Exercises :

4.2.0: True or False .

a. If f, g and h are diffble at a, then $(fgh)'(a) = \bar{f}(a)g(a)h(a) + f(a)\bar{g}(a)h(a) + f(a)g(a)\bar{h}(a)$.

True, proof : Apply the product Rule.

$$(fgh)'(a) = (fg)(a)\,\bar{h}(a) + h(a)\,(fg)'(a)$$

$$= f(a)g(a)\,\bar{h}(a) + h(a)\left(f(a)\bar{g}(a) + g(a)\bar{f}(a)\right)$$

$$= f(a)g(a)\bar{h}(a) + f(a)\bar{g}(a)h(a) + \bar{f}(a)g(a)h(a) .$$

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b. If f is twice diffble at a and g is twice diffble at $f(a)$, then

$(g\circ f)''(a) = \bar{g}(f(a))\,\bar{\bar{f}}(a) + \bar{\bar{g}}(f(a))\,(\bar{f}(a))^2$. True .

By chain Rule and product Rule :

$$(g\circ f)'(a) = \bar{g}(f(a))\,\bar{f}(a)$$

$$\Rightarrow (g\circ f)''(a) = \bar{g}(f(a))\,\bar{\bar{f}}(a) + \bar{f}(a)\,\bar{\bar{g}}(f(a))\,\bar{f}(a)$$

$$= \bar{g}(f(a))\,\bar{\bar{f}}(a) + \bar{\bar{g}}(f(a))(\bar{f}(a))^2 .$$

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c. If the nth order derivatives $f^{(n)}(a)$ and $g^{(n)}(a)$ exist then $(f+g)^n(a) = f^n(a) + g^n(a)$?

True, By induction .

1. $n=1 \Rightarrow (f+g)'(a) = \bar{f}(a) + \bar{g}(a)$ which is true.
2. suppose is true for $n = K \in \mathbb{N}$, that is $(f+g)^K(a) = f^K(a) + g^K(a)$.
3. we need to show that it is true for $n = K+1$ (By step 2) .

$$(f+g)^{K+1}(a) = f^{(K+1)}(a) + g^{(K+1)}(a)$$

✓

d. If the nth order derivatives $f^{(n)}(a)$ and $g^{(n)}(a)$ exist and are Non Zero, then

$$\left(\frac{f}{g}\right)^{n}(a) = \frac{g(a)\, f^{n}(a) + (-1)^{n}\, f(a)\, g^{n}(a)}{g^{n+1}(a)}.$$ False.

let $f(x) = x^3$ and $g(x) = x$ $\rightsquigarrow \left(\frac{f}{g}\right)(x) = x^2 \rightarrow \left(\frac{f}{g}\right)'(x) = 2x$, $\left(\frac{f}{g}\right)''(x) = 2$.

using the formula given: $\left(\frac{f}{g}\right)^{2}(x) = \dfrac{x\,(6x) + (-1)^2\, x^3\,(0)}{(x)^3}$

$$= \frac{6x^2}{x^3} = \frac{6}{x} \neq 2$$

4.2.1 : suppose that f and g are diff'ble at 2 and 3 with $\bar{f}(2) = a$, $\bar{f}(3) = b$, $\bar{g}(2) = c$ and $\bar{g}(3) = d$, If $f(2) = 1$, $f(3) = 2$, $g(2) = 3$, $g(3) = 4$ evaluate each of the following derivative

a. $(fg)'(2) = f(2)\,\bar{g}(2) + g(2)\,\bar{f}(2)$

$= 1\,(c) + 3\,(a)$

$= c + 3a.$

b. $\left(\frac{f}{g}\right)'(3) = \dfrac{\bar{f}(3)\, g(3) - \bar{g}(3)\, f(3)}{(g(3))^2} = \dfrac{b\,(4) - d(2)}{16} = \dfrac{2b - d}{8}$

c. $(g\circ f)'(3) = \bar{g}(f(3))\ \bar{f}(3) = \bar{g}(2)\ \bar{f}(3) = cb$

d. $(f\circ g)'(2) = \bar{f}(g(2))\ \bar{g}(2) = \bar{f}(3)\ \bar{g}(2) = bc$

4.2.2 : suppose that f is diffble at 2 and 4 with $f(2)=2$, $f(4)=3$, $\bar{f}(2)=\pi$, $\bar{f}(4)=e$.

a. If $g(x)=x\,f(x^2)$, find the value of $\bar{g}(2)$.

$$\bar{g}(x) = x\,\bar{f}(x^2)(2x) + f(x^2).1$$

$$\bar{g}(2) = 2\,\bar{f}(4)\,(4) + f(4)$$

$$= 8e + 3 .$$

b. If $g(x) = f^2(\sqrt{x})$, find the value of $\bar{g}(4)$. $(x^{\frac{1}{2}}) = \frac{1}{2} x^{-\frac{1}{2}}$

$$\bar{g}(x) = 2\,f(\sqrt{x})\,\bar{f}(\sqrt{x})\left(\frac{1}{2\sqrt{x}}\right)$$

$$\bar{g}(4) = 2\,f(2)\,\bar{f}(2)\,\frac{1}{4} = \frac{4\pi}{4} = \pi$$

c. If $g(x) = \dfrac{x}{f(x^3)}$, find the value of $\bar{g}(\sqrt[3]{2})$.

$$\bar{g}(x) = \frac{f(x^3).1 - x\,\bar{f}(x^3)\,3x^2}{(f(x^3))^2}$$

$$\bar{g}(\sqrt[3]{2}) = \frac{f(2) - 3(2)\,\bar{f}(2)}{(f(2))^2} = \frac{2-6\pi}{4} = \frac{1-3\pi}{2}$$

4.2.3 : Assume that $(e^x)' = e^x$ for all $x\in\mathbb{R}$ and $(\log x)' = \frac{1}{x}$ for $x>0$. use $x^{\alpha} := e^{\alpha \log x}$ to prove that $(x^{\alpha})' = \alpha x^{\alpha-1}$ for all $x>0$ and all $\alpha\in\mathbb{R}$.

$$(x^{\alpha})' = (e^{\alpha\log x})' = e^{\alpha\log x} \cdot \frac{\alpha}{x} = x^{\alpha} \cdot \frac{\alpha}{x} = \alpha\, x^{\alpha-1}$$

□

4.2.4. By 4.2.2 prove that every polynomial belongs to $C^{\infty}(R)$:

$(x^n)' = n x^{n-1}$ for each $n \in N$

If $p(x) = a_n x^n + \dots + a_0$ then it follows $\bar{p}(x) = n a_n x^{n-1} + \dots + a_1$ exists and is a polynomial.

Hence, By induction $p^{(k)}$ exists for all $k \in N$.

4.2.5 : suppose that f is diffble at a and $f(a) \neq 0$

a. Show that for h sufficiently small, $f(a+h) \neq 0$.

If $f(a) \neq 0$, $|f(a+h)| > \frac{|f(a)|}{2} > 0$ for h small.

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b. using def, prove that $\frac{1}{f(x)}$ is diffble at $x = a$ and $\left(\frac{1}{f}\right)'(a) = \frac{-\bar{f}(a)}{f^2(a)}$.

$$\left(\frac{1}{f}\right)'(a) = \lim_{h \to 0} \frac{\frac{1}{f(h+a)} - \frac{1}{f(a)}}{h} = \lim_{h \to 0} \frac{f(a) - f(a+h)}{h\, f(a)\, f(a+h)}$$

$$= -\frac{1}{f(a)} \lim_{h \to 0} \left(\frac{f(a+h) - f(a)}{h} \cdot \frac{1}{f(a+h)} \right)$$

$$= \frac{-1}{f(a)} \cdot \bar{f}(a) \cdot \frac{1}{f(a)} = \frac{-\bar{f}(a)}{f^2(a)}$$

C. use product Rule and the Reciprocal Rule to prove the Quotient Rule directly.

$$\left(\frac{f}{g}\right)'(x) = \left(f(x) \cdot \frac{1}{g(x)}\right)' = f(x) \cdot \frac{-\bar{g}(x)}{g^2(x)} + \frac{1}{g(x)} \cdot \bar{f}(x) \left(\frac{g(x)}{g(x)}\right)$$

$$= \frac{-f(x)\,\bar{g}(x)}{g^2(x)} + \frac{\bar{f}(x)\, g(x)}{g^2(x)}$$

$$= \frac{\bar{f}(x)\, g(x) - f(x)\, \bar{g}(x)}{g^2(x)}$$

4.2.8: Assuming that e^x is diffble on $\mathbb{R}$, prove that $f(x) = \begin{cases} \frac{x}{1+e^{\frac{1}{x}}} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$

is diffble on $[0, \infty)$. Is diffble at 0 ?

$\rightarrow$ on $(0, \infty) \rightarrow f(x) = \frac{x}{1+e^{\frac{1}{x}}}$ which is diffble on $(0, \infty)$.

$\rightarrow$ on $[0, \infty)$: $f'(0) = \lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^+} \frac{\frac{h}{1+e^{\frac{1}{h}}} - 0}{h}$

$= \lim_{h \to 0^+} \frac{1}{1+e^{\frac{1}{h}}} = 0$

$f'(0^-) = \lim_{h \to 0^-} \frac{1}{1+e^{\frac{1}{h}}} = 1$

$f'(0^+) \neq f'(0^-)$ so f is Not diffble at 0.

4.2.9: Prove each of the following statements:

a. The functions $\sin x$ and $\cos x$ are continuous at 0.

By $-\sin x = \sin(-x)$, $\cos(-x) = +\cos x$ And $0 < x\cos x < \sin x < x$, $0 < x \leq \frac{\pi}{2}$.

$0 \leq |\sin x| < |x|$ for $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ Thus, By squeez Thm and i

$\sin x \rightarrow 0 = \sin 0$ as $x \rightarrow 0$.

Hence, By iii, i , $\cos x = 1 - \sin^2(\frac{x}{2}) \rightarrow 1 - 2\sin^2(0) = 1 = \cos(0)$ as $x \rightarrow 0$

b. The function $\sin x$ and $\cos x$ are continuous on $\mathbb{R}$.

let $x_0 \in \mathbb{R}$, By iv and a.

$\sin((x-x_0)+x_0) = \sin(x-x_0)\cos x_0 + \cos(x-x_0)\sin x_0 \to 0 + \sin x_0 = \sin x_0$

as $x \to x_0$.

Hence, By iii, $\cos x = 1-2\sin^2\left(\frac{x}{2}\right) \to 1-2\sin^2\left(\frac{x_0}{2}\right) = \cos x_0$ as $x \to x_0$. $\blacksquare$