13.3 Arc Length in Space

Def The length of a smooth curve $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$, $a \le t \le b$ that is traced exactly once as t increases from t = a to t = b is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

. Note that $\vec{v} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}$

$$|\vec{v}| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

Hence the Arc length formula becomes

$$L = \int_a^b |\vec{v}|\, dt \qquad a \le t \le b$$

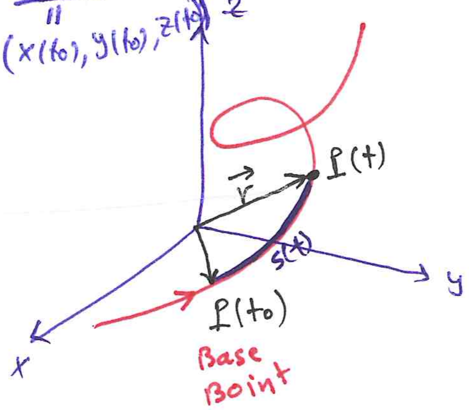
Exp Find the length of the curve $\vec{r}(t) = (2+t)\vec{i} - (t+1)\vec{j} + t\vec{k}$, $0 \le t \le 3$

$\vec{v} = \vec{i} - \vec{j} + \vec{k} \Rightarrow |\vec{v}| = \sqrt{1+1+1} = \sqrt{3}$

$$L = \int_0^3 |\vec{v}|\, dt = \int_0^3 \sqrt{3}\, dt = 3\sqrt{3}$$

Arc length Parameter with Base Point $P(t_0)$ is $= P(x(t_0), y(t_0), z(t_0))$

$$s(t) = \int_{t_0}^{t} |\vec{v}|\, dt$$



Note that $\boxed{L = s(b) - s(a)}$

Exp • Find the arc length parameter along the curve $\vec{r}(t) = (\cos t + t\sin t)\vec{i} + (\sin t - t\cos t)\vec{j}$, considering the Base Point $P(1,0,0)$ and $\frac{\pi}{2} \le t \le \pi$. 55

clearly $t_0 = 0$

$$\vec{V} = (-\sin t + \sin t + t\cos t)\vec{i} + (\cos t - \cos t + t\sin t)\vec{j}$$

$$= (t\cos t)\vec{i} + (t\sin t)\vec{j}$$

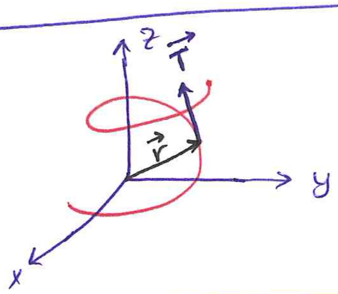
$$|\vec{V}| = \sqrt{t^2\cos^2 t + t^2\sin^2 t} = \sqrt{t^2} = t \quad \text{since } \frac{\pi}{2} \le t \le \pi$$

$$s(t) = \int_0^t |\vec{V}|\,dt = \int_0^t t\,dt = \frac{t^2}{2}\Big|_0^t = \frac{t^2}{2}$$

• Find the length of the curve

$$L = s(\pi) - s\left(\frac{\pi}{2}\right) = \frac{\pi^2}{2} - \frac{\pi^2}{8} = \frac{3\pi^2}{8}$$

Unit Tangent Vector: $\vec{T} = \dfrac{\vec{V}}{|\vec{V}|}$



Exp Find the unit Tangent vector of

$$\vec{r}(t) = (2+t)\vec{i} - (t+1)\vec{j} + t\vec{K}$$

$$\vec{V} = \vec{i} - \vec{j} + \vec{K}$$

$$|\vec{V}| = \sqrt{1+1+1} = \sqrt{3}$$

$$\vec{T} = \frac{\vec{V}}{|\vec{V}|} = \frac{1}{\sqrt{3}}(\vec{i} - \vec{j} + \vec{K})$$

$$= \frac{1}{\sqrt{3}}\vec{i} - \frac{1}{\sqrt{3}}\vec{j} + \frac{1}{\sqrt{3}}\vec{K}$$