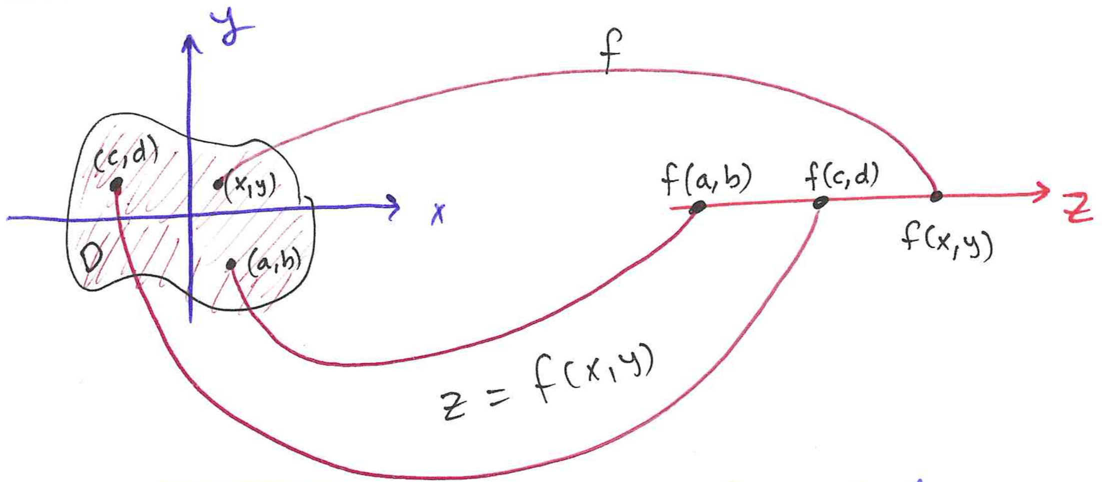
14.1 Functions of Several Variables

Def. Let D be a set of n-tuples of real numbers $(x_1, x_2, \dots, x_n)$. A real-valued function f on D is a rule that assigns a unique real number $w = f(x_1, x_2, \dots, x_n)$ to each element in D.

$$f : D \longrightarrow R$$

$$(x_1, x_2, \dots, x_n) \longrightarrow w = f(x_1, x_2, \dots, x_n)$$

- The domain of f is the set D
- The range of f is the set of w-values "R"
- w is the dependent variable (output)
- $x_1, x_2, \dots, x_n$ are the independent variables (inputs)



- In case where x, y, z are independent variables

$\Rightarrow$ the dependent variable "usually" is w

$$w = f(x, y, z)$$

Exp Let $f(x,y) = x^2 + xy$. Find $f(2,3)$

$$z = f(2,3) = 4 + (2)(3) = 4 + 6 = 10$$

Exp Let $f(x,y,z) = \frac{x-y}{y^2+z^2}$. Find $f(2,1,-3)$

$$w = f(2,1,-3) = \frac{2-1}{1+9} = \frac{1}{10}$$

Exp Find the domain and range of

$\boxed{1}$ $z = \sqrt{y - x^2}$

Domain is the set of all points (x,y) : $y \geq x^2$

Range $= [0, \infty)$

$\boxed{2}$ $z = \frac{1}{xy}$

Domain is the set of all points (x,y) : $xy \neq 0$

Range $= (-\infty, 0) \cup (0, \infty)$

$\boxed{3}$ $z = \sin xy$

Domain is the entire plane

Range $= [-1, 1]$

$\boxed{4}$ $w = \sqrt{x^2 + y^2 + z^2}$

Domain is the entire space

Range $= [0, \infty)$

$\boxed{5}$ $w = \frac{1}{x^2 + y^2 + z^2}$

Domain is the set of all points (x,y,z) except $(x,y,z) = (0,0,0)$

Range $= (0, \infty)$

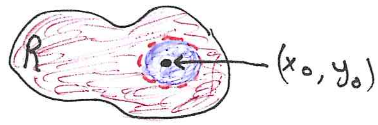
6) $w = xy \ln z$

Domain is the half-space $z > 0$

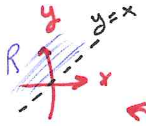
Range $= (-\infty, \infty)$

Function of Two Variables

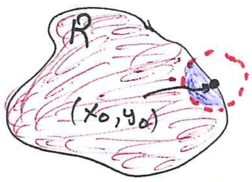
Def * A point (x_0, y_0) in a region R in the xy-plane is an interior point of R if it is the center of a disk that lies entirely in R.



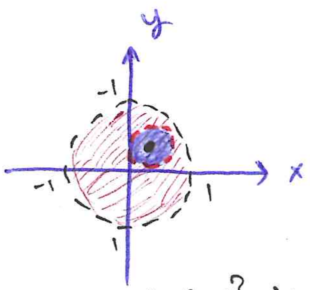
* A point (x_0, y_0) is a boundary point of R if every disk centered at (x_0, y_0) contains points that lie outside R and points as well that lie in R.



(The boundary point itself need not belong to R)

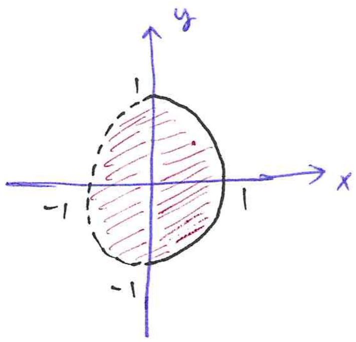


* The interior of R is the set of all interior points of R.
* The boundary of R 〃 〃 〃 〃 〃 boundary 〃 〃 〃.
* The region R is open if it consists entirely of interior points.
* 〃 〃 〃 〃 closed 〃 〃 contains all its boundary points.



$R = \{(x,y): x^2+y^2 < 1\}$ is open

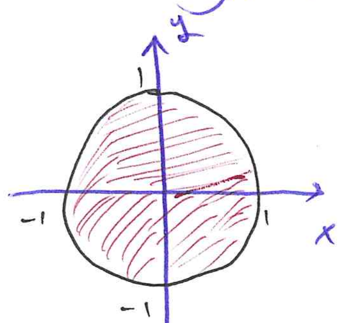
" every point is an interior point"



R is not closed nor open

$(-1,0) \notin R$

$(1,0) \in R$ not interior



- $R = \{(x,y): x^2+y^2 \le 1\}$
- R is closed: contains all its boundary points

* A region in the plane is bounded if it lies inside a disk of fixed radius.

* A region is unbounded if it is not bounded.

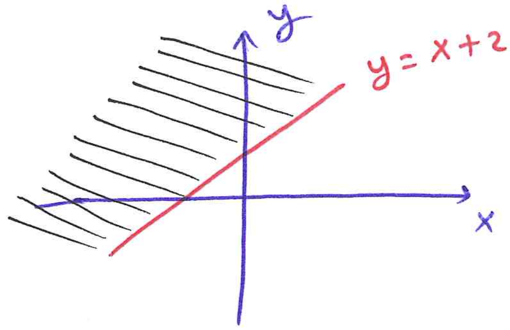
Exp : Bounded sets in the plane : line segments, triangles, rectangles, circles, disks ...

Unbounded sets in the plane : lines, coordinate axes, quadrants, half-planes, the plane.

Exp Find and sketch the domain of

[1] $f(x,y) = \sqrt{y-x-2}$

Domain is the set of all points (x,y) : $y \geq x+2$

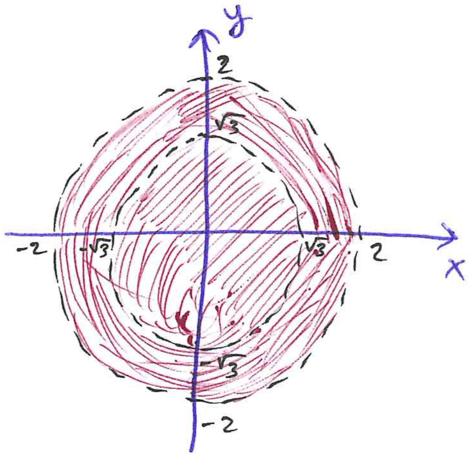


[2] $f(x,y) = \dfrac{1}{\ln(4-x^2-y^2)}$

Domain : all points (x,y) inside the circle $x^2+y^2=4$ such that $x^2+y^2 \neq 3$

because $4-x^2-y^2>0 \Rightarrow x^2+y^2<4$

and $4-x^2-y^2 \neq 1 \Rightarrow x^2+y^2 \neq 3$



Def. The set of points in the plane where a function $f(x,y)$ has a constant value $f(x,y)=c$ is called a level curve. "contour curves"

$\mathbb{R}^3$ • The graph of f "the surface $z=f(x,y)$" is the set of all points $(x,y,f(x,y)) = (x,y,c) = (x,y,z)$ in space, where $(x,y) \in D(f)$.

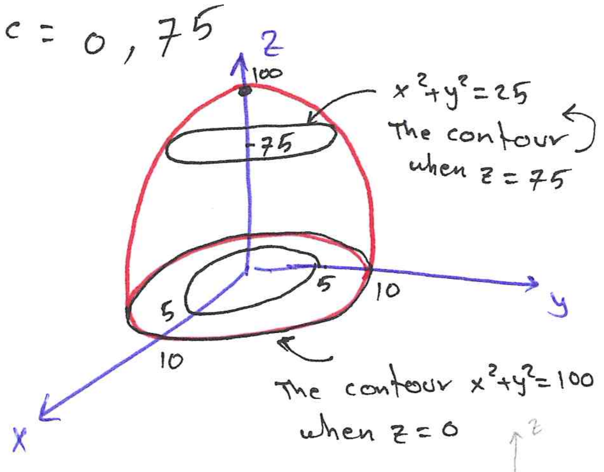
Exp Find and sketch the level curves (contour curves or maps) of

1) $f(x,y) = 100 - x^2 - y^2$ at $c = 0, 75$

$z = 100 - x^2 - y^2$

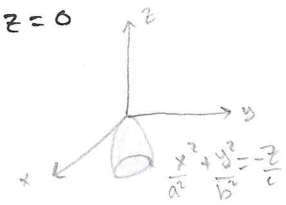
$x^2 + y^2 = 100 - z$

" Paraboloid "



$\boxed{c=0} \Rightarrow 100 - x^2 - y^2 = 0$

$x^2 + y^2 = 100$



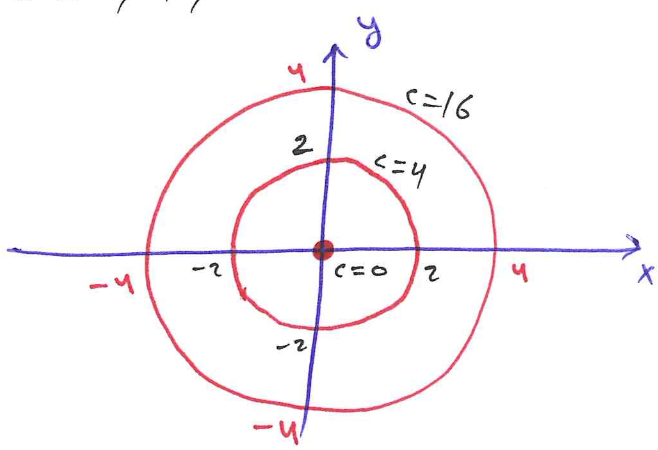
$\boxed{c=75} \Rightarrow 100 - x^2 - y^2 = 75 \Leftrightarrow x^2 + y^2 = 25$

---

2) $f(x,y) = \sqrt{x^2 + y^2}$, $c = 0, 4, 16$

$\boxed{c=0} \Rightarrow 0 = x^2 + y^2$

$\Rightarrow (x,y) = (0,0)$



$\boxed{c=4} \Rightarrow x^2 + y^2 = 4$

$\boxed{c=16} \Rightarrow x^2 + y^2 = 16$

Functions of Three Variables

Def. The set of points (x,y,z) in space where a function of three independent variables has a constant value " $f(x,y,z) = c$ " is called a level surface.

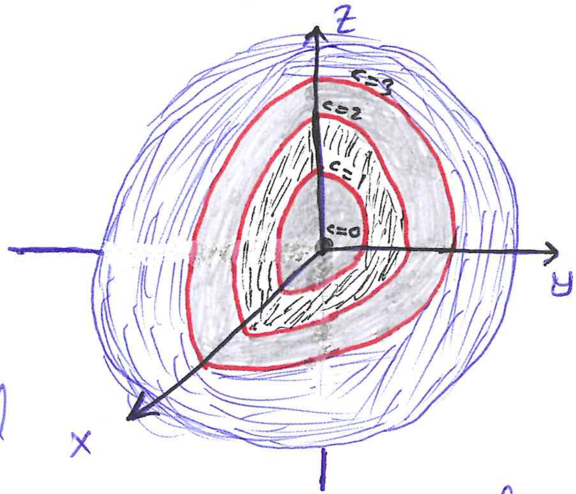
- The graph of f is the set of points $(x,y,z,f(x,y,z)) \in \mathbb{R}^4$ we can not sketch them, but we can look to it's three dimensional level surfaces.

Exp Find and sketch the level surfaces of the function $f(x,y,z) = \sqrt{x^2+y^2+z^2}$ at $c = 1, 2, 3, 0$

$\boxed{c=1}$ $x^2+y^2+z^2 = 1$

$\boxed{c=2}$ $x^2+y^2+z^2 = 4$

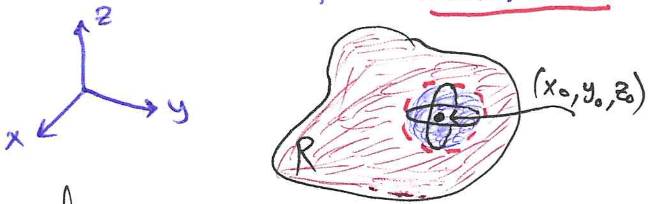
$\boxed{c=3}$ $x^2+y^2+z^2 = 9$



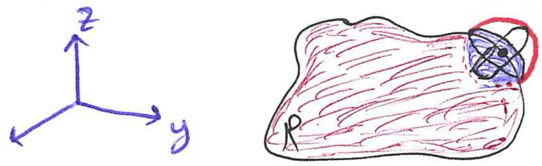
The level surfaces are spheres centered at origin with radius 1, 2, 3

$\boxed{c=0}$ $x^2+y^2+z^2 = 0 \iff (x,y,z) = 0 \Rightarrow$ the level surface is the origin only.

Def * A point (x_0, y_0, z_0) in a region R in space is an interior point of R if it is the center of a solid ball that lies entirely in R.



* A point (x_0, y_0, z_0) is a boundary point of R if every solid ball centered at (x_0, y_0, z_0) contains points that lie outside of R as well as points that lie inside R.



* The interior of R is the set of interior points of R.
* The boundary of R " " " " boundary " " " .
* The region R is open if it consists entirely of interior points.
* " " " " closed if it contains its entire boundary.

Exp: Examples of open sets in space:

[1] interior of a sphere [2] the open half-space $z>0$.
[3] the first octant $(x>0, y>0, z>0)$ [4] space itself

Examples of closed sets in space:

[1] lines [2] planes [3] closed half-space $z \geq 0$

Exp Find an equation for the level surface of the function

$f(x,y,z) = \dfrac{x-y+z}{2x+y-z}$ through the point $(1,0,-2)$

$w = f(1,0,-2) = \dfrac{1-0-2}{2+0+2} = \dfrac{-1}{4}$ $\Leftrightarrow \dfrac{-1}{4} = \dfrac{x-y+z}{2x+y-z}$

$2x - y + z = 0$

Q18 $f(x,y) = \sqrt{y-x}$

[a] Domain is the set of all points (x,y): $y \geq x$

[b] Range: $z \geq 0$

[c] level curves are straight lines $y-x=c$ where $c \geq 0$

[d] The boundary of the domain is the straight line $y=x$

[e] The domain is closed region.

[f] The domain is unbounded.