13.2 Integrals of Vector Functions

- A differentiable vector function $\vec{R}(t)$ is an antiderivative of a vector function $\vec{r}(t)$ on an interval I if $\frac{d\vec{R}}{dt} = \vec{r}$ at each point of I.

Def: • The indefinite integral of $\vec{r}$ w.r.t t is the set of all antiderivatives of $\vec{r}$, denoted by $\int \vec{r}(t)\,dt$.

• If $\vec{R}$ is any antiderivative of $\vec{r}$, then

$$\int \vec{r}(t)\,dt = \vec{R}(t) + \vec{C}$$

constant vector

Exp Find $\int \left((\cos t)\vec{i} + \vec{j} - 2t\,\vec{k} \right) dt$

$$= \left(\int \cos t\,dt\right)\vec{i} + \left(\int dt\right)\vec{j} - \left(\int 2t\,dt\right)\vec{k} \quad \text{skip}$$

$$= (\sin t + C_1)\vec{i} + (t + C_2)\vec{j} - (t^2 + C_3)\vec{k} \quad \text{skip}$$

$$= (\sin t)\vec{i} + t\vec{j} - t^2\vec{k} + C_1\vec{i} + C_2\vec{j} - C_3\vec{k} \quad \text{skip}$$

$$= (\sin t)\vec{i} + t\vec{j} - t^2\vec{k} + \vec{C} \quad \checkmark$$

Def If the components of $\vec{r}(t) = f(t)\,\vec{i} + g(t)\,\vec{j} + h(t)\,\vec{k}$ are integrable on $[a, b]$, then so is $\vec{r}$. The definite integral of $\vec{r}$ from a to b is:

$$\int_a^b \vec{r}(t)\,dt = \left(\int_a^b f(t)\,dt\right)\vec{i} + \left(\int_a^b g(t)\,dt\right)\vec{j} + \left(\int_a^b h(t)\,dt\right)\vec{k}$$

Exp Find $\int_0^{\pi} \left[(\cos t)\vec{i} + \vec{j} - 2t\,\vec{K}\right] dt$

$$= \left[\sin t \Big|_0^{\pi}\right]\vec{i} + \left[t\Big|_0^{\pi}\right]\vec{j} - \left[t^2\Big|_0^{\pi}\right]\vec{K}$$

$$= [0-0]\vec{i} + [\pi - 0]\vec{j} - [\pi^2 - 0]\vec{K}$$

$$= \pi\vec{j} - \pi^2\vec{K}$$

Exp Let $\vec{a}(t) = (-3\cos t)\vec{i} - (3\sin t)\vec{j} + 2\vec{K}$ be the acceleration of a particle. Assume that at time $t=0$, the particle departed from the point $(3,0,0)$ with velocity $\vec{v}(0) = 3\vec{j}$. Find the position of the article at time $t=\pi$.

- $\vec{v}(t) = \int \vec{a}(t)\,dt = (-3\sin t)\vec{i} + (3\cos t)\vec{j} + 2t\vec{K} + \vec{C_1}$

  $\vec{v}(0) = 3\vec{j} = 3\vec{j} + \vec{C_1} \iff \boxed{\vec{C_1} = \vec{0}}$

- $\vec{r}(t) = \int \vec{v}(t)\,dt = (3\cos t)\vec{i} + (3\sin t)\vec{j} + t^2\vec{K} + \vec{C_2}$

  $\vec{r}(0) = 3\vec{i} + 0\vec{j} + 0\vec{K} = 3\vec{i} + 0\vec{j} + 0\vec{K} + \vec{C_2} \iff \boxed{\vec{C_2} = 0}$

- The particle position at any time t is then given by:

$$\boxed{\vec{r}(t) = (3\cos t)\vec{i} + (3\sin t)\vec{j} + t^2\vec{K}}$$

- The position of the article at time $t=\pi$ is

$$\vec{r}(\pi) = -3\vec{i} + 0\vec{j} + \pi^2\vec{K}$$

$$= -3\vec{i} + \pi^2\vec{K}$$
