

Examples:

① $f(x) = x^2 + x^4 + 4$

$f'(x) = 2x + 4x^3$

② $f(x) = \frac{x+1}{x^2-1}$

$f'(x) = \frac{(x^2-1)(1) - (x+1)(2x)}{(x^2-1)^2} = \frac{x^2-1-2x^2-2x}{(x^2-1)^2}$

$= \frac{-x^2-2x-1}{(x^2-1)^2}$

③ $\frac{d}{dx}(x^3+2x)^4 = 4(x^3+2x)^3 \cdot (3x^2+2)$

Example: Find the Horizontal tangent for $f(x) = x^4 - 2x^2 + 2$?

Horizontal tangent occurs at $f'(x) = 0$.

$\Rightarrow f'(x) = 4x^3 - 4x = 4x(x^2-1) = 0$

$\Rightarrow f'(x) = 0$ iff $x = 0, 1, -1$.

3.3 Derivatives of Trigonometric functions:

$$1) (\sin x)' = \cos x$$

$$2) (\cos x)' = -\sin x$$

$$3) (\tan x)' = \sec^2 x.$$

$$4) (\sec x)' = \sec x \tan x$$

$$5) (\csc x)' = -\csc x \cot x$$

$$6) (\cot x)' = -\csc^2 x.$$

Examples: ① $y = x^2 \cdot \sin x$

$$y' = x^2 \cos x + 2x \sin x$$

② $y = \tan(x^3)$

$$y' = \sec^2(x^3) \cdot (3x^2) = 3x^2 \sec^2(x^3).$$

③ $\frac{d}{dx} (\cos(\sqrt{x})) = -\sin(\sqrt{x}) \cdot (\sqrt{x})' = -\sin(\sqrt{x}) \left(\frac{1}{2\sqrt{x}}\right)$
 $= -\frac{1}{2\sqrt{x}} \sin(\sqrt{x})$

$$\begin{aligned}
 \textcircled{4} \quad \frac{d}{dx} (\sec x \tan x) &= \sec x (\tan x)' + \tan x (\sec x)' \\
 &= \sec x \cdot \sec^2 x + \tan x \sec x \tan x \\
 &= \sec^3 x + \sec x \tan^2 x.
 \end{aligned}$$

$$\textcircled{5} \quad y = \sqrt{f(x)} \quad , \quad f(x) > 0$$

$$\begin{aligned}
 y' &= \left[(f(x))^{\frac{1}{2}} \right]' = \frac{1}{2} (f(x))^{\frac{1}{2}-1} (f'(x)) \\
 &= \frac{1}{2} \cdot \frac{1}{\sqrt{f(x)}} \cdot f'(x).
 \end{aligned}$$

$$\textcircled{6} \quad f(x) = \sqrt{x^2+1} = (x^2+1)^{\frac{1}{2}}$$

$$f'(x) = \frac{1}{2} (x^2+1)^{\frac{1}{2}-1} \cdot (2x) = \frac{x}{\sqrt{x^2+1}}$$

$$\textcircled{7} \quad \frac{d}{dt} \left(\frac{\tan t}{1+\sec t} \right) = \frac{(1+\sec t)(\sec^2 t) - (\tan t)(\sec t \tan t)}{(1+\sec t)^2}$$

Example: Find the equation of the tangent line

to the curve $f(x) = \sec x \tan x$ at $x = \frac{\pi}{4}$.

Then find the Normal Line to $f(x)$ at $x = \frac{\pi}{4}$.

So: Equation of the Line is

$$y - y_1 = m(x - x_1), \quad m = \text{slope.}$$

$$x_1 = \frac{\pi}{4} \Rightarrow y_1 = f(x_1) = f\left(\frac{\pi}{4}\right) = \sec \frac{\pi}{4} \tan \frac{\pi}{4} = (\sqrt{2})(1) = \sqrt{2}.$$

$$m = f'\left(\frac{\pi}{4}\right) \Rightarrow f'(x) = \sec^3 x + \tan^2 x \sec x$$

$$\Rightarrow f'\left(\frac{\pi}{4}\right) = \sec^3 \frac{\pi}{4} + \tan^2 \frac{\pi}{4} \sec \frac{\pi}{4}$$

$$= (\sqrt{2})^3 + (1)^2 (\sqrt{2})$$

$$= 2\sqrt{2} + \sqrt{2} = \boxed{3\sqrt{2}}$$

$$\Rightarrow \text{Tangent Line: } y - \sqrt{2} = 3\sqrt{2}\left(x - \frac{\pi}{4}\right)$$

$$\Rightarrow y = 3\sqrt{2}x - 3\sqrt{2}\left(\frac{\pi}{4}\right) + \sqrt{2}$$

$$\text{Normal Line: } y - y_1 = \frac{-1}{m}(x - x_1)$$

$$y - \sqrt{2} = \frac{-1}{3\sqrt{2}}\left(x - \frac{\pi}{4}\right)$$

3.4 Implicit differentiation:

- 1) Differentiate both sides of the equation with respect to x , treating y as a differentiable function of x .
- 2) Collect the terms $\frac{dy}{dx}$ on one side of the equation and solve $\frac{dy}{dx}$.

Example: Find $\frac{dy}{dx}$ if $y^2 = x^2 + \sin xy$.

$$2y y' = 2x + \cos(xy) \left[x \frac{dy}{dx} + y \right]$$

$$2y y' = 2x + x \cos(xy) \frac{dy}{dx} + y \cos(xy)$$

$$\Rightarrow 2y y' - x \cos(xy) y' = 2x + y \cos(xy)$$

$$y' (2y - x \cos(xy)) = 2x + y \cos(xy)$$

$$\Rightarrow y' = \frac{2x + y \cos(xy)}{2y - x \cos(xy)}$$

Example: Find $\frac{d^2y}{dx^2}$ if $2x^3 - 3y^2 = 8$.

$$6x^2 - 6y y' = 0.$$

$$\Rightarrow 6y y' = 6x^2$$

$$\Rightarrow y' = \frac{x^2}{y}$$

Now:

$$y'' = \frac{y(2x) - (x^2)(y')}{y^2} = \frac{2xy - x^2(\frac{x^2}{y})}{y^2}$$

$$= \frac{2x \frac{y^2}{y} - \frac{x^4}{y}}{y^2} = \frac{2xy^2 - x^4}{y^3}$$

Example: Find y' if $xy = \cot(xy)$.

$$y' + 1 \cdot y = -\csc^2(xy)(xy' + y)$$

$$xy' + y = -xy'(\csc^2(xy)) - y \csc^2(xy)$$

$$\Rightarrow (x + x \csc^2(xy))y' = -y \csc^2(xy) - y$$

$$\Rightarrow y' = \frac{-y(1 + \csc^2(xy))}{x(1 + \csc^2(xy))} = \frac{-y}{x}$$

3.5 Linearization and Differentials:

Def: If f is differentiable at $x = a$,

then the approximation function

$$L(x) = f(a) + f'(a)(x-a)$$

is the linearization of f at a .

The approximation $f(x) \approx L(x)$ of f by L is the standard linear approximation of f at a . a is the center of the approximation.

Example: Find the linearization of $f(x) = \sqrt{1+x}$

at $x = 0$.

$$L(x) = f(0) + f'(0)(x-0)$$

$$f(0) = \sqrt{1+0} = \sqrt{1} = 1$$

$$f'(x) = \frac{1}{2\sqrt{1+x}} \Rightarrow f'(0) = \frac{1}{2\sqrt{1+0}} = \frac{1}{2(1)} = \frac{1}{2}$$

$$\Rightarrow L(x) = 1 + \frac{1}{2}x \approx f(x) = \sqrt{1+x}$$

When $x = 0.002 \Rightarrow \sqrt{1+0.002} \approx 1 + \frac{1}{2}(0.002) = 1.001$.

Example: Find the Linearization of $f(x) = \sec x$

at $x = \frac{\pi}{4}$.

$$L(x) = f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right)$$

$$f\left(\frac{\pi}{4}\right) = \sec\left(\frac{\pi}{4}\right) = \sqrt{2}.$$

$$f'(x) = \sec x \tan x \Rightarrow f'\left(\frac{\pi}{4}\right) = \sec \frac{\pi}{4} \tan \frac{\pi}{4} = \sqrt{2}(1) = \sqrt{2}.$$

$$\Delta L(x) = \sqrt{2} + \sqrt{2}\left(x - \frac{\pi}{4}\right)$$

We conclude: $\sec x \approx \sqrt{2} + \sqrt{2}\left(x - \frac{\pi}{4}\right)$ at $x = \frac{\pi}{4}$.

Example: Approximate $\sqrt[3]{8.01}$.

$$f(x) = \sqrt[3]{x} \quad \text{at } a = 8$$

$$f(a) = f(8) = \sqrt[3]{8} = 2$$

$$f'(x) = \frac{1}{3} x^{-2/3} \Rightarrow f'(8) = \frac{1}{3}(8)^{-2/3} = \frac{1}{12}.$$

$$\Rightarrow L(x) = f(8) + f'(8)(x-8) = 2 + \frac{1}{12}(x-8) \approx \sqrt[3]{x}$$

$$\Rightarrow \sqrt[3]{8.01} \approx 2 + \frac{1}{12}(8.01-8) \approx 2.0008$$

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Differentials:

Def: Let $y = f(x)$ be differentiable function.

The differential dx is an Independent Variable

The differential dy is $dy = f'(x) dx$

Note: If the ratio of dy and dx exists, it is equal to the derivative.

$dy = f'(x) dx$ is differential $\approx \Delta f$

while $\frac{dy}{dx} = f'(x)$ is derivative.

Example: (a) Find dy if $y = x^5 + 37x$.

(b) Find the value of dy when $x=1$ and $dx=0.2$

sol:

(a) $dy = (5x^4 + 37) dx$

(b) $dy = (5(1)^4 + 37)(0.2) = (42)(0.2) = 8.4$

Example: The radius r of a circle increases from $r = 10$ to 10.1 . Use dA to estimate the increase in the circle's area.

Estimate the area of the enlarged circle and compare your estimate to the true area found by direct calculation.

Sol: Area of circle:

$$A = \pi r^2.$$

$$\Delta A \approx dA = 2\pi r \underbrace{dr}_{\approx \Delta r} = 2\pi (10)(0.1) = 2\pi.$$

$$\text{When } r=10 \Rightarrow \text{Area} = \pi(10)^2 = 100\pi.$$

$$\text{Exact Area for the enlarged} = \pi(10.1)^2 = 102.01\pi \text{ m}^2$$

$$\text{Approximate area} \approx A + dA = 100\pi + 2\pi = 102\pi \text{ m}^2$$

$$\begin{aligned} \Rightarrow \text{Error} &= | \text{Exact} - \text{Approximation} | \\ &= | 102.01\pi - 102\pi | = 0.01\pi. \end{aligned}$$

Chapter 4 : Applications of derivatives :

4.1 Increasing and decreasing functions :

Def: Let $f(x)$ be a function defined on an Interval I .

(a) f is increasing on I whenever $x_2 > x_1$, then
 $f(x_2) > f(x_1)$, for all $x_1, x_2 \in I$.

(b) f is decreasing on I whenever $x_2 > x_1$, then
 $f(x_2) < f(x_1)$, for all $x_1, x_2 \in I$

Thm: Suppose that f is continuous on $[a, b]$
and differentiable on (a, b) , then:

(a) If $f'(x) > 0$, $\forall x \in (a, b)$, then f is increasing
on $[a, b]$.

(b) If $f'(x) < 0$, $\forall x \in (a, b)$, then f is decreasing
on $[a, b]$.

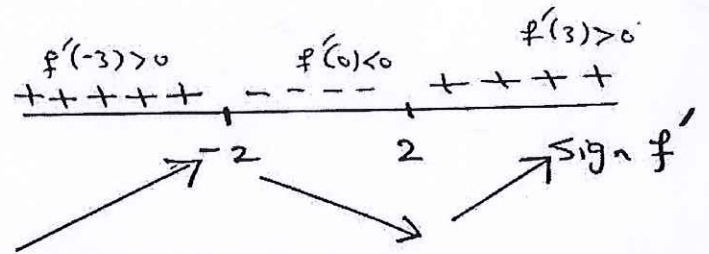
Example: Let $f(x) = x^3 - 12x - 5$.

Identify the intervals on which f is increasing and decreasing.

Sol: $f'(x) = 3x^2 - 12 = 0$

$$3(x^2 - 4) = 3(x-2)(x+2) = 0$$

$$\Rightarrow x = 2, x = -2$$



Then

f is increasing on $(-\infty, -2] \cup [2, \infty)$

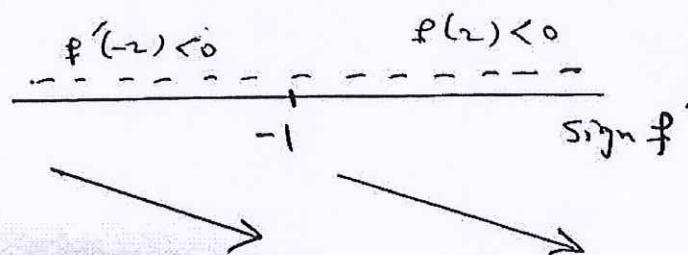
f is decreasing on $[-2, 2]$

Example: Let $f(x) = 1 - (x+1)^3$. Identify

the intervals on which f is increasing and decreasing.

Sol: $f'(x) = 0 - 3(x+1)^2 = -3(x+1)^2 = 0$

$$\Rightarrow \boxed{x = -1}$$



$\Rightarrow f$ is decreasing on

$$(-\infty, -1] \cup [-1, \infty) = (-\infty, \infty).$$

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4.2 Extreme values of functions:

Def: Let f be a function with domain D . Then:

(a) f has an ^{أعلى قيمة} absolute maximum value on D at a point c if $f(x) \leq f(c)$, $\forall x \in D$.

(b) f has an ^{أدنى قيمة} absolute minimum value on D at a point c if $f(x) \geq f(c)$, $\forall x \in D$.

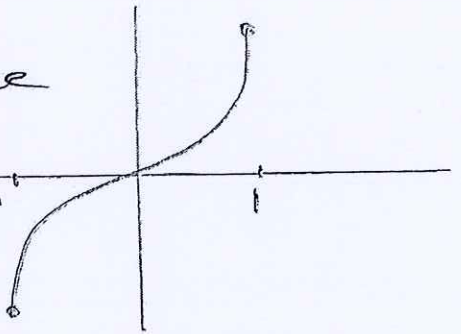
Note: $f(c)$ is called ^{أعلى قيمة} Local maximum (resp. ^{أدنى قيمة} local Min)

if the inequality in (a) (resp. (b)) holds in a small interval around $x = c$.

Example: $f(x) = x^3$, $D = [-1, 1]$

• $f(-1) = -1$ is absolute min value
since $f(x) \geq f(-1)$, $\forall x \in [-1, 1]$

• $f(1) = 1$ is absolute Max value
since $f(x) \leq f(1) = 1$, $\forall x \in [-1, 1]$.



Note: If $f(x) = x^3$, $D = (-1, 1)$, then f has neither a maximum nor minimum.

Thm: If f is continuous function on a closed interval $[a, b]$ then f has both an absolute maximum value and an absolute minimum value.

Example: $f(x) = x^3$, $D = [-1, 1]$.
 f has both absolute maximum $f(1) = 1$ and absolute minimum $f(-1) = -1$

Note: If we want to find the extreme values of a function f on a closed interval, we look for these values at the endpoints of the Interval and at the interior points where $f' = 0$ or f' is undefined. (critical points)

Def: An interior point of the domain of a function f where $f' = 0$ or f' is undefined is called critical point of f .

Example: Find the absolute Max. and Absolute Min.

of $f(x) = x^{2/3}$ on $[-1, 8]$.

sol: critical points: $f'(x) = \frac{2}{3} x^{2/3-1}$

$\Rightarrow f'(x) = \frac{2}{3 x^{1/3}}$ critical point

$\Rightarrow f'(x) \neq 0$ & $f'(x)$ is undefined at $x=0$

$f(-1) = (-1)^{2/3} = \sqrt[3]{(-1)^2} = 1$

$f(0) = (0)^{2/3} = 0$ ←

$f(8) = (8)^{2/3} = \sqrt[3]{(8)^2} = \sqrt[3]{64} = 4$ ←

$\Rightarrow f(0) = 0$ is an absolute minimum value

$f(8) = 4$ is an absolute Maximum value.

Thm: If f is differentiable and has an extreme value at an interior point c , then $f'(c) = 0$

Note: If $f'(c) = 0$, then this does not mean that f has an extreme value (Max or Min) at $x = c$

Example: $f(x) = x^3$, $f'(x) = 3x^2$

$f'(0) = 3(0)^2 = 0$, but 0 is neither Maximum nor Minimum value of $f(x)$.

Thm: First derivative test: suppose that

f has a critical point at $x = c$ & $f'(x)$ exists in an open interval containing $x = c$. Then:

- 1) If f' changes sign from ⁺positive to ⁻negative at $x = c$, then $f(c)$ is a local Maximum.
- 2) If f' changes sign from ⁻negative to ⁺positive at $x = c$, then $f(c)$ is a local Minimum.

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(c) If f' does not change sign at $x=c$, then f does not have an extreme value at $x=c$.

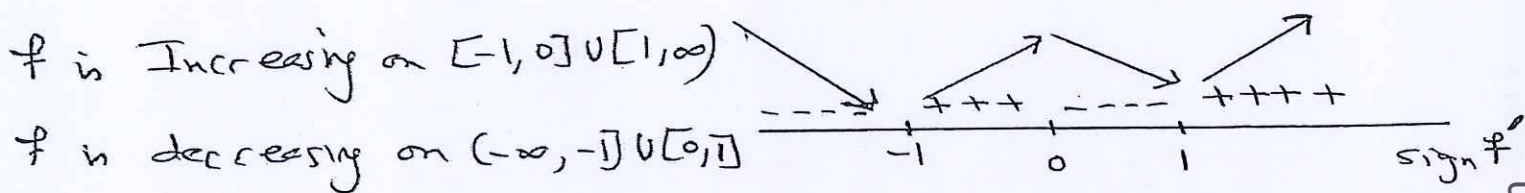
Example: Let $f(x) = x^4 - 2x^2$.

- (a) Find the critical points of f .
- (b) Identify the intervals on which f is increasing and decreasing.
- (c) Find the function's local and absolute extreme values.

Sol: Domain $(-\infty, \infty)$

a) Critical points: $f'(x) = 4x^3 - 4x = 4x(x^2 - 1)$
 $= 4x(x-1)(x+1) = 0$

then $x = 0, x = 1, x = -1$ are critical points of f .



$f(-1) = -1$ is local Min. value \rightarrow (absolute Min)

$f(0) = 0$ is local Max. value

$f(1) = -1$ is local Min. value \rightarrow (absolute Min)

Def: ① If $f''(x) \geq 0, \forall x \in I$ then

f is ^{concave up} Concave up on I

If $f''(x) \leq 0, \forall x \in I$, then f is ^{concave down} Concave down on I

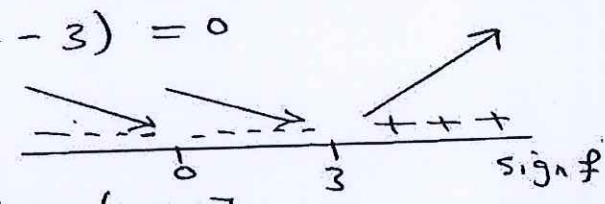
Def: A point where f has tangent line and ^{changes concavity} changes concavity is called an Inflection point of f

Example: Find the Intervals where the function $f(x) = x^4 - 4x^3 + 10$ is increasing, decreasing, concave up, concave down.

Sol: $D = (-\infty, \infty)$

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3) = 0$$

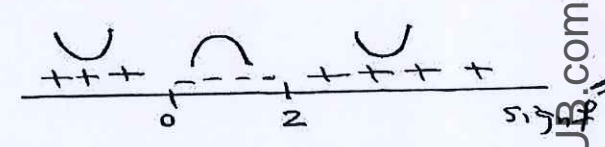
\Rightarrow critical points $x = 0, x = 3$



f is Increasing on $(3, \infty)$ & decreasing on $(-\infty, 3]$

$$f''(x) = 12x^2 - 24x = 12x(x - 2) = 0$$

$x = 2, x = 0$



f is Concave up on $(-\infty, 0] \cup [2, \infty)$
 f is Concave down on $[0, 2]$

$\Rightarrow (0, 10)$ & $(2, -6)$ are Inflection points.

Theorem: Second derivative test

Suppose that $f'(c) = 0$ and f'' is continuous on an open interval containing c . Then:

- (a) If $f''(c) < 0$, then $f(c)$ is a Local Maximum.
- (b) If $f''(c) > 0$, then $f(c)$ is a Local Minimum.
- (c) If $f''(c) = 0$, then the test fails.

Example: Use 2nd derivative test to find an extreme values of $f(x) = x^4 - 2x^2$.

Sol: $f'(x) = 4x^3 - 4x = 4(x)(x^2 - 1) = 0$
 $\Rightarrow x = 0, x = 1, x = -1$

$f''(x) = 12x^2 - 4$

at $x = 0 \Rightarrow f''(0) = -4 < 0 \Rightarrow f(0) = 0$ is Local Max.

at $x = 1 \Rightarrow f''(1) = 8 > 0 \Rightarrow f(1) = -1$ is Local Min.

at $x = -1 \Rightarrow f''(-1) = 8 > 0 \Rightarrow f(-1) = -1$ is Local Min.

Example: Consider $f(x) = \frac{x^2}{x+1}$, sketch f .

$$f'(x) = \frac{x^2 + 2x}{(x+1)^2}, \quad f''(x) = \frac{2}{(x+1)^3}$$

1) Domain: $(-\infty, \infty) \setminus \{-1\}$

2) $\lim_{x \rightarrow +\infty} \frac{x^2}{x+1} = +\infty = \lim_{x \rightarrow +\infty} \frac{x^2}{x(1 + \frac{1}{x})}$

3) $\lim_{x \rightarrow -\infty} \frac{x^2}{x+1} = -\infty$

4) Horizontal Asymptote: None.

5) $\lim_{x \rightarrow -1^-} \frac{x^2}{x+1} = -\infty$

6) $\lim_{x \rightarrow -1^+} \frac{x^2}{x+1} = +\infty$

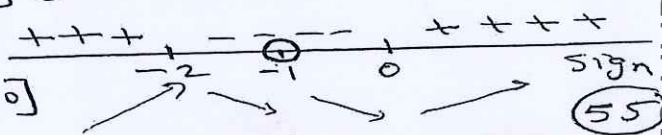
7) Vertical Asymptote: $x = -1$

8) Oblique Asy: $y = x - 1$

9) Critical points: $f'(x) = 0 \Rightarrow x = 0, -2$. $\left. \begin{array}{l} f'(x) \text{ DNE} \\ -1 \notin D \end{array} \right\}$

10) f is Increasing on $(-\infty, -2] \cup [0, \infty)$

f is decreasing on $[-2, -1) \cup (-1, 0]$



11) $f(-2) = -4$ is local Max.

12) $f(0) = 0$ is Local Min.

13) f is concave up on $(-1, \infty)$ & Concave down on $(-\infty, -1)$

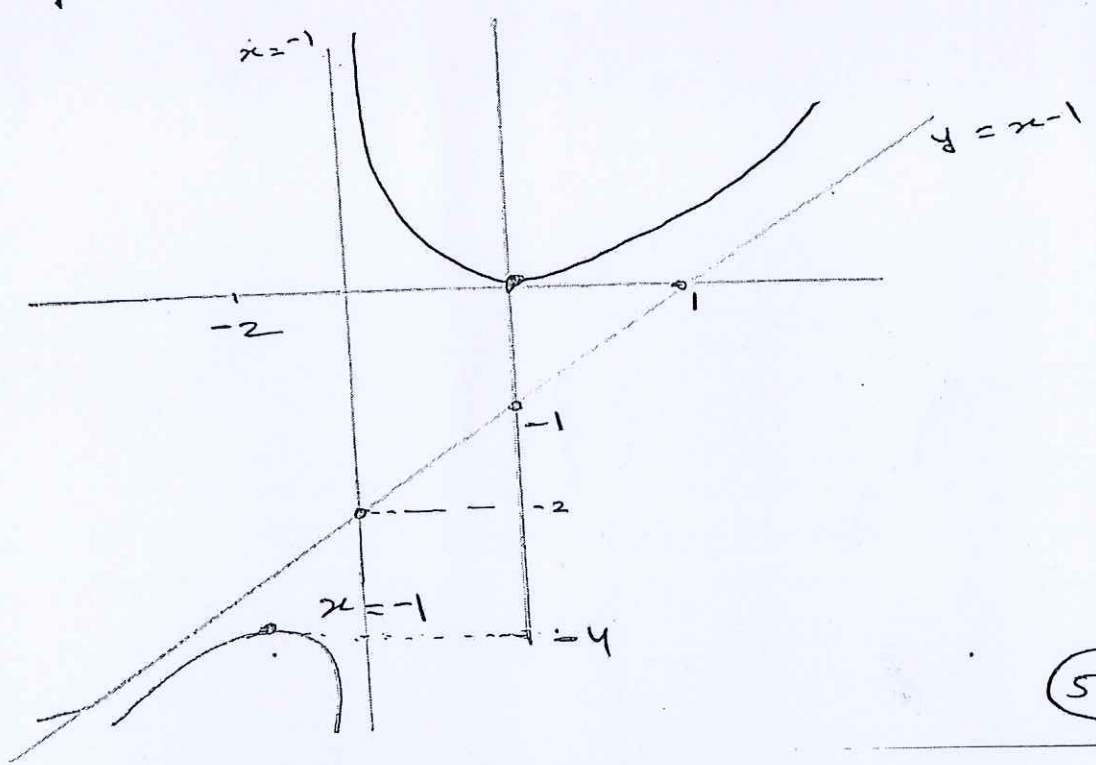
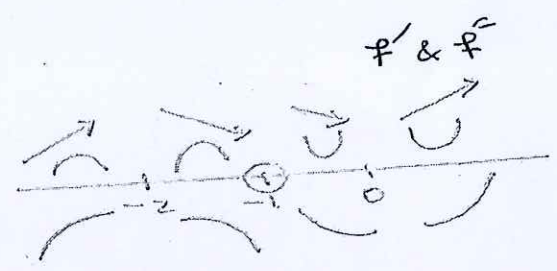
14) NO Absolute Max, NO Absolute Min.

15) NO Inflection points.

16) Range: $(-\infty, -4] \cup [0, \infty)$

17) x -intercept: $y = 0 \Rightarrow x = 0$

18) y -intercept: $x = 0 \Rightarrow y = 0$



Example: $f(x) = \frac{x^2}{x^2-1}$, sketch the graph f .

$$f'(x) = \frac{-2x}{(x^2-1)^2}, \quad f''(x) = \frac{6x^2+2}{(x^2-1)^3}$$

1) Domain: $(-\infty, \infty) \setminus \{\pm 1\}$

2) $\lim_{x \rightarrow \pm\infty} \frac{x^2}{x^2-1} = 1$

3) H.A: $y = 1$

4) $\lim_{x \rightarrow 1^-} \frac{x^2}{x^2-1} = -\infty$

5) $\lim_{x \rightarrow 1^+} \frac{x^2}{x^2-1} = +\infty$

6) $\lim_{x \rightarrow -1^-} \frac{x^2}{x^2-1} = +\infty$

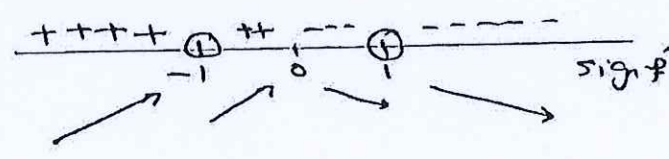
7) $\lim_{x \rightarrow -1^+} \frac{x^2}{x^2-1} = -\infty$

8) V.A: $x = 1$ & $x = -1$

9) critical points $f'(x) = 0 \Rightarrow x = 0$
 $f'(x)$ is Undefined at $x = -1, 1 \notin D$.

10) f is Increasing on $(-\infty, -1) \cup (-1, 0]$

f is decreasing on $[0, 1) \cup (1, \infty)$



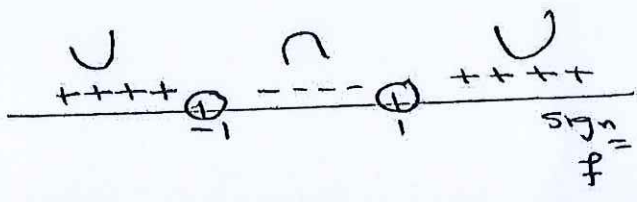
11) $f(0) = 0$ is local Max.

12) NO Local Min.

13) NO Absolute Max, NO Absolute Min.

14) f is concave up on $(-\infty, -1) \cup (1, \infty)$

and concave down on $(-1, 1)$.

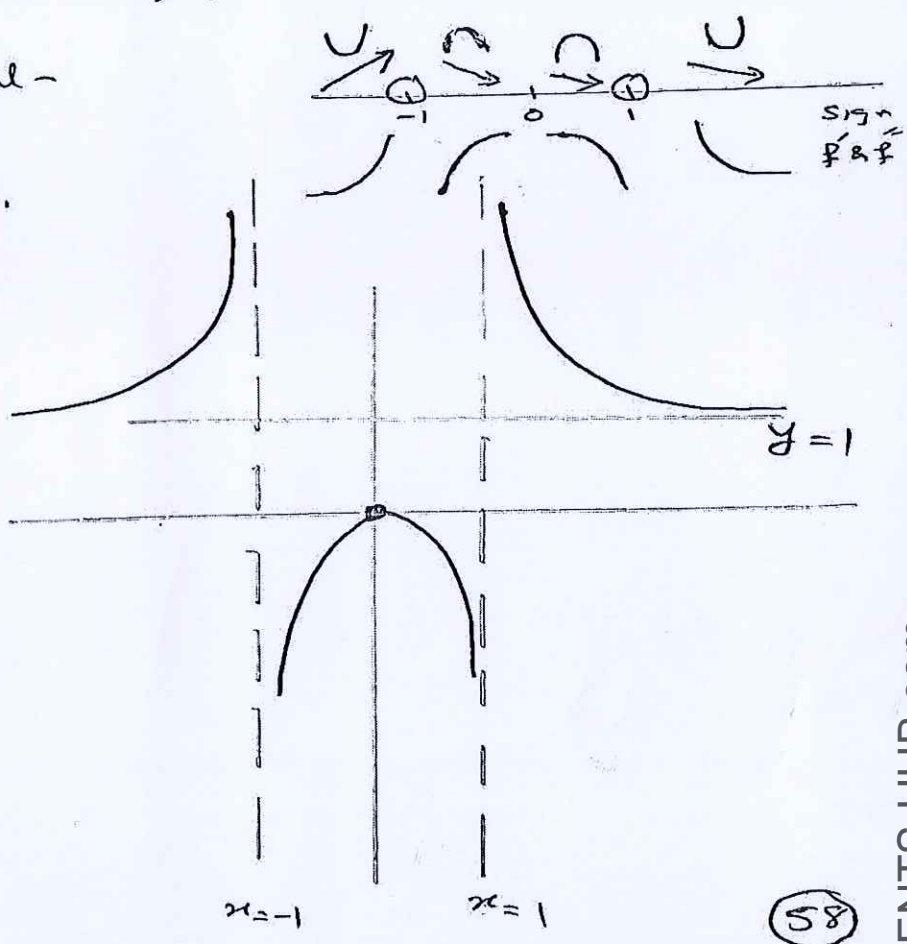


15) NO Inflection point.

16) Range: $(-\infty, 0] \cup (1, \infty)$.

17) x-intercept: $y=0, x=0$

18) y-intercept: $x=0, y=0$



Exempli $f(x) = \frac{x}{x^2+1}$

$$f'(x) = \frac{1-x^2}{(x^2+1)^2}, \quad f''(x) = \frac{2x(x^2-3)}{(x^2+1)^3}$$

1) Domain: $(-\infty, \infty)$

2) $\lim_{x \rightarrow \pm\infty} \frac{x}{x^2+1} = 0$

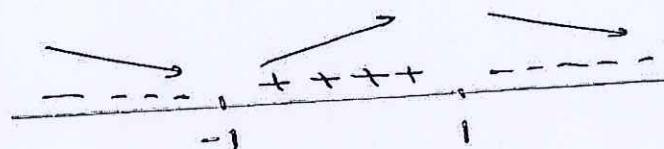
3) Horiz. Asym. $y = 0$

4) V.A: None

5) Oblique Asym. None.

6) Critical points: $x = 1, x = -1$

7) f is increasing on $[-1, 1]$
& decreasing on $(-\infty, -1] \cup [1, \infty)$



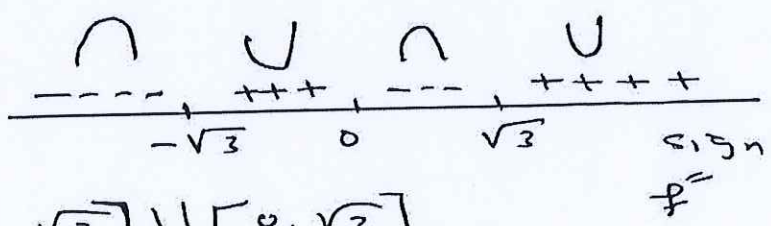
8) Local Max: $f(1) = \frac{1}{2}$

9) Local Min: $f(-1) = -\frac{1}{2}$

10) Absolute Max: $f(1) = \frac{1}{2}$

11) Absolute Min: $f(-1) = -\frac{1}{2}$

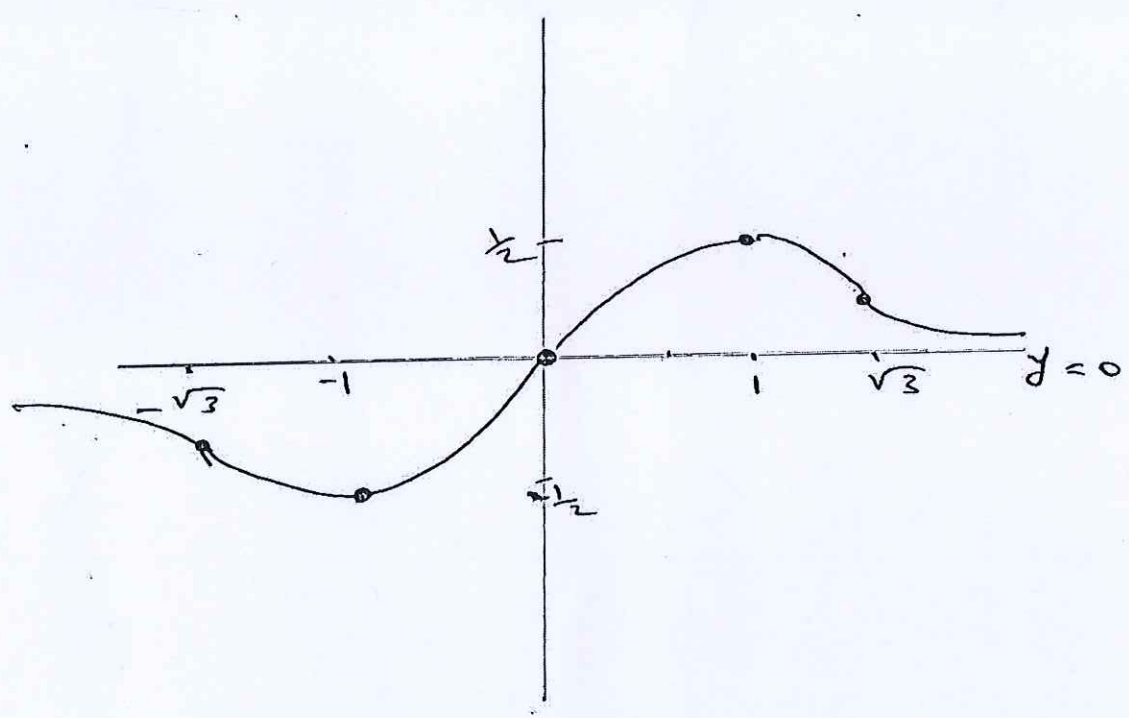
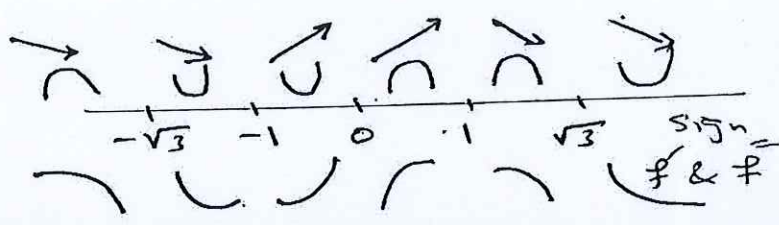
12) f is concave up
on $[-\sqrt{3}, 0] \cup [\sqrt{3}, \infty)$



and concave down $(-\infty, -\sqrt{3}] \cup [0, \sqrt{3}]$

13) Inflection points $(-\sqrt{3}, -\frac{\sqrt{3}}{4}), (0, 0), (\sqrt{3}, \frac{\sqrt{3}}{4})$

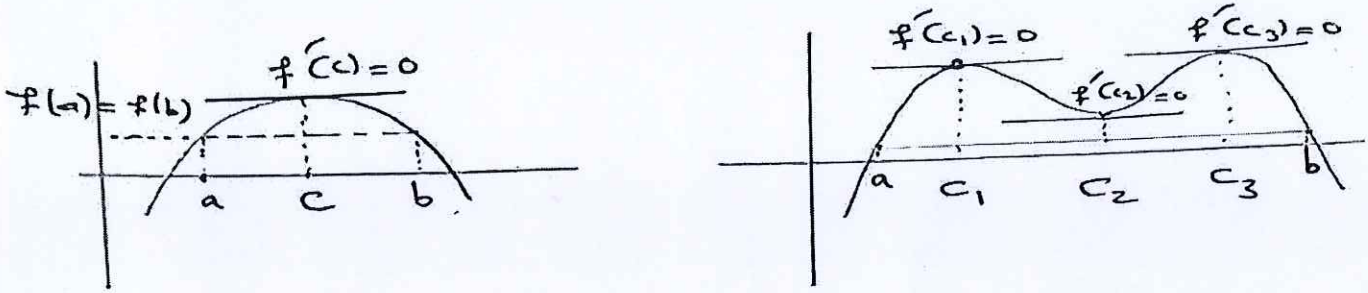
14) Range $[-\frac{1}{2}, \frac{1}{2}]$



4.3 The Mean Value Theorem

Theorem: Rolle's Theorem:

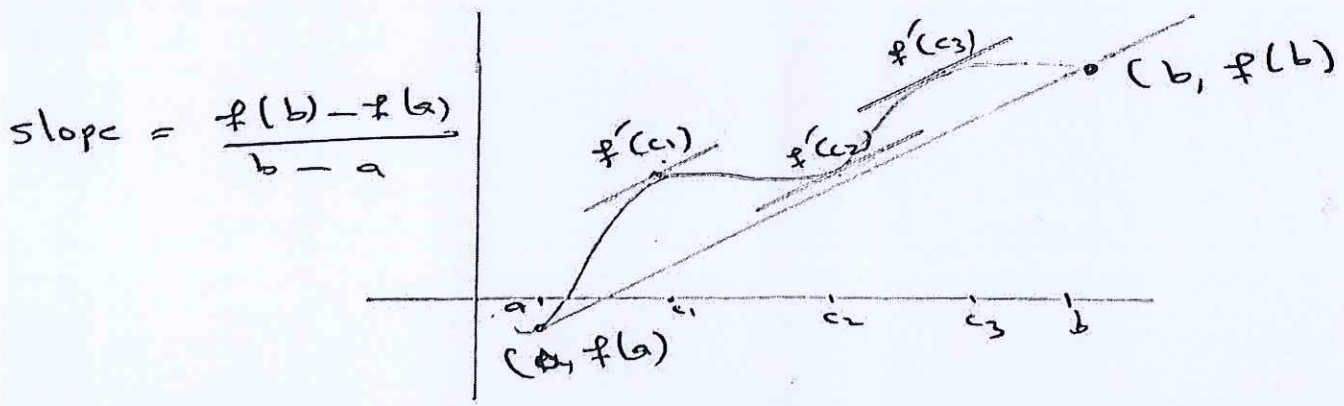
If $f(x)$ is continuous on $[a, b]$ and diff. on (a, b) and $f(a) = f(b)$, then there is at least one point $c \in (a, b)$ such that $f'(c) = 0$.



Theorem: Mean Value Theorem: (MVT)

If f is continuous on $[a, b]$ & Diff. on (a, b) , then there is at least one $c \in (a, b)$ such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



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Example: Let $f(x) = x^2, x \in [1, 4]$

Find the value(s) of c in the conclusion of the Mean Value Theorem.

Sol: 1) f is Cont. on $[1, 4]$ since it's polynomial.

2) f is diff on $(1, 4)$, since $f'(x) = 2x$.

\Rightarrow by MVT, there is $c \in (1, 4)$

such that:

$$f'(c) = \frac{f(4) - f(1)}{4 - 1}$$

$$\Rightarrow 2c = \frac{16 - 1}{3} = \frac{15}{3}$$

$$\therefore c = \frac{15}{6} = \frac{5}{2}$$

Example: Suppose $f'(x) \leq 1, \forall 1 \leq x \leq 4$.

Show that $f(4) - f(1) \leq 3$.

Sol: f is Cont. on $[1, 4]$ & diff. on $(1, 4) \Rightarrow \exists c \in (1, 4)$

$$\text{s.t. } f'(c) = \frac{f(4) - f(1)}{4 - 1} \leq 1 \Rightarrow f(4) - f(1) \leq 3$$

Chapter 5: Integration

الجزء 1

الأقسام السابقة

5.1 Antiderivative and Integration:

Def: A function F is called an antiderivative of a function f on an interval I if

$$F'(x) = f(x), \quad \forall x \in I, \quad \left[\text{i.e.} : \int f(x) dx = F(x) + C \right]$$

The set of all antiderivatives of f is called the indefinite integral of f & is denoted by $\int f(x) dx$.

Example: $F(x) = x^3$ is an antiderivative of $f(x) = 3x^2$

Since $F'(x) = 3x^2 = f(x)$ [or: $\int 3x^2 dx = x^3 + C$]

Example: (a) $\int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$

(b) $\int \sin x dx = -\cos x + C$

(c) $\int \cos x dx = \sin x + C$

$\int \sin(kx) dx = -\frac{\cos(kx)}{k} + C, \quad k \neq 0$

$$(d) \int \sec^2 x \, dx = \tan x + C$$

$$\rightarrow \int \sec^2(kx) \, dx = \frac{\tan(kx)}{k} + C, \quad k \neq 0.$$

$$(e) \int \sec x \tan x \, dx = \sec x + C$$

$$(f) \int \csc x \cot x \, dx = -\csc x + C$$

$$(g) \int \csc^2 x \, dx = -\cot x + C$$

Example: 1) $\int (x^{-2} - x^2 + 1) \, dx = \int x^{-2} \, dx - \int x^2 \, dx + \int 1 \, dx$

$$= \frac{x^{-1}}{-1} - \frac{x^3}{3} + x + C = -\frac{1}{x} - \frac{x^3}{3} + x + C$$

2) $\int \cos^2 \theta \, d\theta = \int \left(\frac{1 + \cos 2\theta}{2} \right) d\theta$

$$= \frac{1}{2} \int (1 + \cos 2\theta) \, d\theta = \frac{1}{2} \left[\theta + \frac{\sin 2\theta}{2} + C \right]$$

$$= \frac{1}{2} \theta + \frac{\sin 2\theta}{4} + C$$

$$3) \int \tan^2 x \, dx = \int (\sec^2 x - 1) \, dx$$

$$= \int \sec^2 x \, dx - \int 1 \, dx = \tan x - x + C.$$

5.2 Definite Integrals and Areas:

We denote for the Definite Integral by:

$$\int_a^b f(x) \, dx$$

← b
 Upper Limit of Integration

$$\int_a^b f(x) \, dx$$

a
 Lower Limit of Integration

Theorem: Fundamental Theorem of Calculus.

(I) Suppose that f is continuous on $[a, b]$ and F is an antiderivative of f on $[a, b]$ then:

$$\int_a^b f(x) \, dx = F(b) - F(a)$$

(II) Suppose that f is continuous on $[a, b]$ and

$$F(x) = \int_a^x f(t) \, dt, \text{ then } F \text{ is continuous on } [a, b]$$

and differentiable on (a, b) and $F'(x) = f(x)$.

$$F(x) = \int_a^x f(t) \, dt \Rightarrow F'(x) = f(x)$$

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Example: Q4) (b) $\int_0^{\frac{\pi}{6}} (\sec x + \tan x)^2 dx$

$$= \int_0^{\frac{\pi}{6}} (\sec^2 x + 2 \tan x \sec x + \underbrace{\tan^2 x}_{\sec^2 x - 1}) dx$$

$$= \int_0^{\frac{\pi}{6}} (2 \sec^2 x + 2 \tan x \sec x - 1) dx$$

$$= \left(\underbrace{2 \tan x + 2 \sec x - x}_{F(x)} \right) \Big|_0^{\frac{\pi}{6}}$$

$$= F\left(\frac{\pi}{6}\right) - F(0) = \left(2 \tan \frac{\pi}{6} + 2 \sec \frac{\pi}{6} - \frac{\pi}{6}\right) - (2 \tan 0 + 2 \sec 0 - 0)$$

$$= 2 \cdot \frac{1}{\sqrt{3}} + 2 \cdot \frac{2}{\sqrt{3}} - \frac{\pi}{6} - 2 = \dots$$

Example: Find the derivative of the following

(a) $\frac{d}{dx} \int_0^x \boxed{\sin t} dt = \sin x$

(You can first find the Integration:

$$\int_0^x \sin t dt = -\cos t \Big|_0^x = -\cos x + 1$$

Now differentiate: $\left(\int_0^x \sin t dt\right)' = (-\cos x + 1)' = \sin x$

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$$(b) \frac{d}{dx} \int_1^{x^2} \frac{dt}{1+t^2} = \frac{1}{1+(x^2)^2} \cdot 2x = \frac{2x}{1+x^4}$$

In General: $\left(\int_a^{g(x)} f(t) dt \right)' = f(g(x)) \cdot g'(x) - 0$

$$\left(\int_{f(x)}^{g(x)} h(t) dt \right)' = h(g(x)) \cdot g'(x) - h(f(x)) \cdot f'(x)$$

Example: $\frac{d}{dx} \int_{\sin x}^{x^3} \frac{1}{1+t^2} dt$

$$= \frac{1}{1+(x^3)^2} \cdot (3x^2) - \frac{1}{1+(\sin x)^2} \cdot (\cos x)$$

$$= \frac{3x^2}{1+x^6} - \frac{\cos x}{1+\sin^2 x}$$

Example: $\frac{d}{dx} \left(\int_a^b f(t) dt \right) = 0$

Note: If $f(x) \geq 0$ is an Integrable function on $[a, b]$, then $\int_a^b f(x) dx$ is the area enclosed between the Curve $f(x)$ and the x -axis.

Example: Find the area enclosed between $f(x)$ and the x -axis, where $f(x) = 2x\sqrt{x^2+1}$, $x \in [0, 1]$

$$\text{Area} = \int_0^1 2x\sqrt{x^2+1} dx$$

Let $u = x^2 + 1$, then $du = 2x dx$.

when $x = 0$, then $u = 1$ & when $x = 1$, then $u = 2$.

$$\begin{aligned} \Rightarrow \text{Area} &= \int_1^2 \sqrt{u} du = \left. \frac{2}{3} u^{\frac{3}{2}} \right|_1^2 \\ &= \frac{2}{3} \left((2)^{\frac{3}{2}} - (1)^{\frac{3}{2}} \right) = \frac{2}{3} (2\sqrt{2} - 1) = \frac{4\sqrt{2} - 2}{3} \end{aligned}$$

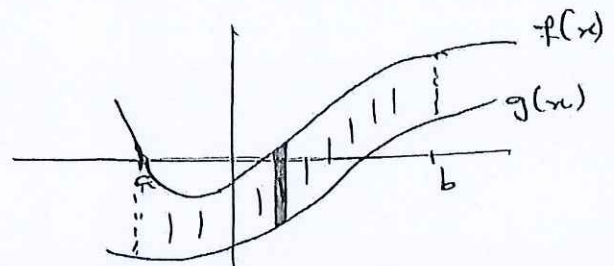
Area Between Curves:

We can find the area enclosed between two functions

$f(x)$ and $g(x)$ in some Interval $[a, b]$, where

$f(x) \geq g(x)$ using the formula:

$$A = \int_a^b (f(x) - g(x)) dx$$



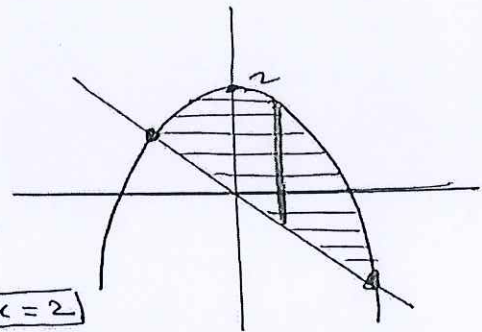
Example: Find the area enclosed between the Curves

$$f(x) = 2 - x^2 \quad \& \quad y = -x.$$

We find the Intersection points:

$$2 - x^2 = -x \Leftrightarrow x^2 - x - 2 = 0$$

$$\Leftrightarrow (x+1)(x-2) = 0 \Leftrightarrow \boxed{x = -1} \quad \& \quad \boxed{x = 2}$$



$$\Rightarrow \text{Area} = A = \int_{-1}^2 (2 - x^2) - (-x) dx = \int_{-1}^2 (2 - x^2 + x) dx.$$

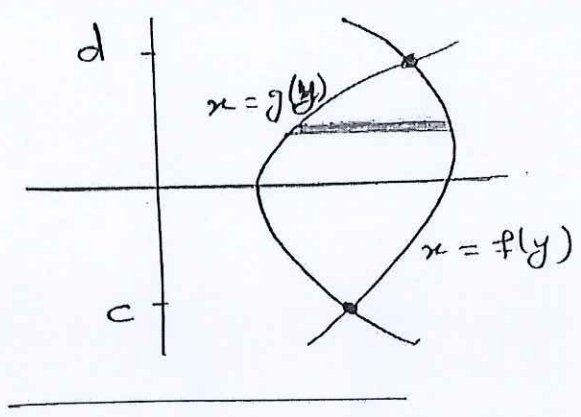
$$= 2x - \frac{x^3}{3} + \frac{x^2}{2} \Big|_{-1}^2 = \left(4 - \frac{8}{3} + \frac{4}{2}\right) - \left(-2 + \frac{1}{3} + \frac{1}{2}\right)$$
$$= \boxed{4.5}.$$

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Note: Sometimes, the functions are expressed in terms of y in some Interval $[c, d]$, so the area

in this case is $A = \int_c^d (f(y) - g(y)) dy$.

\downarrow right \downarrow left



Example: Find the area enclosed between $y = \sqrt{x}$, the x -axis and the line $y = x - 2$.

For $y = x - 2$

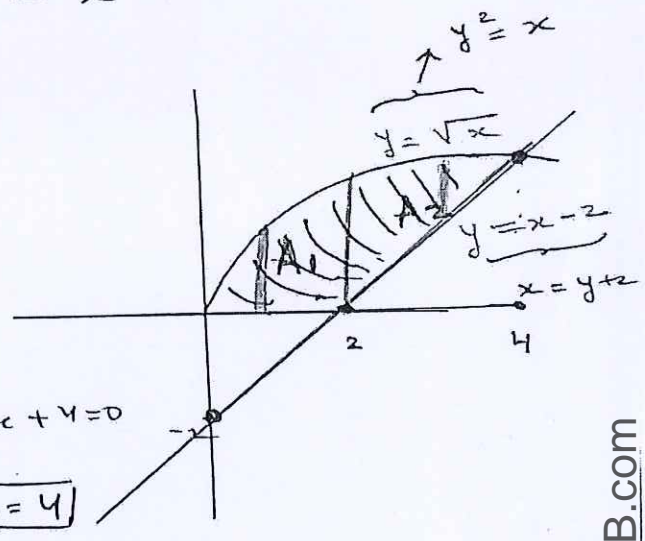
when $x = 0$, then $y = -2$
 when $y = 0$, then $x = 2$

Intersection points: $\sqrt{x} = x - 2$

$$\Leftrightarrow x = x^2 - 4x + 4 \Leftrightarrow x^2 - 5x + 4 = 0$$

$$\Leftrightarrow (x - 1)(x - 4) = 0 \Leftrightarrow \boxed{x = 1} \text{ \& } \boxed{x = 4}$$

apart



$$\Rightarrow \text{Area} = A_1 + A_2 = \int_0^2 \sqrt{x} dx + \int_2^4 (\sqrt{x} - (x - 2)) dx$$

$$= \left[\frac{2}{3} x^{3/2} \right]_0^2 + \left[\frac{2}{3} x^{3/2} - \frac{x^2}{2} + 2x \right]_2^4$$

$$= \frac{2}{3} (\sqrt{2^3}) + \left(\frac{2}{3} (\sqrt{4^3}) - \frac{16}{2} + 8 \right) - \left(\frac{2}{3} \sqrt{2^3} - \frac{4}{2} + 4 \right)$$

$$= \frac{2}{3} (8) - 8 + 8 + 2 - 4 = \frac{16}{3} - 2 = \boxed{\frac{10}{3}}$$

OR:

$$\text{Area} = \int_0^2 (y+2) - y^2 dy = \left[\frac{y^2}{2} + 2y - \frac{y^3}{3} \right]_0^2 = \boxed{\frac{10}{3}}$$

OR:

$$\text{Area} = \int_0^4 (\sqrt{x}-0) dx - \int_2^4 (x-2) - (0) dx = \boxed{\frac{10}{3}}$$

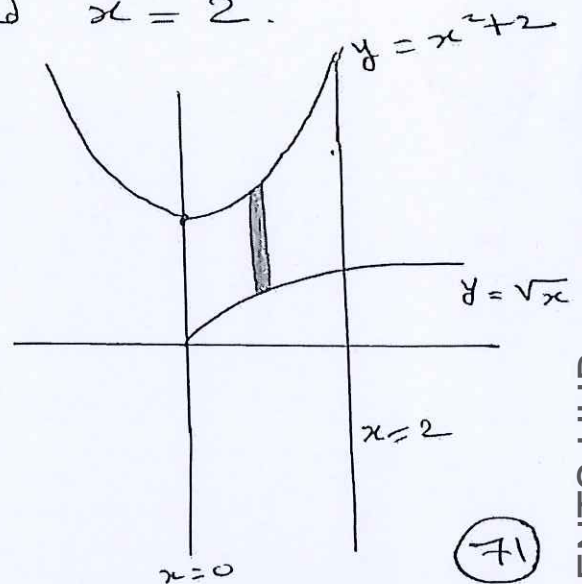
Example: Find the area enclosed between the Curves

$$y = x^2 + 2, y = \sqrt{x}, x = 0 \text{ and } x = 2.$$

$$\text{Area} = \int_0^2 (x^2+2) - (\sqrt{x}) dx$$

$$= \left[\frac{x^3}{3} + 2x - \frac{2}{3} x^{3/2} \right]_0^2$$

$$= \dots$$



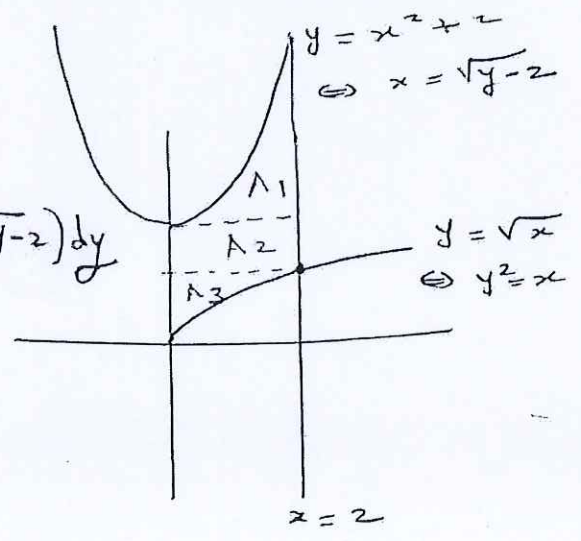
(71)

OR:

$$\text{Area} = A_3 + A_2 + A_1$$

$$= \int_0^{\sqrt{2}} (y^2 - 0) dy + \int_{\sqrt{2}}^2 2 dy + \int_2^6 (2 - \sqrt{y-2}) dy$$

$$= \dots$$



Theorem: The Substitution Rule: If $u = g(x)$ is

a differentiable function whose range is I , and

f is cont. on I , then:

$$\int f(g(x)) \cdot g'(x) dx = \int f(u) du$$

Example: $\int x^2 \sin(x^3) dx$

$$= \int \frac{\sin u}{3} du$$

Let $u = x^3$
 $du = 3x^2 dx$
 $\frac{du}{3} = x^2 dx$

$$= -\frac{1}{3} \cos u + C = -\frac{1}{3} \cos(x^3) + C.$$

Example: $\int \frac{1}{\sqrt{x}} \left(\frac{1}{(1+\sqrt{x})^2} \right) dx$.

Let $u = 1 + \sqrt{x}$

$$du = \frac{1}{2\sqrt{x}} dx \iff 2du = \frac{dx}{\sqrt{x}}$$

$$\Rightarrow \int \frac{2}{u^2} du = 2 \frac{u^{-1}}{-1} + C = \frac{-2}{1+\sqrt{x}} + C.$$