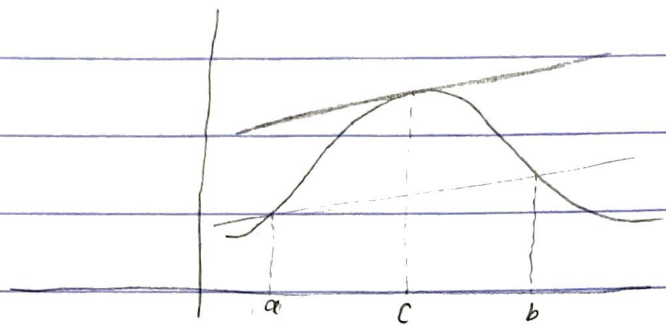
4.3 : The mean Value Theorem .



- f is cont. on $[a,b]$, $\exists c \in (a,b)$ and diffble on (a,b) : $f'(c) = \dfrac{f(b) - f(a)}{b-a}$.
- If $f(a) = f(b) \rightsquigarrow f'(c) = 0$

Rolle's Thm

lemma : Rolle's Thm .

suppose that $a,b \in \mathbb{R}$ with $a<b$. If f is continuous on $[a,b]$, diffble on (a,b), and if $f(a) = f(b)$, then $f'(c) = 0$ for some $c \in (a,b)$.

PF :

By the Extreme value thm, f has a finite Max. M and a finite min. m on $[a,b]$.

ثابت → If $M = m$, then f is constant on (a,b) and $f'(x) = 0$, $\forall x \in (a,b)$.

يوجد نقطة (a,b) تكون عبارة اما max أو min

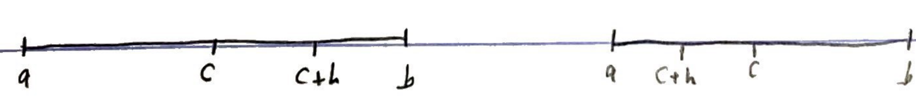
Suppose that $M \neq m$. since $f(a) = f(b)$, f must assume one of the values M or m at some $c \in (a,b)$.

we may suppose that $f(c) = M$ (similar proof when $f(c) = m$) .

Since M is the max. of f on $[a,b]$, we have

f(c) Max

$$f(c+h) - f(c) \le 0 \ , \ \forall h \text{ satisfy } c+h \in (a,b) .$$



$\hookrightarrow h>0$. $\qquad \hookrightarrow h<0$

case 1 : $h > 0$:

$$f'(c) = \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} \leq 0$$

case 2 : $h < 0$

$$f'(c) = \lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h} \geq 0$$

It follows that $f'(c) = 0$

RMK :

1. The continuity hypothesis in Rolle's Thm cannot be relaxed at even one point in $[a,b]$.

exp :
$$f(x) = \begin{cases} x , & 0 \leq x < 1 \\ 0 , & x = 1 \end{cases}$$

is cont. on $[0,1)$ and diffble on $(0,1)$ and $f(0) = f(1)$ But $f'(x)$ is never zero on $(0,1)$.

2. The differentiability hypothesis in Rolle's Thm cannot be relaxed at even one point in (a,b).

exp : $f(x) = |x|$ is cont. on $[-1,1]$, diffble on $(-1,1) \setminus \{0\}$

$f(-1) = f(1) = 1$

But $f'(x)$ is never zero.

Thm 6 : suppose that $a, b \in \mathbb{R}$ with $a < b$.

i. [Generalized Mean Value Thm] :

If f, g are continuous on $[a,b]$ and diffble on (a,b), then there is a $c \in (a,b)$ s.t $g'(c)\,(f(b) - f(a)) = f'(c)\,(g(b) - g(a))$.

ii. Mean Value Theorem :

If f is continuous on $[a,b]$ and diffble on (a,b) then there is a $c \in (a,b)$ s.t $f(b) - f(a) = f'(c)(b-a)$.

proof :

i. set $h(x) = f(x)\,(g(b) - g(a)) - g(x)\,(f(b) - f(a))$

since $h'(x) = f'(x)\,(g(b) - g(a)) - g'(x)\,(f(b) - f(a))$,

It is clear that h is conti on $[a,b]$

and diffble on (a,b) and $h(a) = h(b) = 0$.

Thus, by Rolle's Thm, $h'(c) = 0$ for some $c \in (a,b)$

That is their is a $c \in (a,b)$ s.t

$$f'(c)\,(g(b) - g(a)) - g'(c)\,(f(b) - f(a)) = 0.$$

ii. set $g(x) = x$ and apply (i), Then

$\exists$ a $c \in (a,b)$ s.t $f(b) - f(a) = f'(c)(b-a)$.

→ Prove $f'(x) = g'(x)$ has a solution.

let $h = f(x) - g(x)$

↳ $h'(x) = 0$

Rolle's → Mean → generalized

RMK :

1. The generalized Mean Value Thm is also called cauchy's Mean Value Thm.

2. For a geometric interpretation of (ii), see the opening graph (p. 19 Note).

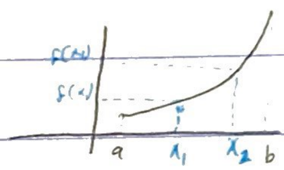
3. The mean Value Thm is most often used to extract information about f from f' as follows.

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Def: let E be a nonempty subset of $\mathbb{R}$ and $f : E \to \mathbb{R}$

i. f is said to be increasing (resp. strictly increasing) on E iff

$x_1, x_2 \in E$ and $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$ (resp. $f(x_1) < f(x_2)$).

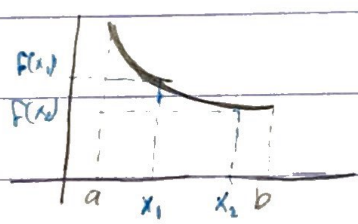


stri. increasing

$x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

ii. f is said to be decreasing (resp. strictly decreasing) on E iff

$x_1, x_2 \in E$ and $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$ (resp. $f(x_1) > f(x_2)$).



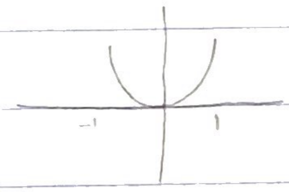
stric. decreasing.

$x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$.

iii. f is said to be <u>Monotone</u> (resp. strictly Monotone) on E iff f is either decreasing or increasing (resp. either strictly decreasing or strictly increasing) on E.

لو كان الاقتران Monotonic على 2-sets is Monotone to their union? False → exp.

exp: $f(x) = x^2$ is strictly monotone on $[0,1]$, and on $[-1,0]$, it is Not Monotone on $[-1,1]$.



Thm 7: suppose that $a, b \in \mathbb{R}$, with $a < b$, that f is continuous on $[a,b]$ and that f is diffble on (a,b);

i. If $f'(x) > 0$ (resp. $f'(x) < 0$) $\forall x \in (a,b)$, then f is strictly increasing (resp. strictly decreasing) on $[a,b]$.

تطبيق على Mean Value Thm

ii. If $f'(x) = 0$, $\forall x \in (a,b)$, then f is constant on $[a,b]$.

iii. If g is continuous on $[a,b]$ and diffble on (a,b), and if $f'(x) = g'(x)$, $\forall x \in (a,b)$ then $f - g = c$ is constant on $[a,b]$.

proof:

i. let $a \leq x_1 < x_2 \leq b$. By the <u>mean value</u> Thm $[x_1, x_2]$, $\exists$ a $c \in (a,b)$ s.t

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$. Thus

If $f'(c) > 0$ then $f(x_2) - f(x_1) > 0$ (i.e, $f(x_1) < f(x_2)$) implies f is strictly increasing.

If $f'(c) < 0$ then $f(x_2) - f(x_1) < 0$ (i.e $f(x_2) < f(x_1)$)

$\Rightarrow$ f is strictly decreasing.

ii. If $\bar{f}(x)=0$ then By the prove of part i, f is both increasing and decreasing, and hence constant on $[a,b]$.

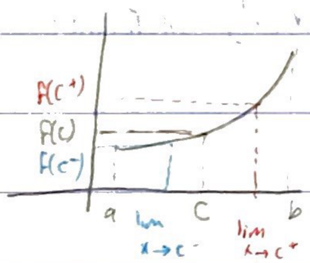
iii. set $h(x) = f(x) - g(x)$ on $[a,b]$, and apply (ii):

$$\bar{h}(x) = \bar{f}(x) - \bar{g}(x) = 0 \quad \forall\ x \in (a,b).$$

By part (ii) $h = f - g$ is constant on $[a,b]$. لازم المشتقة صفر طول الفترة

Thm 8: suppose that f is increasing on $[a,b]$:

i. If $c \in [a,b)$, then $f(c^+)$ exists and $f(c) \le f(c^+) = \lim\limits_{x \to c^+} f(x)$.



$f(c^+) > f(c)$.   Proof X

$f(c^-) < f(c)$.

ii. If $c \in (a,b]$ then $f(c^-)$ exists and $f(c^-) = \lim\limits_{x \to c^-} f(x) \le f(c)$.

Thm 9: If f is Monotone on an interval I, then f has at most countably many points of discontinuity on I.

same $\sin x \leq x \rightarrow x - \sin x \geq 0$

نقدر اثبتها جهاز السكشن

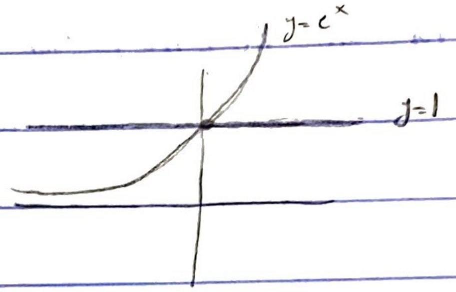
By $\uparrow, \downarrow$, mean value Thm.

كثير مهم

Application (Thm 7 i).

exp. prove that $1 + x < e^x \quad \forall x > 0$.

pf.



let $f(x) = e^x - x \rightarrow$ We need to prove $e^x - x > 1$

$f'(x) = e^x - 1 > 0 \quad \boxed{\forall x > 0} \rightarrow$

من الرسمة

$\Rightarrow f$ is strictly increasing on $(0, \infty)$

لما أثبتنا انه $e^x - 1 > 0$ على كل $x > 0$ هذا يعني انه $f(x) > f(0)$.

Thus, As $x > 0$ then $f(x) > f(0)$ — By Def and Thm

i.e $e^x - x > e^0 - 0$

$e^x > 1 + x$

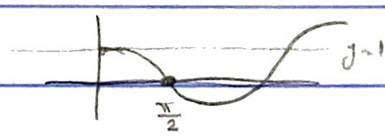
OR, $f(x) = e^x - x - 1$

$f'(x) = e^x - 1 > 0$

$x > 0 \rightarrow f(x) > f(0)$

$e^x - x - 1 > 0 \rightarrow e^x > x + 1$

exp: $\sin x < x \quad , \quad x \geq 0$.



let $f(x) = x - \sin x$

$f'(x) = 1 - \cos x > 0$

$\Rightarrow f$ is increasing on $(0, \infty)$, Thus,

$x \geq 0 \Rightarrow f(x) \geq f(0)$

$x - \sin x > 0 - \sin 0$

$\sin x \leq x$

Thm 6: Bernoulli's inequality.

let α be a positive real number. If $0 < \alpha \leq 1$, then $(1+x)^{\alpha} \leq 1 + \alpha x$ for all $x \in [-1, \infty)$ and if $\alpha \geq 1$, then $(1+x)^{\alpha} \geq 1 + \alpha x$ for all $x \in [-1, \infty)$.

exp: $(1+x)^{\frac{1}{2}} \leq 1 + \frac{1}{2} x$

$x \in [-1, \infty)$ $\leftarrow$ $(1+x)^2 \geq 1 + 2x$.

$x \geq 0$

proof: case 1 : $0 < \alpha \leq 1$

Fix $x \geq -1$ and $f(t) = t^{\alpha}$, $t \in [0, \infty)$.

since $f'(t) = \alpha t^{\alpha - 1}$, By MVT (applied to $a = 1$, $b = 1+x$)

$f(1+x) - f(1) = \underbrace{\alpha x c^{\alpha-1}}_{= f'(c)(1+x-1)}$ * for some c between 1 and $1+x$.

subcases 1.1 : $x > 0$ Then $c > 1$

since $0 < \alpha \leq 1 \Rightarrow \alpha - 1 \leq 0$

$\Rightarrow c^{\alpha - 1} \leq 1 \Rightarrow x c^{\alpha - 1} \leq x$

By (*), $(1+x)^{\alpha} = f(1+x) = f(1) + \alpha x c^{\alpha - 1}$

$\leq f(1) + \alpha x$

$= 1 + \alpha x$ as required.

subcase 1.2 : $-1 \leq x < 0$ Then $c \leq 1$

So $c^{\alpha - 1} \geq 1$, But $x \leq 0 \Rightarrow x c^{\alpha - 1} \leq x$

By (*), $(1+x)^{\alpha} = f(1+x) = f(1) + \alpha x c^{\alpha - 1}$

$\leq f(x) + \alpha x = 1 + \alpha x$.

$\therefore (1+x)^{\alpha} \leq 1 + \alpha x$, $x \geq -1$, $0 < \alpha \leq 1$,

case 2 exercise.

exp: prove that the sequence $(1+\frac{1}{n})^n$ is increasing as $n\to\infty$, and its limit L satisfies $2 < L \leq 3$ (The limit L turns out to be an irrational number, the natural base $e = 2.718281828\ldots$). Recall, $(1+\frac{x}{n})^n \to e^x$ as $n\to\infty$

تطبيق على Bernoli

proof:

$X_n \leq X_{n+1}$:

علشان increasing

$$X_n = \left(1+\frac{1}{n}\right)^n = \left[\left(1+\frac{1}{n}\right)\frac{n}{n+1}\right]^{n+1}$$

x $\quad \alpha \leq 1$

$$\leq \left[1+\frac{1}{n}\left(\frac{n}{n+1}\right)\right]^{n+1} \quad \text{(Bernoulli's ineq)}$$

$$= \left(1+\frac{1}{n+1}\right)^{n+1} = X_{n+1}.$$

→ To prove $\lim_{n\to\infty} X_n = L$: $2 < L \leq 3$

$$\left(1+\frac{1}{n}\right)^n = \sum_{K=0}^{n}\binom{n}{K}\left(\frac{1}{n}\right)^K (1)^{n-K}$$

$$= \sum_{K=0}^{n}\binom{n}{K}\left(\frac{1}{n}\right)^K$$

$$\text{Now, } \binom{n}{K}\frac{1}{n^K} = \frac{n!}{K!\,(n-K)!\,n^K}$$

$$= \frac{n(n-1)\ldots(n-(K-1))\cancel{(n-K)!}}{K!\,\cancel{(n-K)!}\,n^K} \to \leq 1$$

$$\leq \frac{1}{K!} \leq \frac{1}{2^{K-1}}, \quad \forall K \in \mathbb{N}$$

(check by induction: $K! \geq 2^{K-1}$)

→

cont.

Bernoelli's ineq.

x $\quad$ $\alpha = n > 1$

$$\underline{2} = 1 + \frac{1}{n} \cdot n < \left(1 + \boxed{\frac{1}{n}}\right)^{n} = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k}$$

$$= 1 + \sum_{k=1}^{n} \binom{n}{k} \frac{1}{n^k}$$

$$\leq 1 + \sum_{k=1}^{n} \frac{1}{2^{k-1}} \longrightarrow \text{finite Geometric series.}$$

$$= 1 + \frac{1\left(1 - \left(\frac{1}{2}\right)^n\right)}{1 - \frac{1}{2}}$$

$$= 1 + 2\left(1 - \left(\frac{1}{2}\right)^n\right)$$

$$= 3 - \frac{1}{2^{n-1}} \rightarrow < 1$$

$$\leq 3 \quad \text{for } n \geq 1.$$

Hence, By the MCT, the limit L exists and satisfies $2 < L \leq 3$.

Thm 11: Intermediate Value Theorem For Derivatives.

suppose that f is differentiable on $[a,b]$ with $f'(a) \neq f'(b)$. If y_0 is a real number which lies between $f'(a)$ and $f'(b)$, then there is an $x_0 \in (a,b)$ s.t $f'(x_0) = y_0$.

proof: suppose that y_0 lies between $f'(a)$ and $f'(b)$.

By symmetry assume $f'(a) < y_0 < f'(b)$.

$F(x) = \int f'(x) - y_0 \, dx$. Set $F(x) = f(x) - y_0 x$, for $x \in [a,b]$.

observe that F is diffble on $[a,b]$.

$F'(x_0) = 0$

Hence, By the extreme value Thm, F has an absolute min say $F(x_0)$ on $[a,b]$.

$F'(a) = f'(a) - y_0 < 0$ (F is decreasing at a).

So $F(a+h) - F(a) < 0$ for $h > 0$.

Hence, $F(a)$ is Not the absolute min. of F on $[a,b]$.

similarly, $F(b)$ is Not the absolute min of F.

Hence, the absolute min $F(x_0)$ must occur on (a,b)

i.e. $x_0 \in (a,b)$ s.t $F'(x_0) = 0$

$0 = y_0 - f'(x_0)$

Hence, $f'(x_0) = y_0$, $x_0 \in (a,b)$. ∎

H.W: 0-6, 8, 9.