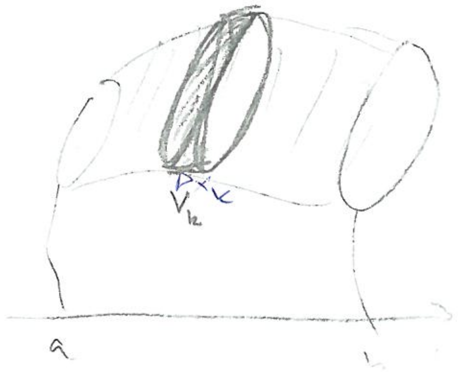
6.1 Volumes by Slicing and Rotation about an axis

1

The volume of the small slice $V_k = \underbrace{A(x_k)}_{\text{Cross Section Area}} \cdot \Delta x_k$



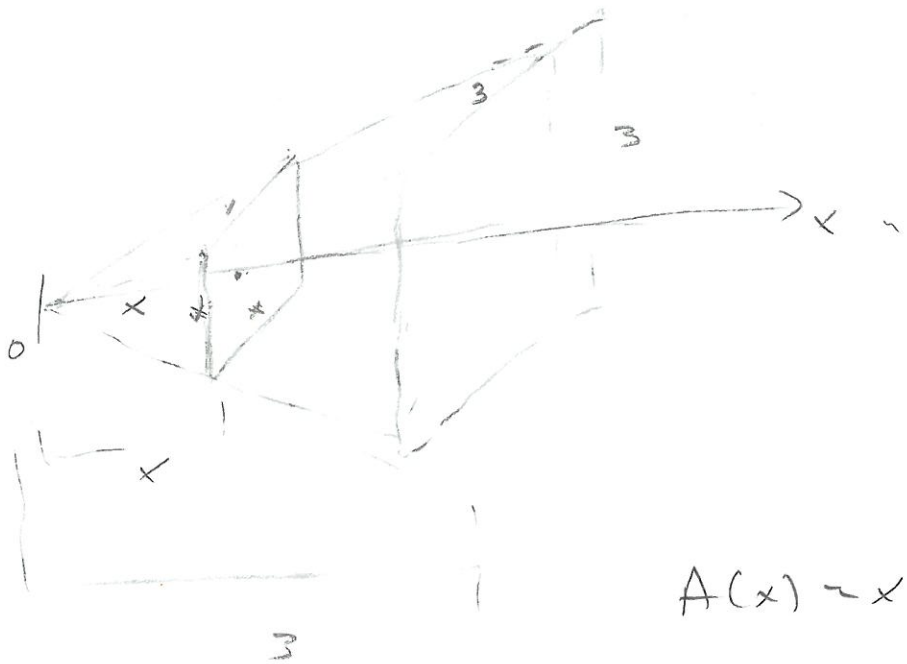
The Sum $\sum V_k = \sum A(x_k)\,\Delta x_k$ will estimate the volume of the solid so

$$V = \lim_{n\to\infty} \sum_{k=1}^{n} A(x_k)\,\Delta x_k$$

$$= \int_a^b A(x)\,dx$$ when $A(x)$ is the cross section Area

Example 1 : Volume of a Pyramid :

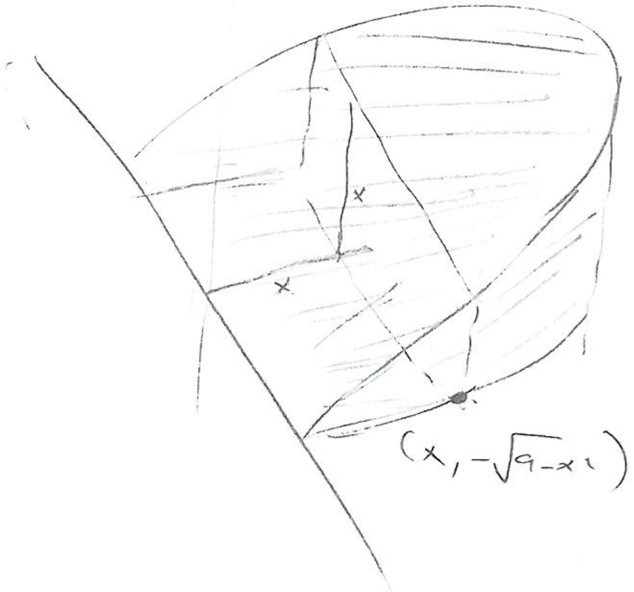
A pyramid 3 m high has a square base that is 3 m on a side. The cross section of the Pyramid perpendicular to the altitude x m down from the vertex is a square x m on a side. Find the Volume of the pyramid.



$A(x) = x^2$

$$V = \int_0^3 x^2 \, dx = \frac{x^3}{3} = 9 \text{ m}^3$$

E.X A curved wedge is cut from a cylinder of radius 3 by two planes. One plane is perpendicular to the axis of the cylinder. The second plan crosses the first plane at 45° angle at the center of the cylinder. Find the volume of the wedge.



$A(x) = \text{hight (width)}$

$= x\left(2\sqrt{9-x^2}\right)$

$$V = \int_a^b A(x)\,dx$$

$$= \int_0^3 2x\sqrt{9-x^2}\,dx.$$

let $u = 9-x^2 \Rightarrow du = -2x\,dx$

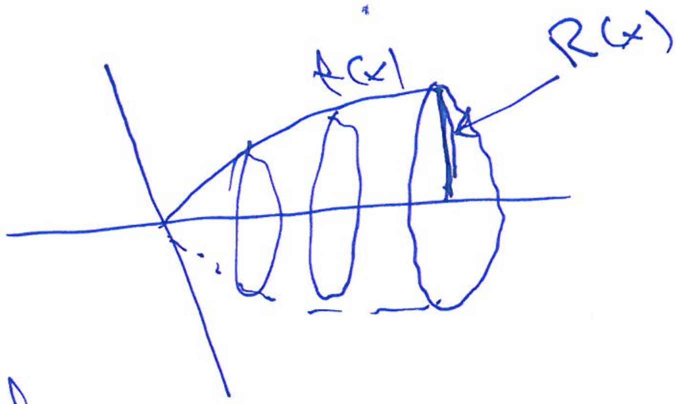
$$= -\frac{2}{3}(9-x^2)^{3/2}\Big|_{x=0}^{x=3} = 18$$

Solids of Revolution

Disk method

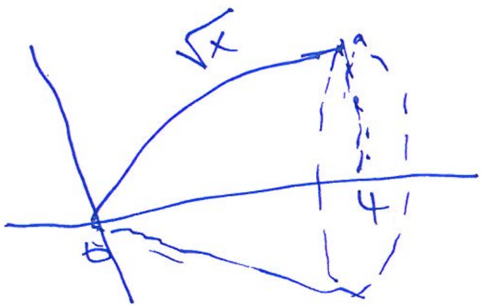
Most common method of slicing is solids of revolution

Cross section Area is $\pi R^2(x)$



$$\text{Volume} = \int_a^b \pi (R^2(x))\, dx$$

Ex1 Find the volume of the solid generated by revolving the region bounded by the curve $y=\sqrt{x}$, $0 \le x \le 4$ and the x-axis

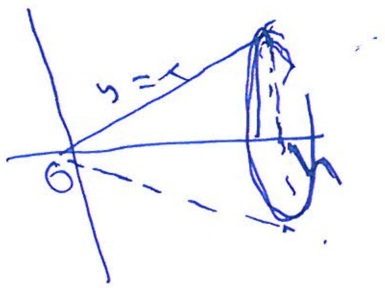


$$V = \pi \int_0^4 R^2(x)\, dx$$

$$= \pi \int_0^4 (\sqrt{x})^2\, dx$$

$$= \pi \int_0^4 x\, dx = \pi \left.\frac{x^2}{2}\right|_0^4 = \pi\left(\frac{16}{2} - 0\right) = 8\pi$$

Ex 2 Find the volume of the cone generated by revolving $y = x$ between 0 and h

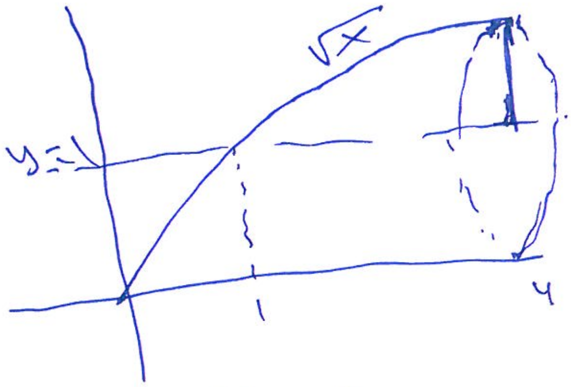


$$V = \pi \int_0^h R^2(x)\,dx$$

$$= \pi \int_0^h x^2\,dx$$

$$= \pi \left.\frac{x^3}{3}\right|_0^h = \frac{\pi}{3} h^3$$

Ex 3 Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$, the lines $y = 1$, $x = 4$ about the line $\boxed{y = 1}$



$$V = \pi \int_1^4 R^2(x)\,dx$$

$$= \pi \int_1^4 (\sqrt{x} - 1)^2\,dx$$

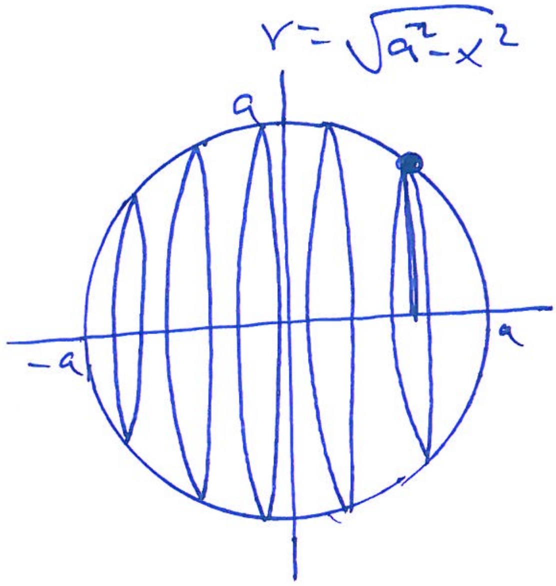
$\sqrt{x} = 1$

$x = 1$

$$= \pi \int_1^4 (x - 2\sqrt{x} + 1)\,dx$$

$$= \pi \left(\frac{x^2}{2} - 2 \cdot \frac{2}{3} x^{3/2} + x \Big|_1^4 \right) = \frac{7\pi}{6}$$

Ex 4 Find the volume of the sphere of radius a



Area of the circle

$= \pi r^2$

$= \pi\left(\sqrt{a^2 - x^2}\right)^2$

$$V = \pi \int_{-a}^{a} \left(\sqrt{a^2 - x^2}\right)^2 dx$$

$$= \pi \int_{-a}^{a} (a^2 - x^2)\, dx$$

$$= 2\pi \int_{0}^{a} (a^2 - x^2)\, dx$$

$$= 2\pi \left(a^2 x - \frac{x^3}{3} \Big|_0^a \right)$$

$$= 2\pi \left(a^3 - \frac{a^3}{3} - 0 \right)$$

$$= 2\pi \left(\frac{2}{3} a^3 \right)$$

$$= \frac{4}{3} \pi a^3$$

Revolution about y-axis

$$V = \pi \int_c^d (R(y))^2 \, dy$$

Ex1 Find the volume of the solid generated by revolving the region between the y-axis and the curve $x = \frac{2}{y}$, $1 \le y \le 4$ about the y-axis

$$V = \pi \int_1^4 \left(\frac{2}{y}\right)^2 dy$$

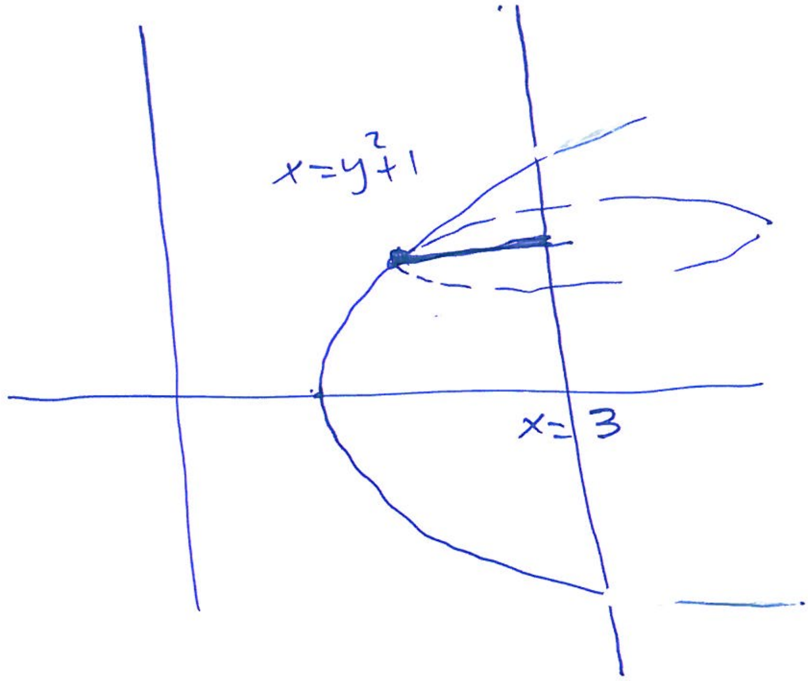
$$= \pi \int_1^4 \frac{4}{y^2} \, dy$$

$$= \pi \int_1^4 4y^{-2} \, dy = \pi \left(\frac{4y^{-1}}{-1} \Big|_1^4 \right)$$

$$= \pi \left[\frac{-4}{y} \Big|_1^4 \right] = \pi \left[\frac{-4}{4} - \left(\frac{-4}{1} \right) \right] = \pi(-1+4)$$

$$= \underline{3\pi}$$

Ex2 Find the volume of the solid generated by revolving the region between the parabola $x = y^2 + 1$ and the line $x = 3$ about the line $x = 3$



$$V = \pi \int_c^d R^2(y)\, dy$$

$$R(y) = 3 - (y^2 + 1) = 2 - y^2$$

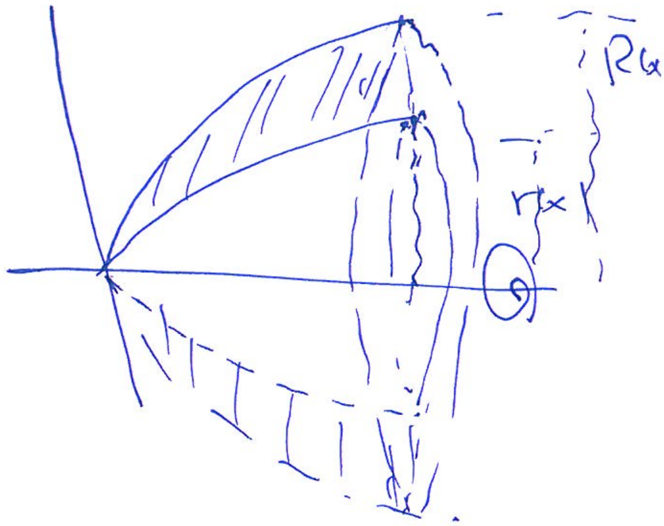
$$= \pi \int_{-\sqrt{2}}^{\sqrt{2}} (2 - y^2)^2\, dy$$

$$3 = y^2 + 1$$
$$y^2 = 2$$
$$y = \pm\sqrt{2}$$

Washer Cross Section

If the region we revolve to generated the solid does not border or cross the axis of revolution

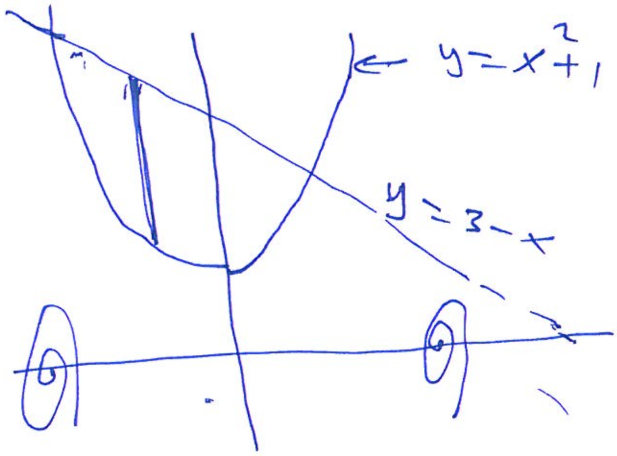
Outer radius $R(x)$



$$A(x) = \pi\left(R^2(x) - r^2(x)\right)$$

$$V = \pi\int_a^b \left(R^2(x) - r^2(x)\right) dx$$

Ex Revolving the region bounded by $y = 3 - x$, $y = x^2 + 1$ about the x-axis



$$V = \pi\int_{-2}^{1}\left[(3-x)^2 - (x^2+1)^2\right] dx$$

$$3 - x = x^2 + 1$$
$$x^2 + x - 2 = 0$$
$$(x+2)(x-1) = 0$$
$$x = -2,\ x = 1$$

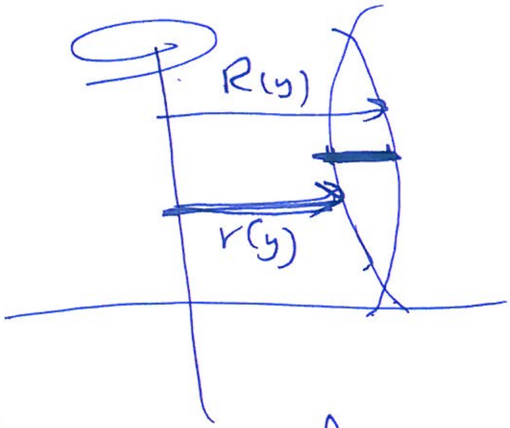
$$= \pi\int_{-2}^{1}\left[(9 - 6x + x^2) - (x^4 + 2x^2 + 1)\right]$$

$$= \pi\int_{-2}^{1}(8 - 6x - x^2 - x^4)\, dx$$

$$= \pi\left(8x - 3x^2 - \frac{x^3}{3} - \frac{x^5}{5}\Big|_{-2}^{1}\right) = \frac{117}{5}\pi \text{ units}$$

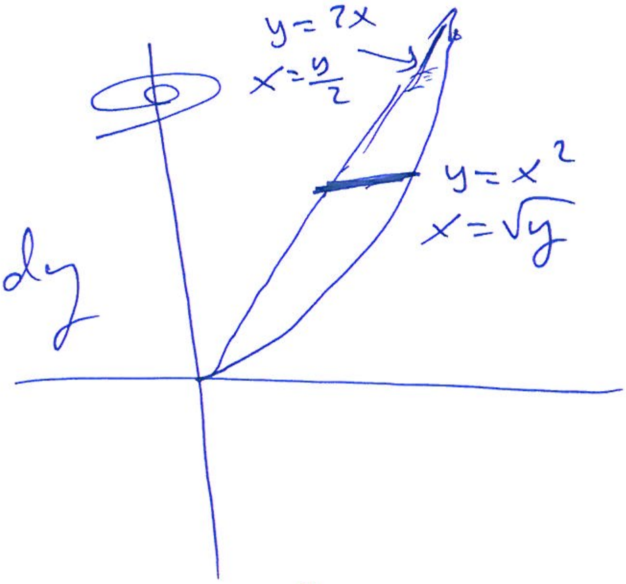
Similarly Washeres about y-axis

$$V = \pi \int_c^d \left(R^2(y) - r^2(y) \right) dy$$



Ex Find the volume of the solid generated by revolving the region bounded by the parabola $y = x^2$ and $y = 2x$ in the 1st qudrant about y-axis

$$V = \pi \int_0^4 \left[(\sqrt{y})^2 - \left(\frac{y}{2}\right)^2 \right] dy$$



$$= \frac{8\pi}{3}$$

$x^2 = 2x$

$x^2 - 2x = 0$

$x(x-2) = 0$

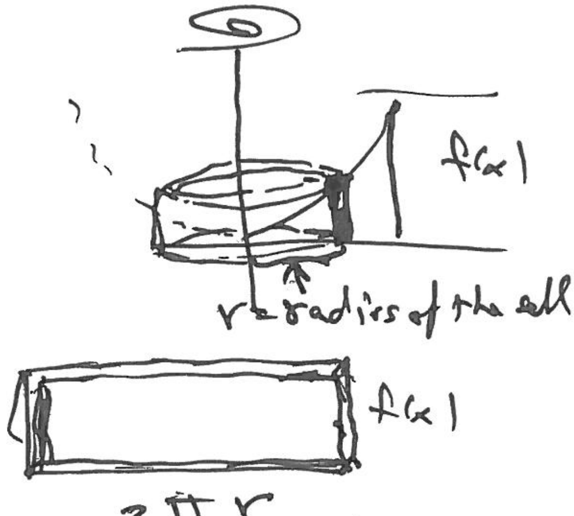
$x = 0 \ , \ x = 2$

$\downarrow \qquad \downarrow$

$y = 0 \qquad y = 4$

Volumes by shells:

Lets find the volume of the solid generated by revolving the region between the x-axis and the graph of a continous function $y = f(x) \geq 0$, $0 < a \leq x \leq b$ about a vertical line



volume of the shell

$2\pi r f(x) \Delta x$

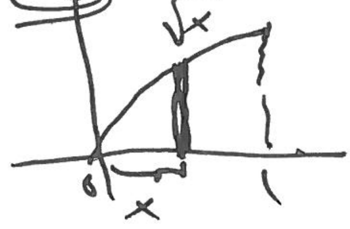
$$V = \lim_{\Delta x_k \to 0} \sum 2\pi r_k f(x_k) \Delta x_k$$

$$\Rightarrow V = \int_a^b 2\pi \begin{pmatrix} \text{shell} \\ \text{radius} \end{pmatrix} \begin{pmatrix} \text{shell} \\ \text{hight} \end{pmatrix} dx$$

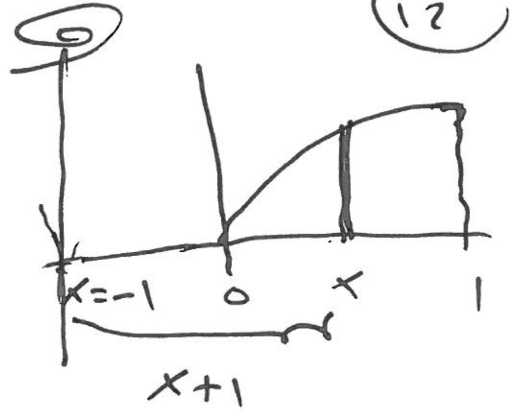
Ex1 Find the volume of the solid generated by revolving $y = \sqrt{x}$, $0 \leq x \leq 1$ about the y-axis, Using Shell method

$$V = 2\pi \int_0^1 \begin{pmatrix} \text{shell} \\ \text{radius} \end{pmatrix} \begin{pmatrix} \text{shell} \\ \text{hight} \end{pmatrix} dx$$

$$= 2\pi \int_0^1 (x) \sqrt{x} \, dx = \frac{4\pi}{5}$$



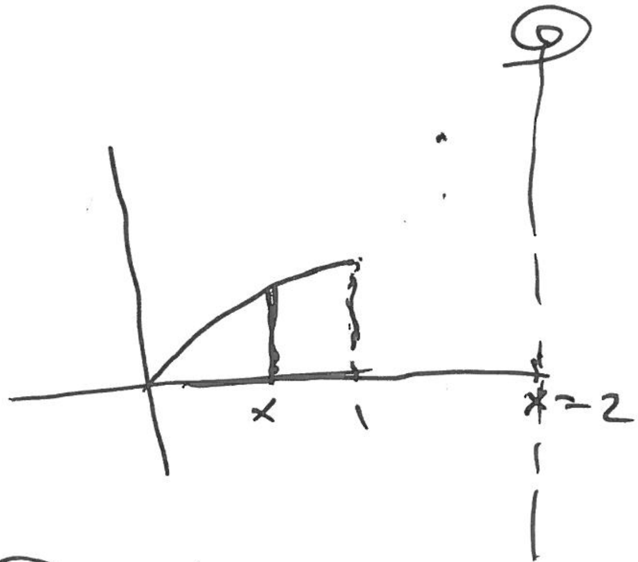
Ex 2



$$V = 2\pi \int_0^1 \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{hight}\end{smallmatrix}\right) dx$$

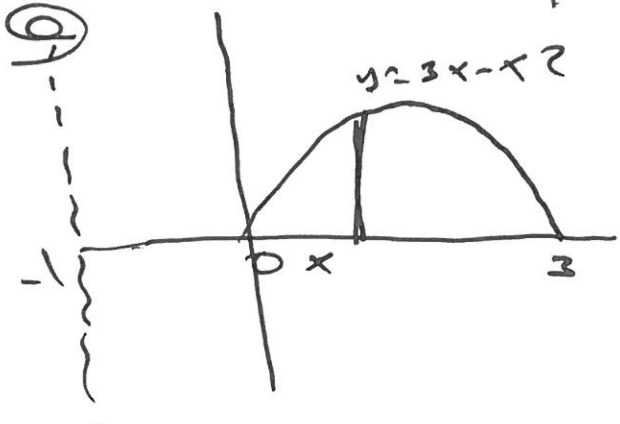
$$= 2\pi \int_0^1 (x+1)\sqrt{x}\, dx$$

Ex 3



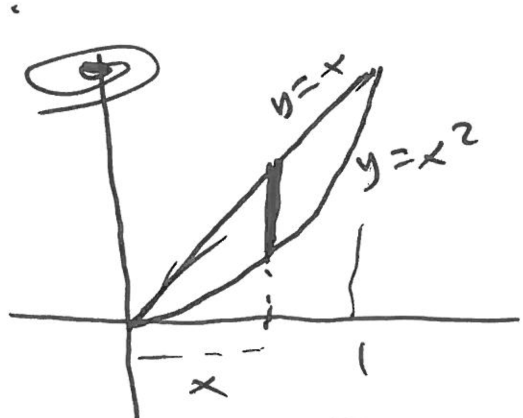
$$V = 2\pi \int_0^1 (2-x)\sqrt{x}\, dx$$

Ex 4



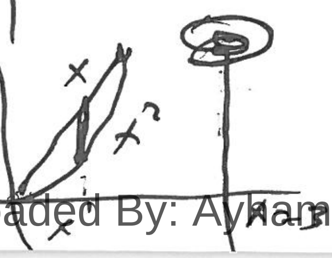
$$V = 2\pi \int_0^1 (x+1)(3x-x^2)\, dx$$

Ex 5



$$V = 2\pi \int_0^1 x\,(x-x^2)\, dx$$

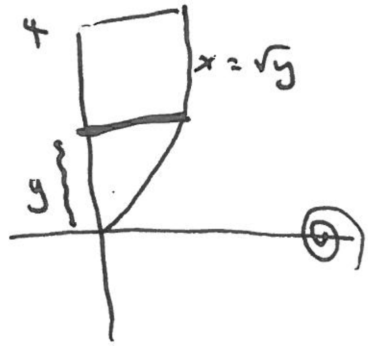
Ex 6



$$V = 2\pi \int_0^1 (3-x)(x-x^2)\, dx$$

Similarly, we can do it about a horizontal line $y = a$

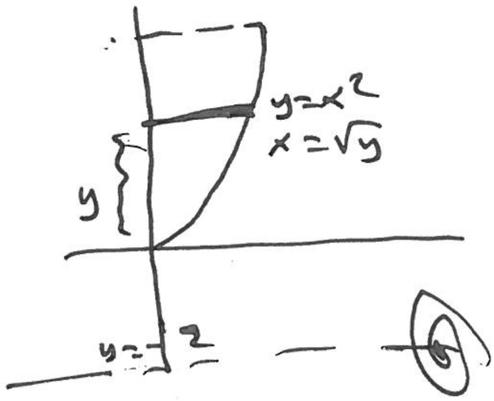
$$V = 2\pi \int_c^d \begin{pmatrix}\text{shell}\\ \text{radius}\end{pmatrix}\begin{pmatrix}\text{shell}\\ \text{hight}\end{pmatrix} dy$$



<u>Ex1</u> $V = 2\pi \int_0^4 y \sqrt{y}\, dy.$

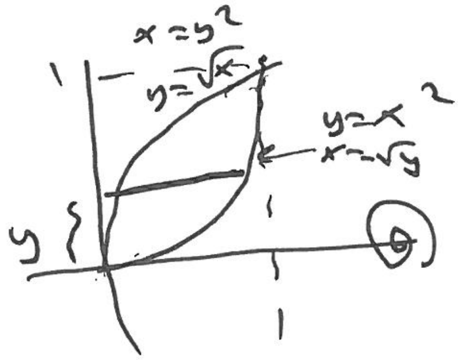
<u>Ex2</u>

$$V = 2\pi \int_0^4 (y+2)\sqrt{y}\, dy$$



<u>Ex3</u>

$$V = 2\pi \int_0^1 y\left(\sqrt{y} - y^2\right) dy$$



<u>Ex4</u>

$$V = 2\pi \int_0^1 (3-y)\left(\sqrt{y} - y^2\right) dy$$