int_{\boxed{0}}^{\boxed{2}} 2\pi \left(\begin{matrix}\text{shell}\\ \text{radius}\end{matrix}\right)\left(\begin{matrix}\text{shell}\\ \text{Height}\end{matrix}\right) dx$$

$$V = 2\pi \int_0^2 x\left(2 - \frac{x^2}{4}\right) dx$$

$$= 2\pi \int_0^2 2x - \frac{x^3}{4} \, dx$$

$$= 2\pi \left[x^2 - \frac{x^4}{16}\right]_0^2 = 2\pi\left[4 - 1\right] = \boxed{6\pi}$$

$\boxed{Q_4}$ Use the shell Method to find the volume of the solid generated by revolving the region ~~in~~ about the x-axis



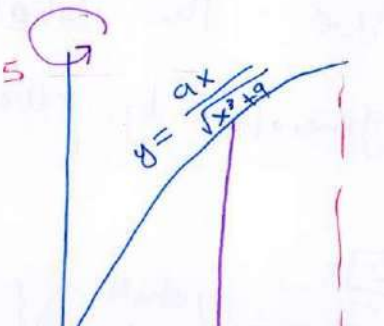
$$V = \int_{\boxed{0}}^{\boxed{\sqrt{3}}} 2\pi \begin{pmatrix}\text{shell}\\ \text{radius}\end{pmatrix}\begin{pmatrix}\text{shell}\\ \text{length}\end{pmatrix} dy$$

$$= \int_0^{\sqrt{3}} 2\pi(y)\left(3 - 3 + y^2\right) dy$$

$$= \int_0^{\sqrt{3}} 2\pi\left(y^3\right) dy = 2\pi \left.\frac{y^4}{4}\right]_0^{\sqrt{3}}$$

$$= \frac{\pi}{2}\left[9 - 0\right] = \boxed{\frac{9\pi}{2}}$$

$\boxed{Q_6}$ About y-axis



$$V = \int_{\boxed{0}}^{\boxed{3}} 2\pi \begin{pmatrix}\text{Shell}\\ \text{radius}\end{pmatrix}\begin{pmatrix}\text{Shell}\\ \text{height}\end{pmatrix} dx$$

$$= 2\pi \int_0^3 (x)\left(\frac{9x}{\sqrt{x^2+9}}\right) dx$$

By substitution

$U = x^3 + 9$ $\quad x = 3 \longrightarrow U = 36$

$dU = 3x^2\, dx$ $\quad x = 0 \longrightarrow U = 9$

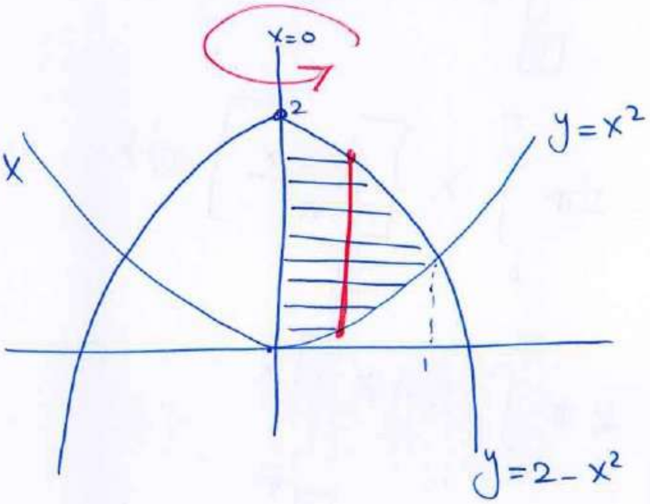
$$= 2\pi \int_0^3 \frac{9x^2}{\sqrt{x^3+9}} dx = 2\pi \int_9^{36} \frac{3\cancel{9x^2}}{\sqrt{U}} \frac{dU}{\cancel{3x^2}} = 6\pi \int_9^{36} U^{-1/2} dU$$

$$= 6\pi \left[2\sqrt{U}\right]_9^{36}$$

$$= 12\pi \left[6 - 3\right] = \boxed{36\pi}$$

$\boxed{Q_{10}}$ Use the Shell Method to find the volume of the solids generated by revolving the region bounded by the curves and lines about the y-axis

$y = 2 - x^2$ $\qquad$ $y = x^2$ $\qquad$ $x = 0$



$$V = \int_{\boxed{0}}^{\boxed{1}} 2\pi \binom{\text{shell}}{\text{radius}} \binom{\text{shell}}{\text{height}} dx$$

$$2\pi \int_0^1 x \left(2 - x^2 - x^2\right) dx$$

$$2\pi \int_0^1 2x - 2x^3 \quad dx$$

$$2\pi \left[x^2 - \frac{x^4}{2} \right]_0^1 = \pi$$

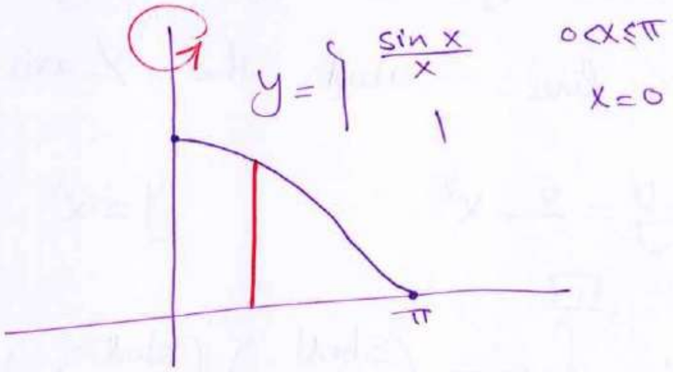
$\boxed{Q_{13}}$ $f(x) = \begin{cases} \dfrac{\sin x}{x} & 0 < x \leq \pi \\ 1 & x = 0 \end{cases}$

(a) Show that $x f(x) = \sin x \quad \forall \quad 0 \leq x \leq \pi$

$$x f(x) = \begin{cases} \sin x & 0 < x \leq \pi \\ x & x = 0 \end{cases}$$

$$x f(x) = \sin x \quad \forall \ x \in [0, \pi]$$

(b) Find the Volume of the Solid generated by revolving the shaded region about the y-axis (Shell Method)



$$V = 2\pi \int_{\boxed{0}}^{\boxed{\pi}} \begin{pmatrix}\text{shell}\\ \text{radius}\end{pmatrix}\begin{pmatrix}\text{shell}\\ \text{height}\end{pmatrix} dx$$

$$= 2\pi \int_0^{\pi} x \left[\frac{\sin x}{x}\right] dx$$

$$= 2\pi \int_0^{\pi} \sin x \, dx$$

$$= 2\pi \; -\cos x \Big]_0^{\pi}$$

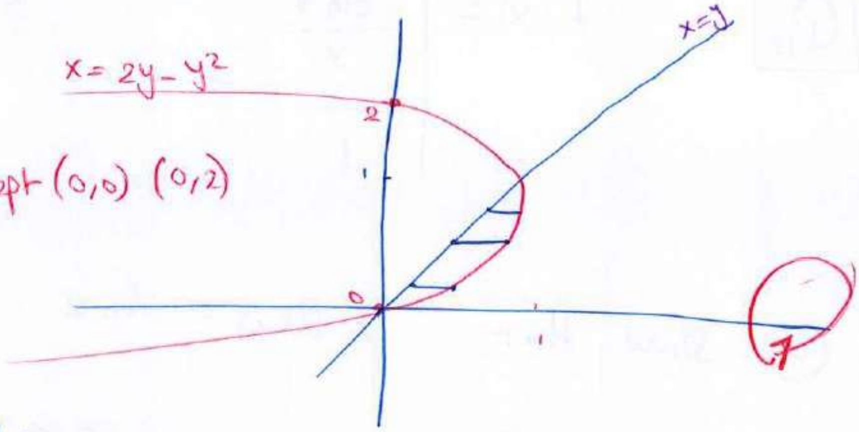
$$= -2\pi \left[-1 - 1\right] = \boxed{4\pi}$$

$\boxed{Q_{18}}$ $\quad x = 2y - y^2 \quad , \quad x = y \quad$ about x-axis

for the graph $x = 2y - y^2$

vertex $(1, 1)$

x-intercept $(0,0)$ $\quad$ y-intercept $(0,0)$ $(0,2)$



$$V = 2\pi \int_{\boxed{0}}^{\boxed{1}} \begin{pmatrix}\text{shell}\\ \text{radius}\end{pmatrix}\begin{pmatrix}\text{shell}\\ \text{lenght}\end{pmatrix} dy$$

$$= 2\pi \int_0^1 y\,(2y - y^2 - y)\, dy \quad = 2\pi \int_0^1 (y^2 - y^3)\, dy$$

$$2\pi \left[\frac{y^3}{3} - \frac{y^4}{4}\right]_0^1 = \boxed{\frac{\pi}{6}}$$

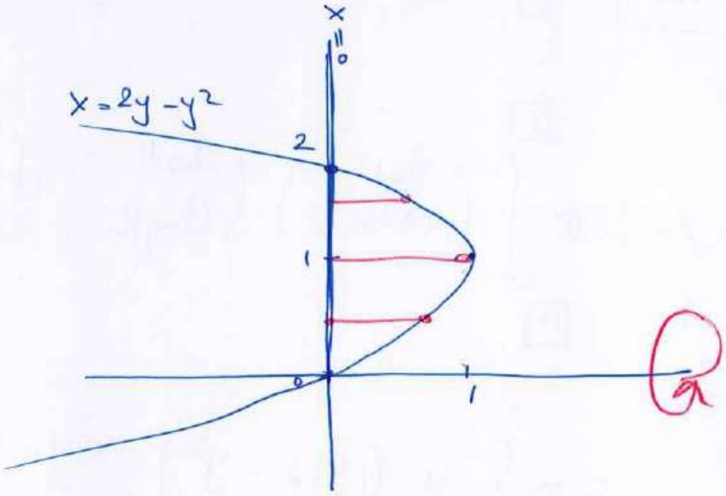
$\boxed{Q_{A}}$ $x = 2y - y^2$ $x=0$ about x-axis

for the graph $x = 2y - y^2$

x-intercept $\begin{matrix} x=0 \\ y=0 \end{matrix}$ $(0,0)$

y-intercept $(0,0)$ $(0,2)$

Vertex $(1,1)$



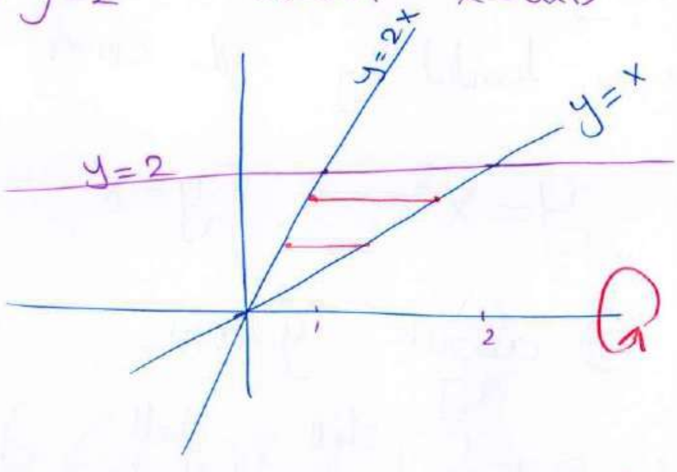
$$V = \int_{\boxed{0}}^{\boxed{2}} 2\pi \begin{pmatrix} \text{shell} \\ \text{radius} \end{pmatrix} \begin{pmatrix} \text{shell} \\ \text{length} \end{pmatrix} dy$$

$$= 2\pi \int_0^2 (y)(2y - y^2) \, dy = 2\pi \int_0^2 2y^2 - y^3 \, dy$$

$$= 2\pi \left[\frac{2y^3}{3} - \frac{y^4}{4} \right]_0^2 = 2\pi \left[\frac{16}{3} - 4 \right] = \boxed{\frac{8\pi}{3}}$$

$\boxed{Q_{20}}$ $y = x$, $y = 2x$, $y = 2$ about x-axis

$$V = 2\pi \int_{\boxed{}}^{\boxed{}} \begin{pmatrix} \text{shell} \\ \text{radius} \end{pmatrix} \begin{pmatrix} \text{shell} \\ \text{length} \end{pmatrix} dy$$

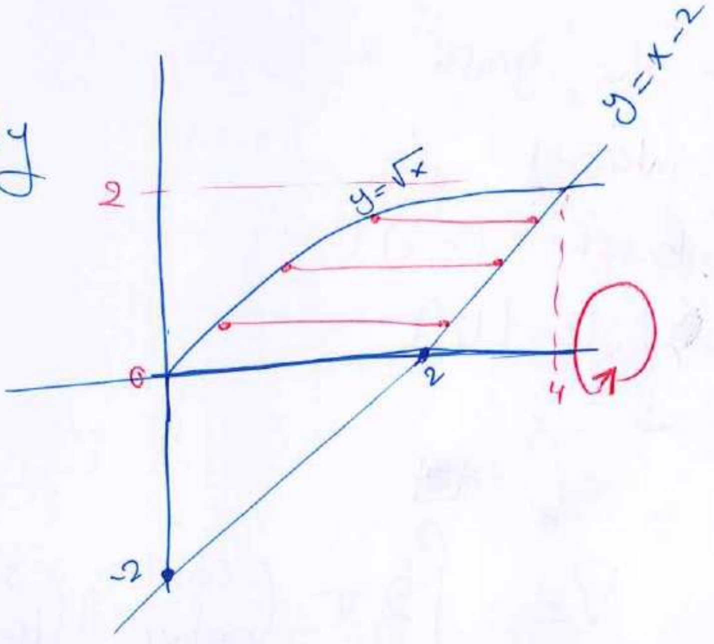


$$= 2\pi \int_0^2 y \left(y - \frac{y}{2} \right) dy$$

$$= 2\pi \int_0^2 \left(y^2 - \frac{y^2}{2} \right) dy = 2\pi \int_0^2 \frac{y^2}{2} \, dy = 2\pi \left[\frac{y^3}{6} \right]_0^2$$

$$= \boxed{\frac{8\pi}{3}}$$

$\boxed{Q_{21}}$ $y=\sqrt{x}$ $\quad y=0 \quad y=x-2$ about x-axis



$$V = 2\pi \int_{\boxed{0}}^{\boxed{2}} \begin{pmatrix}\text{shell}\\\text{radius}\end{pmatrix}\begin{pmatrix}\text{shell}\\\text{length}\end{pmatrix} dy$$

$$= 2\pi \int_0^2 y\,(y+2-y^2)\,dy$$

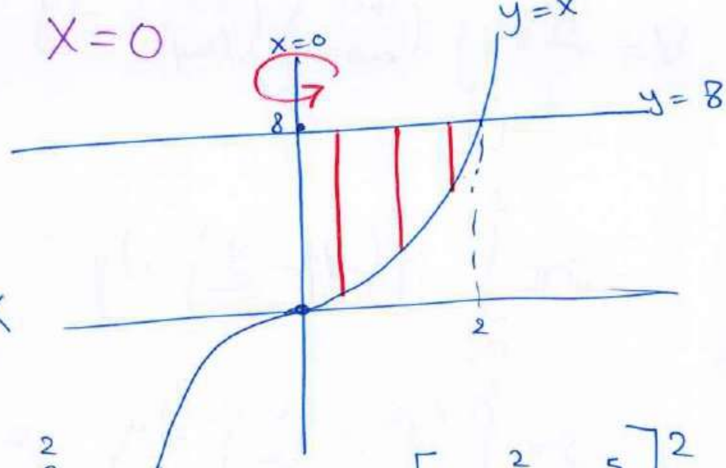
$$= 2\pi \int_0^2 y^2+2y-y^3\,dy$$

$$= 2\pi\left[\frac{y^3}{3}+\frac{2y^2}{2}-\frac{y^4}{4}\right]_0^2$$

$$= 2\pi\left[\frac{8}{3}+4-4\right] = \boxed{\frac{16\pi}{3}}$$

$\boxed{Q_{24}}$ Find the volume of the solids generated by revolving the region bounded by the curves about the given lines

$y=x^3 \quad y=8 \quad x=0$



ⓐ about y-axis

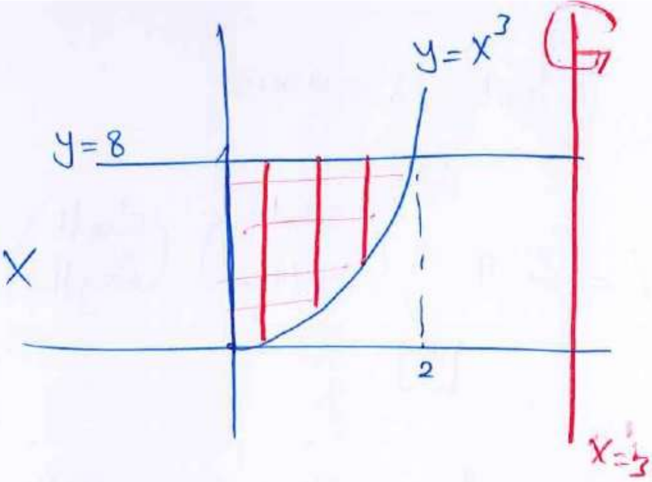
$$V = 2\pi \int_{\boxed{0}}^{\boxed{2}} \begin{pmatrix}\text{shell}\\\text{radius}\end{pmatrix}\begin{pmatrix}\text{shell}\\\text{Height}\end{pmatrix} dx$$

$$= 2\pi\int_0^2 x\,(8-x^3)\,dx = 2\pi\int_0^2 (8x-x^4)\,dx = 2\pi\left[4x^2-\frac{x^5}{5}\right]_0^2$$

$$= \boxed{\frac{96\pi}{5}}$$

(b) about $x=3$

$$V = 2\pi \int_{\boxed{0}}^{\boxed{2}} \binom{\text{shell}}{\text{radius}} \binom{\text{shell}}{\text{height}} \, dx$$



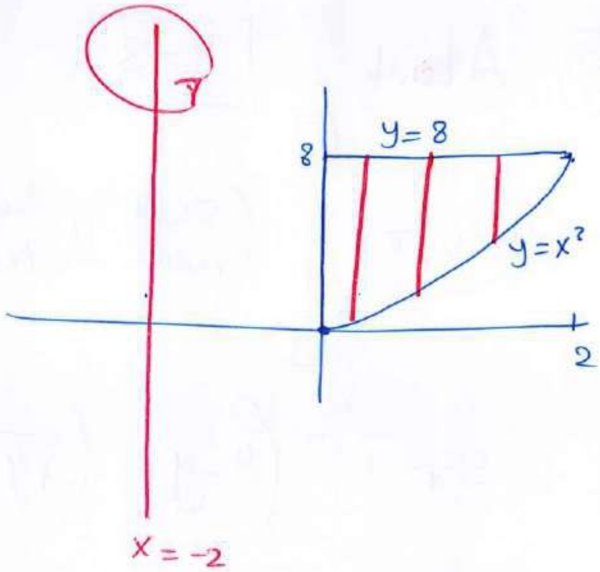
$$= 2\pi \int_0^2 (3-x)(8-x^3)\, dx$$

$$= 2\pi \int_0^2 (24 - 3x^3 - 8x + x^4)\, dx$$

$$= 2\pi \left[24x - \frac{3x^4}{4} - \frac{8x^2}{2} + \frac{x^5}{5} \right]_0^2 = \boxed{\frac{264\pi}{5}}$$

(c) about $x=-2$

$$V = 2\pi \int_{\boxed{0}}^{\boxed{2}} \binom{\text{shell}}{\text{radius}} \binom{\text{shell}}{\text{height}} \, dx$$



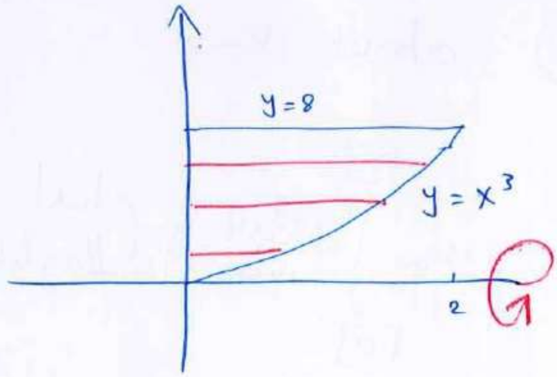
$$= 2\pi \int_0^2 (x+2)(8-x^3)\, dx$$

$$= 2\pi \int_0^2 (8x - x^4 + 16 - 2x^3)\, dx$$

$$= 2\pi \left[\frac{8x^2}{2} - \frac{x^5}{5} + 16x - \frac{2x^4}{4} \right]_0^2 = \boxed{\frac{336\pi}{5}}$$

(d) About x-axis

$$V = 2\pi \int_{\boxed{0}}^{\boxed{8}} \begin{pmatrix}\text{shell}\\ \text{radius}\end{pmatrix}\begin{pmatrix}\text{shell}\\ \text{length}\end{pmatrix} dy$$



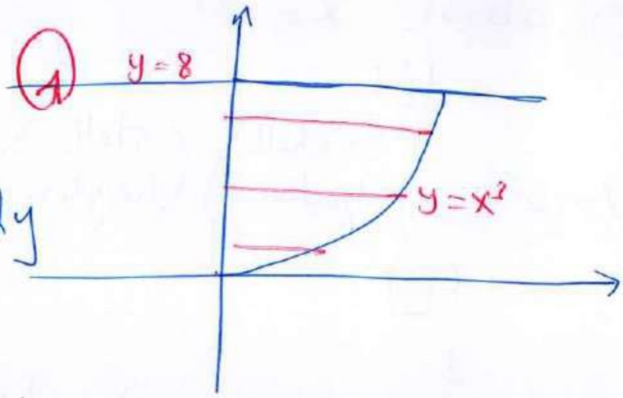
$$= 2\pi \int_0^8 (\ y\ )\left(\sqrt[3]{y}\right) dy$$

$$= 2\pi \int_0^8 y^{\frac{4}{3}}\, dy \quad = 2\pi \tfrac{3}{7}\, y^{7/3}\Big]_0^8$$

$$= \frac{6\pi}{7}\left[2^7 - 0\right] = \boxed{\frac{768\,\pi}{7}}$$

(e) About $\boxed{y=8}$

$$V = 2\pi \int_{\boxed{0}}^{\boxed{8}} \begin{pmatrix}\text{shell}\\ \text{radius}\end{pmatrix}\begin{pmatrix}\text{shell}\\ \text{length}\end{pmatrix} dy$$

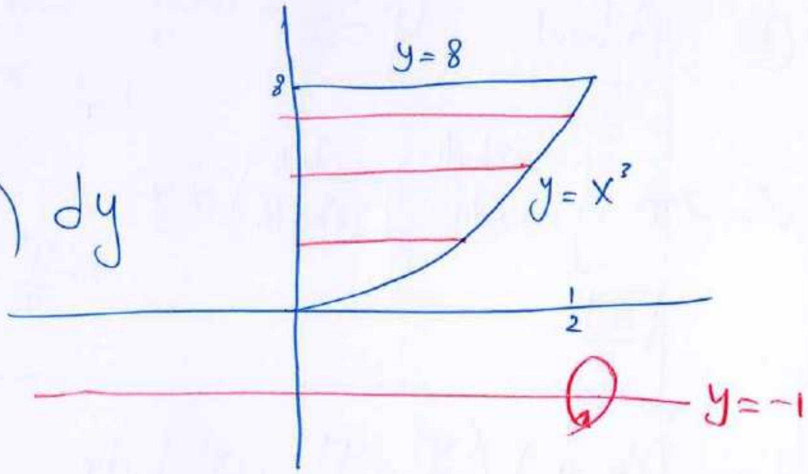


$$= 2\pi \int_0^8 (8-y)\ \left(\sqrt[3]{y}\right)\ dy$$

$$= 2\pi \int_0^8 \left(8\sqrt[3]{y} - y^{\frac{4}{3}}\right) dy$$

$$= 2\pi \int_0^8 8y^{1/3} - y^{4/3}\, dy \quad = 2\pi\left[8\cdot\tfrac{3}{4}\, y^{4/3} - \tfrac{3}{7} y^{7/3}\right]_0^8 = \boxed{\frac{576\,\pi}{7}}$$

(f) About $\boxed{y=-1}$



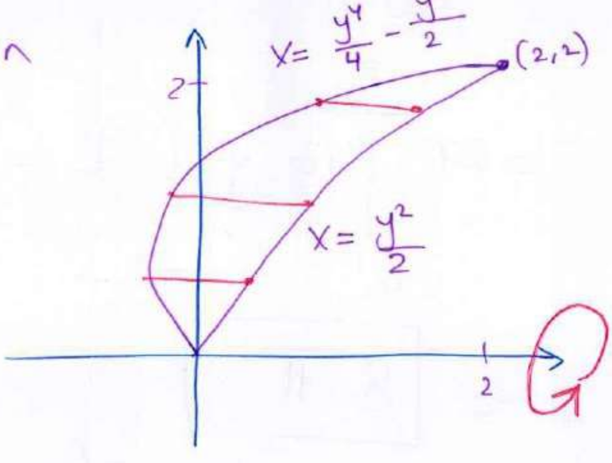
$$V = 2\pi \int_{\boxed{0}}^{\boxed{8}} \begin{pmatrix} \text{shell} \\ \text{radius} \end{pmatrix} \begin{pmatrix} \text{shell} \\ \text{Length} \end{pmatrix} dy$$

$$= 2\pi \int_0^8 (y+1)(\sqrt[3]{y} - 0)\, dy$$

$$= 2\pi \int_0^8 \left(y^{\frac{4}{3}} + y^{1/3}\right) dy = 2\pi \left[\frac{3}{7} y^{7/3} + \frac{3}{4} y^{4/3}\right]_0^8$$

$$= \boxed{\frac{936\,\pi}{7}}$$

$\boxed{Q_{28}}$ Find the Volume by using shell Method of the solid generated by revolving the shaded region



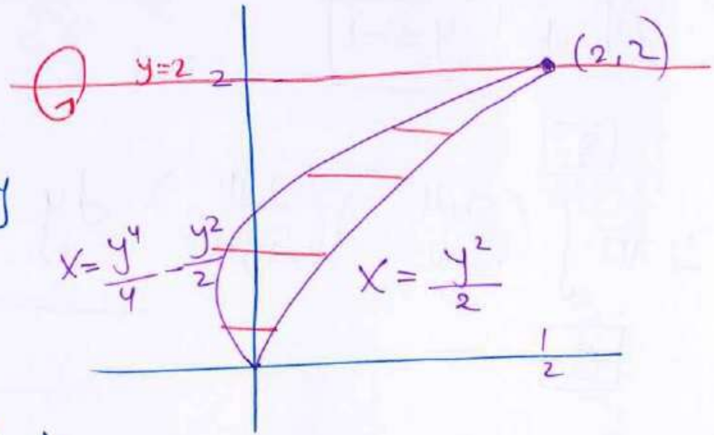
(a) About x-axis

$$V = 2\pi \int_{\boxed{0}}^{\boxed{2}} \begin{pmatrix} \text{shell} \\ \text{radius} \end{pmatrix} \begin{pmatrix} \text{shell} \\ \text{length} \end{pmatrix} dy$$

$$= 2\pi \int_0^2 (y)\left(\frac{y^2}{2} - \frac{y^4}{4} + \frac{y^2}{2}\right) dy$$

$$= 2\pi \int_0^2 \left(y^3 - \frac{y^5}{4}\right) dy = \boxed{\frac{8\pi}{3}}$$

(b) About $y=2$

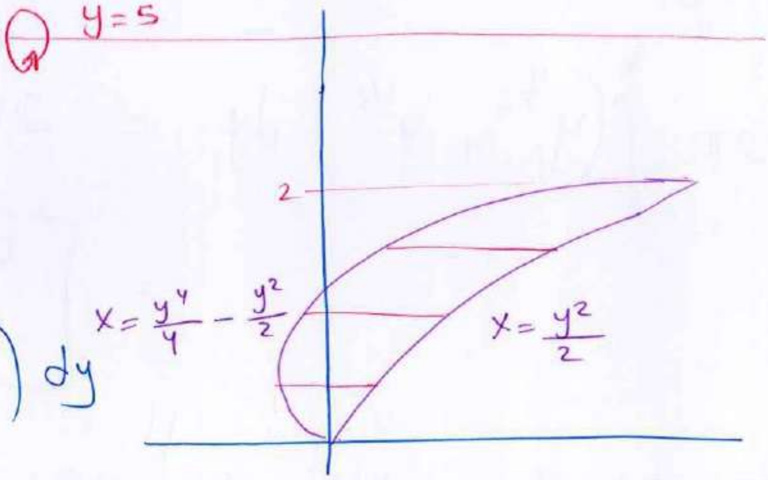


$$V = 2\pi \int_{\boxed{0}}^{\boxed{2}} \begin{pmatrix}\text{shell}\\ \text{radius}\end{pmatrix}\begin{pmatrix}\text{shell}\\ \text{length}\end{pmatrix} dy$$

$$= 2\pi \int_0^2 (2-y)\left(\frac{y^2}{2}-\frac{y^4}{4}+\frac{y^2}{2}\right) dy$$

$$= \boxed{\frac{8\pi}{5}}$$

(c) About $y=5$

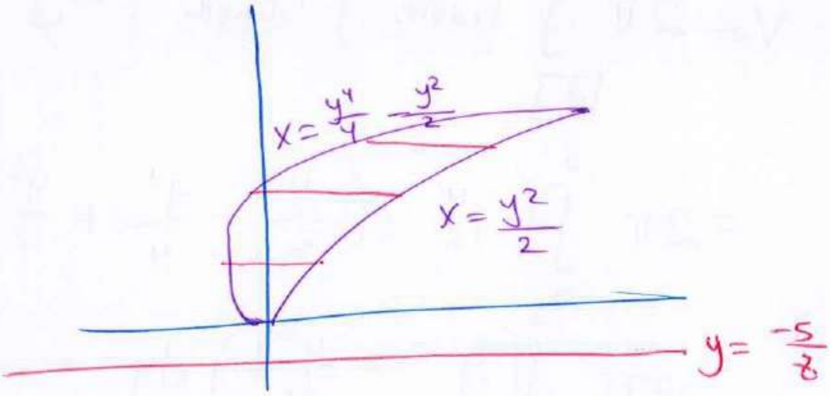


$$V = 2\pi \int_{\boxed{0}}^{\boxed{2}} \begin{pmatrix}\text{shell}\\ \text{radius}\end{pmatrix}\begin{pmatrix}\text{shell}\\ \text{length}\end{pmatrix} dy$$

$$= 2\pi \int_0^2 (5-y)\left(\frac{y^2}{2}-\frac{y^4}{4}+\frac{y^2}{2}\right) dy$$

$$= \boxed{8\pi}$$

(e) About $y=\frac{-5}{8}$



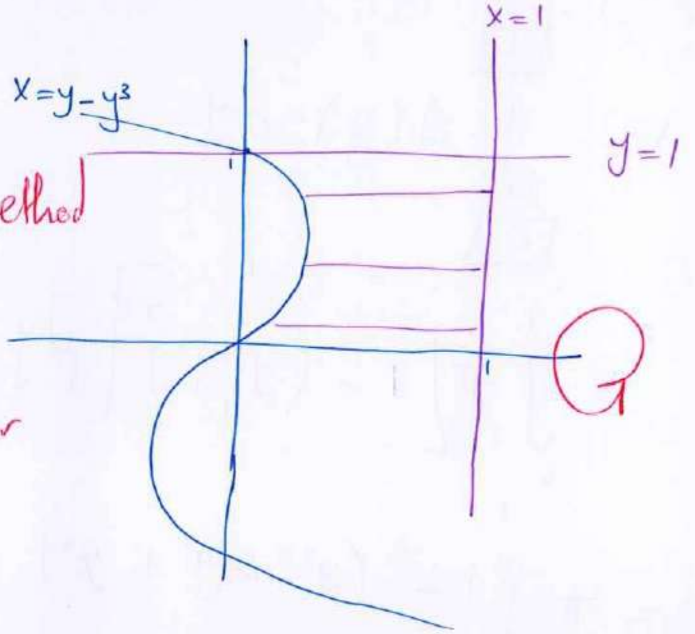
$$V = 2\pi \int_{\boxed{0}}^{\boxed{2}} \begin{pmatrix}\text{shell}\\ \text{radius}\end{pmatrix}\begin{pmatrix}\text{shell}\\ \text{length}\end{pmatrix} dy$$

$$= 2\pi \int_0^2 \left(y+\frac{5}{8}\right)\left(\frac{y^2}{2}-\frac{y^4}{4}+\frac{y^2}{2}\right) dy$$

$$\boxed{4\pi}$$

$\boxed{34}$ Find the volume in the First quadrant bounded by

$x = y - y^3$, $x = 1$, $y = 1$



(a) About x_axis

(b) About y-axis

(c) About $x=1$

(d) About $y=1$

Shell Method

Cross-section Disk or Washer

(a) About x_axis (Shell Method)

$$V = 2\pi \int_{\boxed{0}}^{\boxed{1}} \begin{pmatrix}\text{Shell}\\ \text{radius}\end{pmatrix} \begin{pmatrix}\text{shell}\\ \text{length}\end{pmatrix} dy$$

$$= 2\pi \int_0^1 y\left(1 - y + y^3\right) dy = \boxed{\frac{11\ \pi}{15}}$$

Note: in part (a) We can't use Disk Method, since we have to write $y = f(x)$ which is hard

(d) About $y=1$ (Shell Method)

$$V = 2\pi \int_0^1 \begin{pmatrix}\text{shell}\\ \text{radius}\end{pmatrix} \begin{pmatrix}\text{shell}\\ \text{length}\end{pmatrix} dy$$

$$= 2\pi \int_0^1 (1-y)\left(1 - y + y^3\right) dy$$

$$= \boxed{\frac{23}{30}\pi}$$

(b) About y-axis
(By washer Method)

$$V = \int_{\boxed{0}}^{\boxed{1}} A(y)\, dy$$

$$= \int_0^1 \pi\left[1 - (y-y^3)^2\right] dy$$

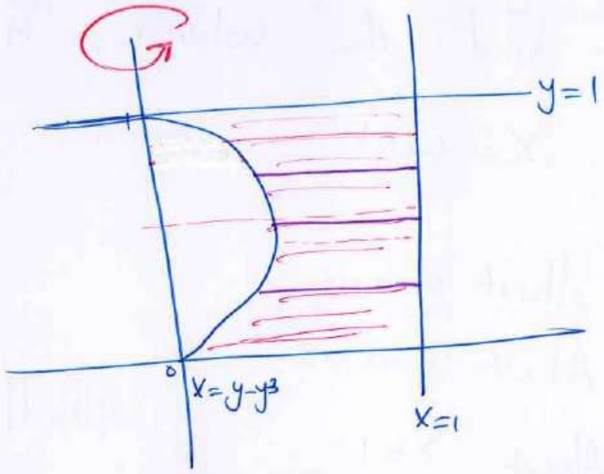
$R(y) = 1$

$r(y) = y - y^3$

$$= \pi \int_0^1 1 - (y^2 - 2y^4 + y^6)\ dy$$

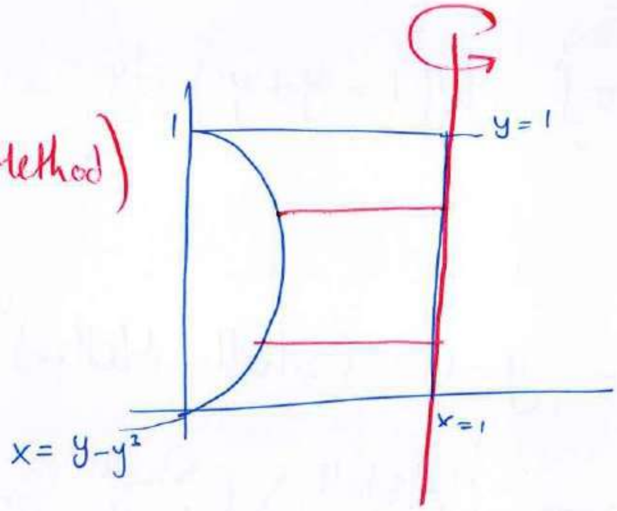
$$= \pi \int_0^1 (1 - y^2 + 2y^4 - y^6)\ dy$$

$$= \boxed{\frac{97\pi}{105}}$$



(c) About $\boxed{x=1}$ (Disk Method)

$$V = \int_{\boxed{\ }}^{\boxed{1}} A(y)\, dy$$



$$V = \int_0^1 \pi\, R^2(y)\, dy$$

$R(y) = 1 - (y - y^3)$

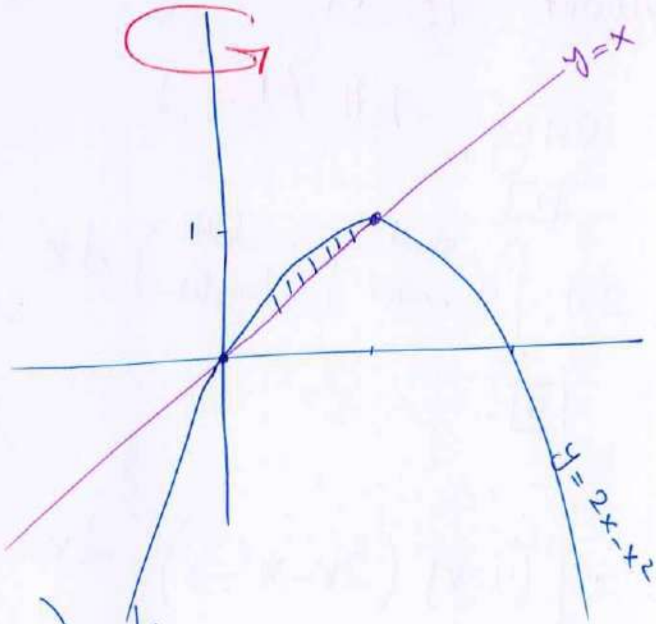
$$\pi \int_0^1 (1 - y + y^3)^2\ dy = \pi \int_0^1 1 - 2(y-y^3) + (y-y^3)^2\, dy$$

$$= \boxed{\frac{121\pi}{210}}$$

$\boxed{36}$ — The region bounded by $y = 2x - x^2$ & $y = x$

ⓐ About the y-axis.

ⓑ About $x = 1$.



ⓐ About y-axis

First: By shell Method

$$V = 2\pi \int_{\boxed{0}}^{\boxed{1}} \left(\begin{matrix}\text{shell}\\ \text{radius}\end{matrix}\right)\left(\begin{matrix}\text{shell}\\ \text{height}\end{matrix}\right) dx$$

$$= 2\pi \int_0^1 x\,(2x - x^2 - x)\,dx$$

$$= 2\pi \int_0^1 2x^2 - x^3 - x^2 \; dx \quad = 2\pi \int_0^1 (x^2 - x^3)\,dx$$

$$= 2\pi \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{\pi}{6}$$

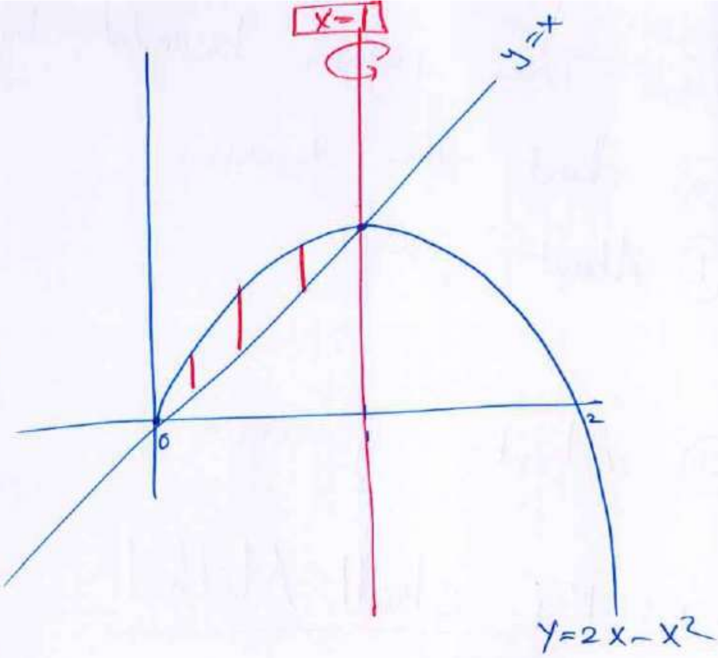
Second: By Washer Method

$\cancel{V = \int A(y)\,dy}$ Washer Method can't use since we have to write $x = f(y)$ which is hard.

(b) About $\boxed{x=1}$

By using shell Method

$$V = 2\pi \int_{\boxed{0}}^{\boxed{1}} \left(\begin{matrix}\text{shell}\\ \text{radius}\end{matrix}\right) \left(\begin{matrix}\text{shell}\\ \text{height}\end{matrix}\right) dx$$



$$= 2\pi \int_0^1 (1-x)\left(2x-x^2-x\right) dx$$

$$= 2\pi \int_0^1 (1-x)\left(x-x^2\right) dx = 2\pi \int_0^1 x - x^2 - x^2 + x^3 \, dx$$

$$= 2\pi \int_0^1 \left(x - 2x^2 + x^3\right) dx$$

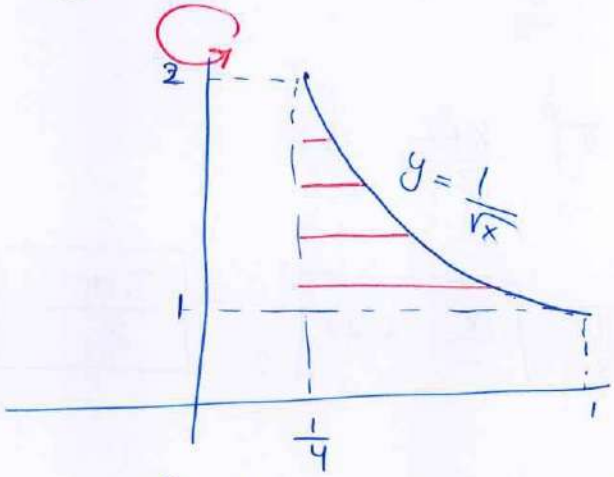
$$= 2\pi \cdot \frac{1}{12} = \boxed{\frac{\pi}{6}}$$

$\boxed{38}$ The region in the First quadrant that is bounded above by $y = \frac{1}{\sqrt{x}}$, on the left by $x = \frac{1}{4}$, and below by $y = 1$ is revolved about the y-axis to generate a solid

Find the volume of the solid by

ⓐ Washer Method

ⓑ Shell Method



ⓐ Washer Method

$$V = \int_{\boxed{1}}^{\boxed{2}} A(y)\, dy$$

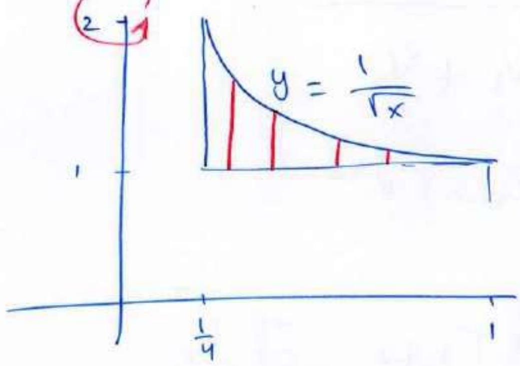
$$= \int_1^2 \pi \left[\frac{1}{y^4} - \frac{1}{16} \right] dy$$

$$= \pi \left[\frac{y^{-3}}{-3} - \frac{1}{16} y \right]_1^2$$

$$= \boxed{\frac{11\pi}{48}}$$

$R(y) = \frac{1}{y^2} - 0 = \frac{1}{y^2}$

$r(y) = \frac{1}{4} - 0 = \frac{1}{4}$



ⓑ shell Method

$$V = 2\pi \int_{\boxed{1/4}}^{\boxed{1}} \left(\text{shell radius}\right)\left(\text{shell height}\right) dx$$

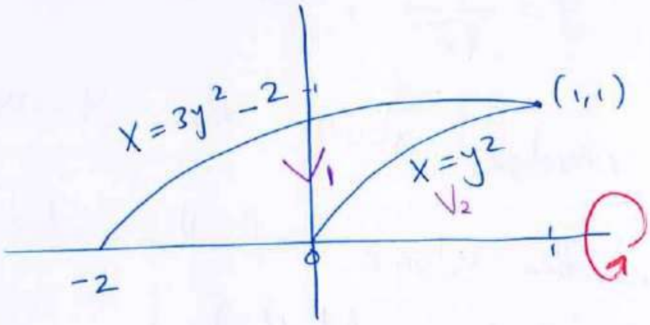
$$= 2\pi \int_{1/4}^{1} (x)\left(\frac{1}{\sqrt{x}} - 1\right) dx = 2\pi \int_{1/4}^{1} \left(x^{1/2} - x\right) dx$$

$$= 2\pi \left[\frac{2}{3} x^{3/2} - \frac{x^2}{2}\right]_{1/4}^{1} = \frac{11\pi}{48}$$

39 The region is revolved about x-axis

* Disk Method

$V = V_1 - V_2$



$V_1 = \int_{-2}^{1} A(x)\, dx$

$= \pi \int_{-2}^{1} \frac{x+2}{3}\, dx$

$= \frac{\pi}{3} \left[\frac{x^2}{2} + 2x\right]_{-2}^{1} = \boxed{\frac{3\pi}{2}}$

$V_1 \longrightarrow R(x) = \sqrt{\frac{x+2}{3}}$

$V_2 \longrightarrow R(x) = \sqrt{x}$

$V_2 = \int_{0}^{1} A(x)\, dx$

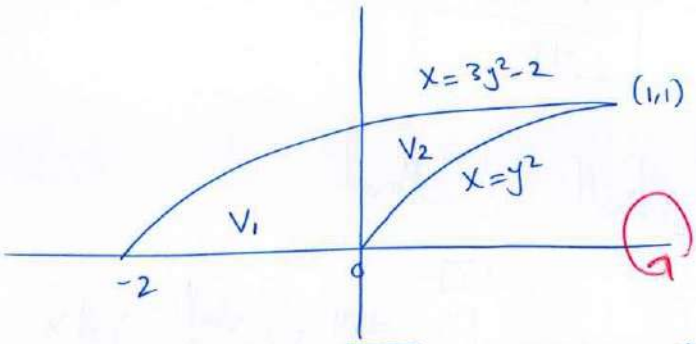
$= \pi \int_{0}^{1} x\, dx = \boxed{\frac{\pi}{2}}$

$\xrightarrow{\text{Then}}$ $\boxed{V = \frac{3\pi}{2} - \frac{\pi}{2} = \pi}$

Disk : 2 integrals

* Washer Method

$V = V_1 + V_2$



$V_1 = \int_{-2}^{0} A(x)\, dx$

$= \pi \int_{-2}^{0} \left[\frac{x+2}{3} - 0\right] dx$

$= \frac{\pi}{3} \left[\frac{x^2}{2} + 2x\right]_{-2}^{0} = \frac{2\pi}{3}$

$V_1 \longrightarrow R(x) = \sqrt{\frac{x+2}{3}}$ $\quad r(x) = 0$

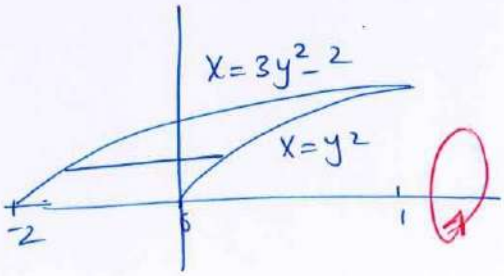
$V_2 \longrightarrow R(x) = \sqrt{\frac{x+2}{3}}$ $\quad r(x) = \sqrt{x}$

$V_2 = \int_{0}^{1} A(x)\, dx = \pi \int_{0}^{1} \left[\frac{x+2}{3} - x\right] dx = \frac{\pi}{3}$

$\xrightarrow{\text{then}}$ $V = V_1 + V_2 = \pi$ Washer : 2 integrals

* Shell Method:

$$V = 2\pi \int_{2}^{1} \begin{pmatrix} \text{shell} \\ \text{radius} \end{pmatrix} \begin{pmatrix} \text{shell} \\ \text{Length} \end{pmatrix} dy$$



$$= 2\pi \int_0^1 y\left(y^2 - 3y^2 + 2\right) dy$$

$$= 2\pi \int_0^1 -2y^3 + 2y \; dy = \pi$$

$$\Longrightarrow \boxed{V = \pi}$$

Shell Method: 1 Integral