2.2] Linear Time Invariant (LTI) system

- If $x(t) = \delta(t) \Rightarrow y(t) = h(t)$

where $h(t)$ is called the "impulse response" of the system

- $\int_{-\infty}^{\infty} x(t)\,\delta(t - t_0) = x(t_0)$

replace t by τ and t_0 by t $\Rightarrow \int_{-\infty}^{\infty} x(\tau)\,\delta(\tau - t)\,d\tau = x(t)$

since $\delta(t)$ is even $\Rightarrow x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t - \tau)\,d\tau$

$\Rightarrow$ $x(t)$ is the sum of weighted and shifted delta functions

$$x(t) = \int_{-\infty}^{\infty} x(\lambda)\,\delta(t-\lambda)\,d\lambda$$

Since the system is linear and the integral can be seen as an infinite sum, we can write

$y(t) = T[x(t)]$ where $T[\cdot]$ is the transformation operator over parameter t

$$y(t) = T\left[\int_{-\infty}^{\infty} x(\lambda)\,\delta(t-\lambda)\,d\lambda\right]$$

$$y(t) = \int_{-\infty}^{\infty} x(\lambda)\,T[\delta(t-\lambda)]\,d\lambda$$

Since the system is TI $\Rightarrow$ $T[\delta(t-\lambda)] = h(t-\lambda)$

$$\Rightarrow \quad y(t) = \int_{-\infty}^{\infty} x(\lambda)\,h(t-\lambda)\,d\lambda$$

$$y(t) = \int_{-\infty}^{\infty} x(\lambda)\, h(t-\lambda)\, d\lambda$$

It is called convolution integral equation

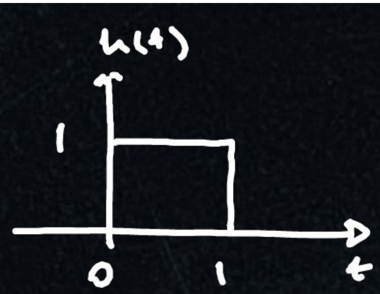
$$y(t) = x(t) \overset{\text{convolution}}{\ast} h(t) = \int_{-\infty}^{\infty} x(\lambda)\, h(t-\lambda)\, d\lambda$$

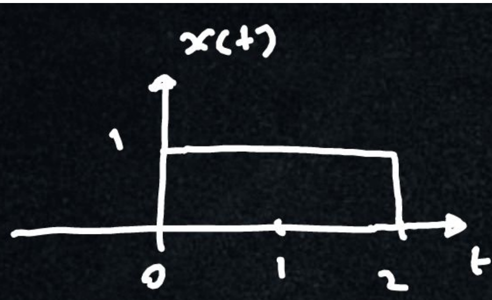
let $\tau = t - \lambda \qquad d\tau = -d\lambda$

$$y(t) = \int_{\infty}^{-\infty} h(\tau)\, x(t-\tau)\,(-d\tau) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

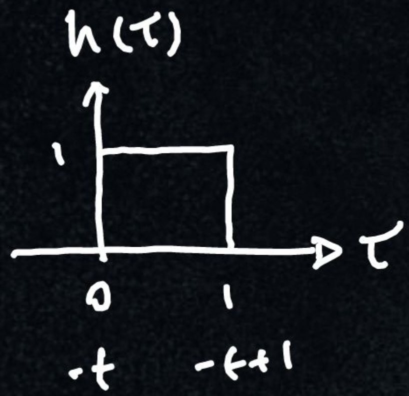
$$\therefore \boxed{y(t) = h(t) \ast x(t) = x(t) \ast h(t)}$$

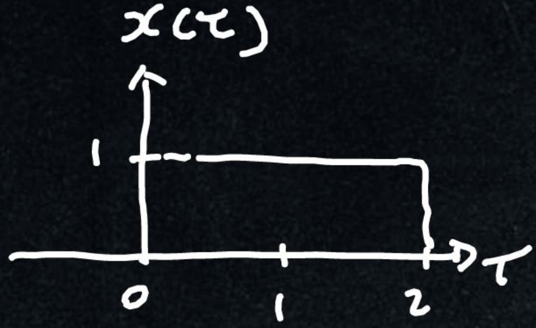
Ex:- Find $y(t)$





$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

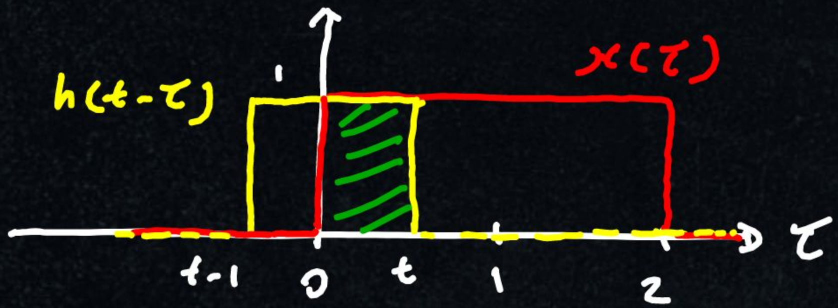




Case 1: $t < 0$

$$\boxed{y(t) = 0}$$

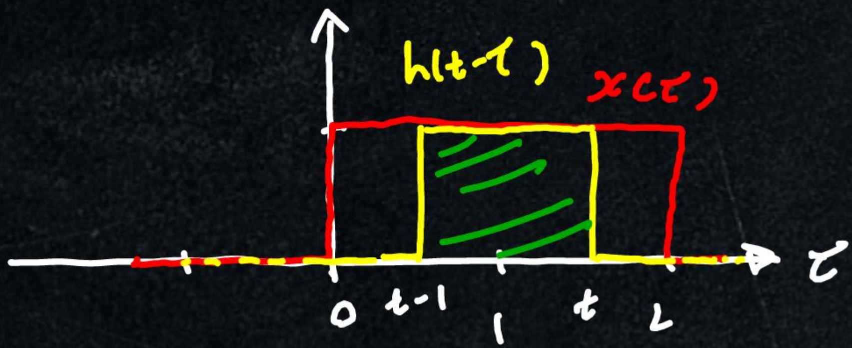
Case 2: $0 < t < 1$



$$y(t) = \int_0^t 1 \, d\tau = \tau \Big|_0^t = t$$

$$\boxed{y(t) = t}$$

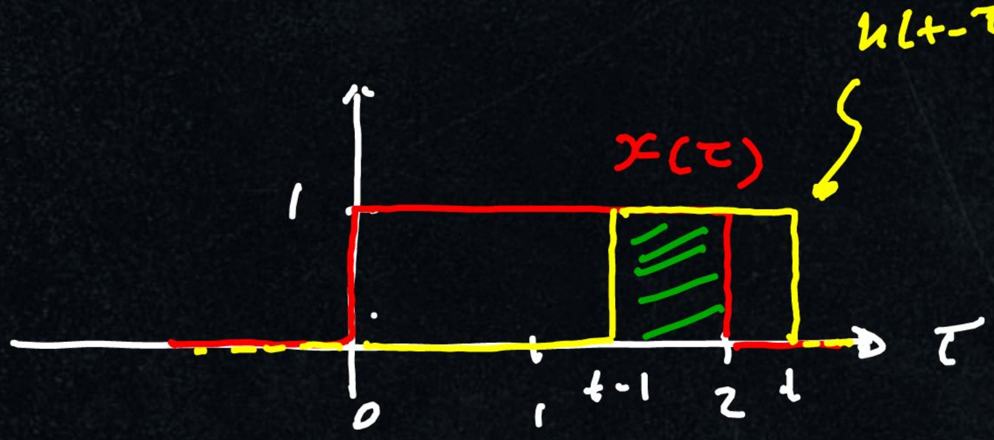
Case 3: $1 < t < 2$



$$y(t) = \int_{t-1}^{t} 1 \, d\tau = t - (t-1) = 1$$

$$\boxed{y(t) = 1}$$

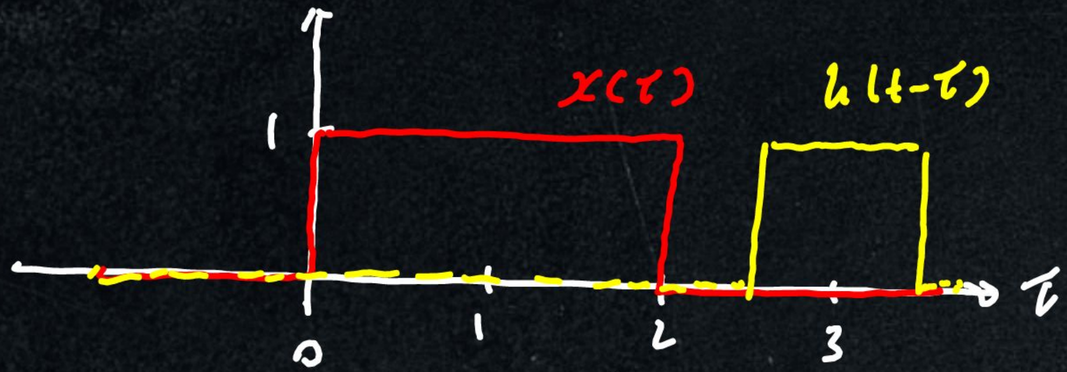
Case 4: $2 < t < 3$



$$y(t) = \int_{t-1}^{2} 1\, d\tau = 2-(t-1)$$

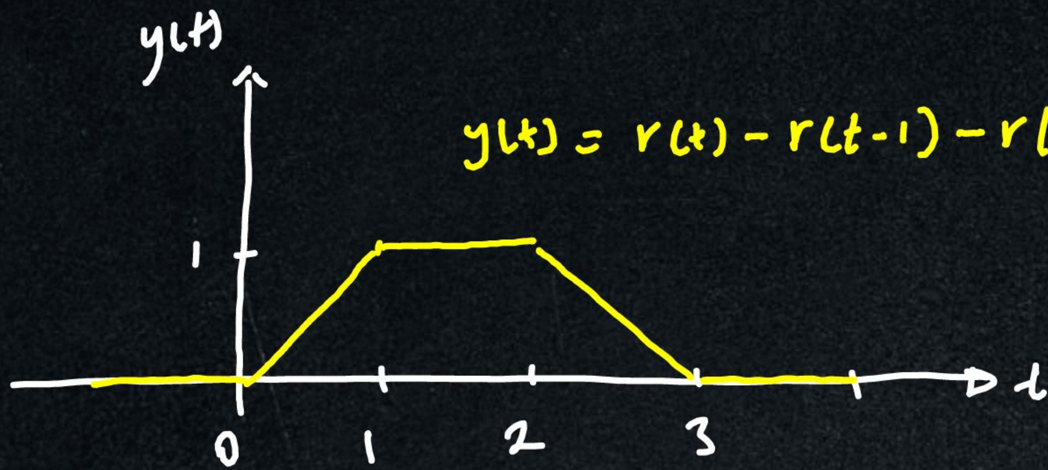
$$\boxed{y(t) = 3-t}$$

Case 5: $t > 3$



$$\boxed{y(t) = 0}$$

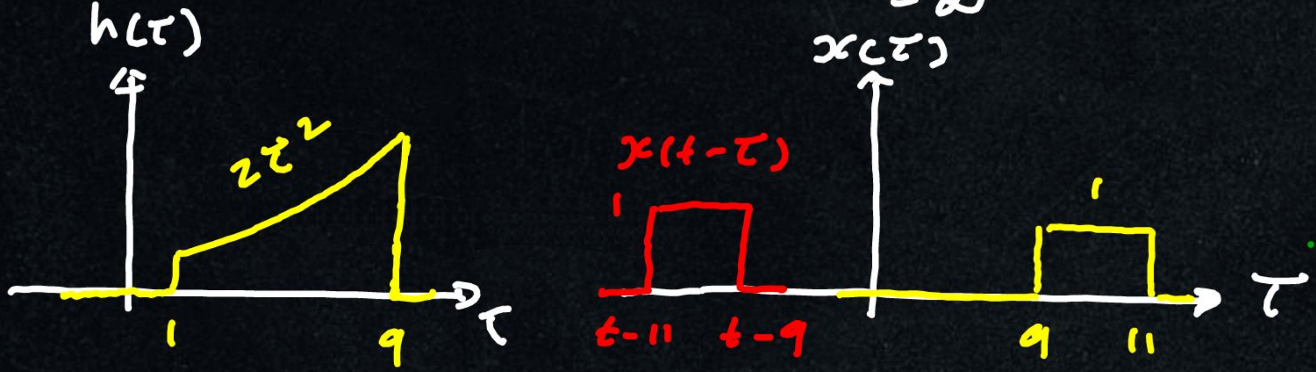
$$
y(t) = \begin{cases} 0 & t < 0 \\ t & 0 < t < 1 \\ 1 & 1 < t < 2 \\ 3 - t & 2 < t < 3 \\ 0 & t > 3 \end{cases}
$$



$$
y(t) = r(t) - r(t-1) - r(t-2) + r(t-3)
$$

EX :- $h(t) = 2t^2\,\pi\left(\frac{t-5}{8}\right)$, $x(t) = \pi\left(\frac{t-10}{2}\right)$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(t-\tau)\,h(\tau)\,d\tau$$



Case 1 : $t < 10 \Rightarrow y(t) = 0$

Case 2: $10 < t < 12 \Rightarrow y(t) = \int_{1}^{t-9} 2\tau^2\,d\tau = \frac{2}{3}\left[(t-9)^3 - 1\right]$

Case 3: $12 < t < 18 \Rightarrow y(t) = \int_{t-11}^{t-9} 2\tau^2\,d\tau = \frac{2}{3}\left[(t-9)^3 - (t-11)^3\right]$

Case 4: $18 < t < 20 \Rightarrow y(t) = \int_{t-11}^{9} 2\tau^2 \, d\tau = \frac{2}{3}\left[(9)^3 - (t-11)^3\right]$

Case 5: $t > 20 \Rightarrow y(t) = 0$

$$y(t) = \begin{cases} 0 & t \leq 10 \\ \frac{2}{3}\left[(t-9)^3 - 1\right] & 10 \leq t \leq 12 \\ \frac{2}{3}\left[(t-9)^3 - (t-11)^3\right] & 12 \leq t \leq 18 \\ \frac{2}{3}\left[9^3 - (t-11)^3\right] & 18 \leq t \leq 20 \\ 0 & t \geq 20 \end{cases}$$

Convolution properties

① Commutative property

$$x(t) * h(t) = h(t) * x(t)$$

② Associative property

$$x_1(t) * [x_2(t) * x_3(t)] = [x_1(t) * x_2(t)] * x_3(t)$$

③ Distributive property

$$x_1(t) * [x_2(t) + x_3(t)] = x_1(t) * x_2(t) + x_1(t) * x_3(t)$$