

Math141-Calculus I: Review of differentiation and integration  
Lecture notes based on Thomas Calculus Book Chapter 1 to  
Chapter 5

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# Chapter 1

## Functions

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### 1.1 Functions

In this lecture, we review some important functions with their domains, ranges and graphs.

**Definition 1.1.1** *A function  $f$  is a rule that assigns to each point  $x$  in the domain a unique point  $y = f(x)$  in the range of  $f$ . We write  $f : D \rightarrow R$  where  $D$  is the domain of  $f$  and  $R$  is its range.*

**Remark 1.1.1** The set of  $x$ -values at which  $f(x)$  is defined forms the domain of  $f$  while the set of  $y$ -values (the set of the images of the  $x$ -values) forms the range of  $f$ . The domain of  $x$  appears on the horizontal axis (the  $x$ -axis), while the range of  $f$  appears on the vertical axis (the  $y$ -axis).

Now, we give some important basic functions with their domains, ranges and graphs.

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<sup>1</sup>This part is a review of chapter 1 in the textbook

**Example 1.1.1** (a)  $f(x) = x^2$ ,  $D = (-\infty, \infty)$ ,  $R = [0, \infty)$ . If we let  $y = x^2$  then  $x \in (-\infty, \infty)$ ,  $y \in [0, \infty)$ .

(b)  $f(x) = \sqrt{x}$ ,  $D = R = [0, \infty)$ , hence  $x, y \in [0, \infty)$ .

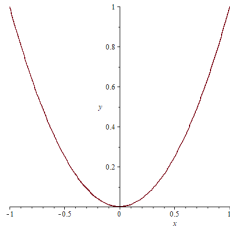


Figure 1.1: Graph of  $y = x^2$

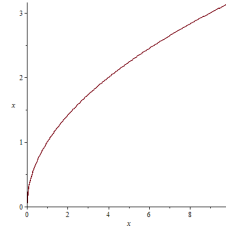


Figure 1.2: Graph of  $y = \sqrt{x}$

(c) The absolute value function  $f(x) = |x| = \sqrt{x^2}$ ,  $D = (-\infty, \infty)$ ,  $R = [0, \infty)$ . Then,  $x \in (-\infty, \infty)$ ,  $y \in [0, \infty)$ .

(d)  $f(x) = \sqrt{1 - x^2}$ . The domain of  $f$  is the set of values of  $x$  such that  $1 - x^2 \geq 0$ , so we must have  $x^2 \leq 1$ . Taking the square root of both sides, we get  $\sqrt{x^2} \leq 1$  which implies that  $|x| \leq 1$ . The last inequality is equivalent to  $-1 \leq x \leq 1$ . We find that  $x \in [-1, 1]$ ,  $y \in [0, 1]$ . So,  $D = [-1, 1]$ ,  $R = [0, 1]$ .

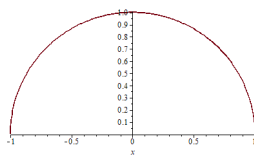


Figure 1.3: Graph of  $y = \sqrt{1 - x^2}$

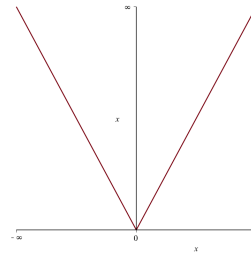
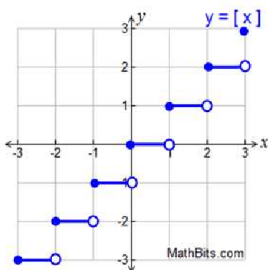


Figure 1.4: Graph of  $y = |x|$

(e) The greatest integer function  $f(x) = \lfloor x \rfloor$ ,  $D = (-\infty, \infty)$ ,  $R = 0, \pm 1, \pm 2, \dots$

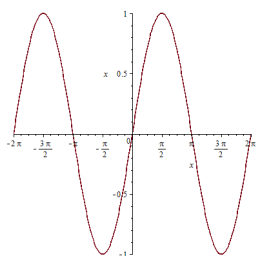
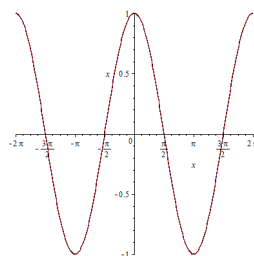
Figure 1.5: Graph of  $y = [x]$ 

## 1.2 Trigonometric functions

In this section, we review the six trigonometric functions:  $\sin x$ ,  $\cos x$ ,  $\tan x$ ,  $\cot x$ ,  $\sec x$  and  $\csc x$ . You are supposed to know the values of these functions at the main values  $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \dots$

(a)  $y = \sin x$ ,  $D = (-\infty, \infty)$ ,  $R = [-1, 1]$ .

(b)  $y = \cos x$ ,  $D = (-\infty, \infty)$ ,  $R = [-1, 1]$ .

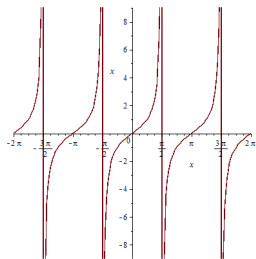
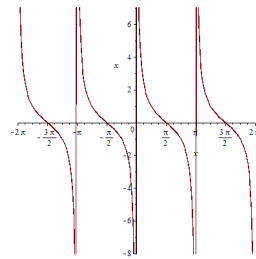
Figure 1.6: Graph of  $y = \sin x$ Figure 1.7: Graph of  $y = \cos x$ 

Note that

$$\cos x = 0 \text{ if } x = \frac{\pi}{2} \pm n\pi \text{ and } \sin x = 0 \text{ if } x = \pm n\pi, n = 0, 1, 2, \dots$$

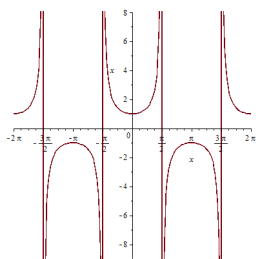
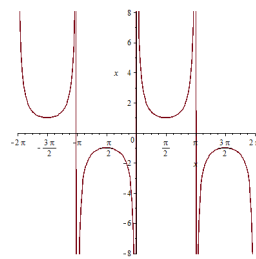
(c)  $y = \tan x = \frac{\sin x}{\cos x}$ ,  $D = (-\infty, \infty) \setminus \{\frac{\pi}{2} \pm n\pi\}$ ,  $n = 0, 1, 2, \dots$ ,  $R = (-\infty, \infty)$

(d)  $y = \cot x = \frac{\cos x}{\sin x}$ ,  $D = (-\infty, \infty) \setminus \{\pm n\pi\}$ ,  $n = 0, 1, 2, \dots$ ,  $R = (-\infty, \infty)$

Figure 1.8: Graph of  $y = \tan x$ Figure 1.9: Graph of  $y = \cot x$ 

(e)  $y = \sec x = \frac{1}{\cos x}$ ,  $D = (-\infty, \infty) \setminus \{\frac{\pi}{2} \pm n\pi\}$ ,  $n = 0, 1, 2, \dots$ ,  
 $R = (-\infty, -1] \cup [1, \infty)$

(f)  $y = \csc x = \frac{1}{\sin x}$ ,  $D = (-\infty, \infty) \setminus \{\pm n\pi\}$ ,  $n = 0, 1, 2, \dots$ ,  
 $R = (-\infty, -1] \cup [1, \infty)$

Figure 1.10: Graph of  $y = \sec x$ Figure 1.11: Graph of  $y = \csc x$ 

**Remark 1.2.1** We have the following results

- Since  $\sin(x + 2\pi) = \sin x$ ,  $\cos(x + 2\pi) = \cos x$ ,  $\sec(x + 2\pi) = \sec x$  and  $\csc(x + 2\pi) = \csc x$ , the functions  $\sin x$ ,  $\cos x$ ,  $\sec x$  and  $\csc x$  are called periodic with period  $2\pi$ .

- Since  $\tan(x + \pi) = \tan x$  and  $\cot(x + \pi) = \cot x$  then  $\tan x$  and  $\cot x$  are periodic with period  $\pi$ .

### 1.2.1 Trigonometric identities

1.  $\sin^2 x + \cos^2 x = 1$ .
2.  $\sin(2x) = 2 \sin x \cos x$ .
3.  $\cos(2x) = \cos^2 x - \sin^2 x$ .
4.  $\cos^2 x = \frac{1 + \cos(2x)}{2}$ .
5.  $\sin^2 x = \frac{1 - \cos(2x)}{2}$ .
6.  $\sec^2 x = 1 + \tan^2 x$ .
7.  $\csc^2 x = 1 + \cot^2 x$ .
8.  $\cos(A + B) = \cos A \cos B - \sin A \sin B$ .
9.  $\sin(A + B) = \sin A \cos B + \cos A \sin B$ .

**Example 1.2.1** Using the above identities, we find the following:

$$(a) \sin(x + \pi) = \sin(x) \underbrace{\cos(\pi)}_{-1} + \cos(x) \underbrace{\sin(\pi)}_0 = -\sin x,$$

$$(b) \cos(x + \pi) = \cos(x) \underbrace{\cos(\pi)}_{-1} - \sin(x) \underbrace{\sin(\pi)}_0 = -\cos x.$$

$$(c) \sin\left(x + \frac{\pi}{2}\right) = \sin(x) \underbrace{\cos\left(\frac{\pi}{2}\right)}_0 + \cos(x) \underbrace{\sin\left(\frac{\pi}{2}\right)}_1 = \cos x,$$

$$(d) \cos\left(x + \frac{\pi}{2}\right) = \cos(x) \underbrace{\cos\left(\frac{\pi}{2}\right)}_0 - \sin(x) \underbrace{\sin\left(\frac{\pi}{2}\right)}_1 = -\sin x$$

### 1.3 Even and odd functions

**Definition 1.3.1** Let  $f$  be a function defined on an interval  $I = [-a, a]$ , where  $a$  is a positive real number. Then

- $f(x)$  is called even if  $f(-x) = f(x)$ . If  $f$  is even then its graph is symmetric about the  $y$ -axis.
- $f(x)$  is called odd if  $f(-x) = -f(x)$ . If  $f$  is odd then its graph is symmetric about the origin.

**Example 1.3.1**  $x^2, x^4, x^6, \dots, \cos x, \sec x$  are even functions.  $x, x^3, x^5, \dots, \sin x, \tan x, \csc x, \cot x$  are odd functions.

**Example 1.3.2** Determine whether the functions  $f(x) = x^2 + |x|$ ,  $g(x) = x^3 + x^5$ ,  $h(x) = x + x^2$  are even, odd or neither.

$$f(-x) = (-x)^2 + |-x| = x^2 + |x| = f(x), \quad \text{so } f \text{ is even}$$

$$g(-x) = (-x)^3 + (-x)^5 = -x^3 - x^5 = -(x^3 + x^5) = -g(x) \text{ so } g \text{ is odd}$$

$$h(-x) = (-x) + (-x)^2 = -x + x^2 \text{ then } h(-x) \neq h(x), \quad h(-x) \neq -h(x)$$

we conclude that  $h$  is neither even nor odd.

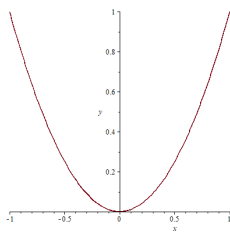


Figure 1.12: Graph of  $y = x^2$

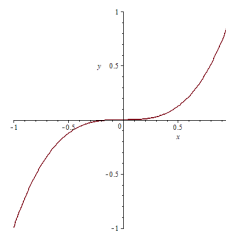


Figure 1.13: Graph of  $y = x^3$



## 1.4 Exercises

(1) Find the domain and the range of the following functions:

(a)  $f(x) = \frac{1}{\sqrt{x}}$ .

(b)  $f(x) = \tan(\pi x)$ .

(c)  $f(x) = 1 + |x|$ .

(d)  $f(x) = \sec^2 x$ .

(e)  $g(x) = \frac{1}{x^2}$ .

(f)  $h(x) = \frac{1}{\sqrt{1-x^2}}$ .

(2) Sketch the following functions:

(a)  $y = \sin(\pi x)$

(b)  $y = |x - 1|$

(c)  $y = \cos(x) + 1$

(3) Determine whether the following functions are even, odd or neither:

(a)  $f(x) = x^2 + 1$ .

(b)  $f(x) = x^3 + x$ .

(c)  $g(t) = \frac{1}{t-1}$ .

(d)  $h(x) = \frac{x}{x^2-1}$ .

(4) Prove the following:

(a) If  $f(x)$  is even and  $g(x)$  is odd then  $(g \circ f)(x)$  is even.

(b) If  $f(x)$  is even and  $g(x)$  is odd then  $\frac{f(x)}{g(x)}$  is odd.



## Chapter 2

# Limits and continuity

1

### 2.1 Limits of functions

When a function  $f$  approaches a certain limit  $L$  as  $x$  approaches  $a$ , we write

$$\lim_{x \rightarrow a} f(x) = L$$

This limit means that *the function gets arbitrarily close to  $L$  when  $x$  is sufficiently close to  $a$* . Notice that  $a$  or  $L$  or both of them can be  $+\infty$  or  $-\infty$ . The function  $f$  may or may not be defined at  $x = a$ . As you know,

$$\lim_{x \rightarrow a} f(x) = L \quad \text{if and only if} \quad \lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x) = L$$

where  $\lim_{x \rightarrow a^+} f(x)$  is the limit of  $f(x)$  as  $x$  approaches  $a$  from the right (also called the right-hand limit) and  $\lim_{x \rightarrow a^-} f(x)$  is the limit of  $f(x)$  as  $x$  approaches  $a$  from the left (also called the left-hand limit).

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<sup>1</sup>This is a review of chapter two in the textbook

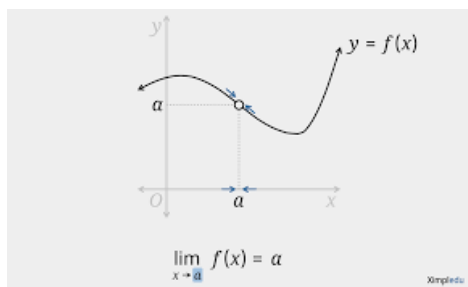


Figure 2.1: Limit of a function

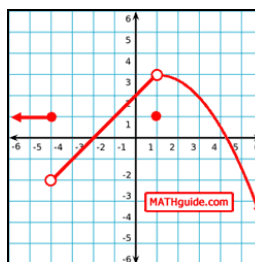


Figure 2.2: Example of limits

**Example 2.1.1** We can use simple techniques to find the following limits:

$$(a) \lim_{x \rightarrow 1} \frac{x-1}{x+1} = 0.$$

$$(b) \lim_{x \rightarrow 1} \frac{x^2-1}{x-1} = \lim_{x \rightarrow 1} \frac{(x-1)(x+1)}{(x-1)} = 2.$$

$$(c) \lim_{x \rightarrow +\infty} \frac{1}{x} = 0.$$

$$(d) \lim_{x \rightarrow 0^+} \frac{1}{x} = +\infty.$$

$$(e) \lim_{x \rightarrow 1} \frac{x^2+x-2}{x^2-x} = \lim_{x \rightarrow 1} \frac{(x+2)(x-1)}{x(x-1)} = 3.$$

$$(f) \lim_{x \rightarrow -1} \frac{\sqrt{x^2+8}-3}{x+1} = \lim_{x \rightarrow -1} \frac{\sqrt{x^2+8}-3}{x+1} \frac{\sqrt{x^2+8}+3}{\sqrt{x^2+8}+3}$$

$$(\sqrt{x^2+8}-3)(\sqrt{x^2+8}+3) = (\sqrt{x^2+8})^2 - 3\sqrt{x^2+8} + 3\sqrt{x^2+8} - 9 = x^2 + 8 - 9 = x^2 - 1 = (x-1)(x+1)$$

$$\lim_{x \rightarrow -1} \frac{\sqrt{x^2+8}-3}{x+1} = \lim_{x \rightarrow -1} \frac{(x-1)(x+1)}{(x+1)\sqrt{x^2+8}+3} = \frac{-2}{6} = -\frac{1}{3}.$$

**Theorem 2.1.1** (*The Sandwich Theorem*) Suppose that

$$g(x) \leq f(x) \leq h(x)$$

for all  $x$  in some open interval containing  $c$ , except possibly at  $x = c$  and that

$$\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} h(x) = L \quad \text{then} \quad \lim_{x \rightarrow c} f(x) = L$$

**Example 2.1.2** Suppose that  $f(x)$  is a function that satisfies

$$1 - x^2 \leq f(x) \leq 1 + x^2$$

Then  $\lim_{x \rightarrow 0} f(x) = 1$  since  $\lim_{x \rightarrow 0} (1 - x^2) = \lim_{x \rightarrow 0} (1 + x^2) = 1$ .

**Example 2.1.3** Find  $\lim_{x \rightarrow +\infty} \frac{\sin x}{x}$ . Since

$$-\frac{1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}$$

and  $\lim_{x \rightarrow \infty} \frac{1}{x} = 0$ , then, by the sandwich theorem

$$\lim_{x \rightarrow \infty} \frac{\sin x}{x} = 0$$

**Remark 2.1.1** Please do not confound the previous limit with  $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ .

**Example 2.1.4** Consider the function

$$f(x) = \begin{cases} x + 1 & , \quad x \leq 0 \\ -x & , \quad x > 0 \end{cases}$$

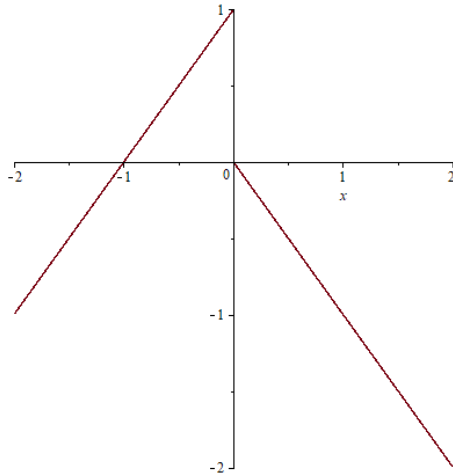
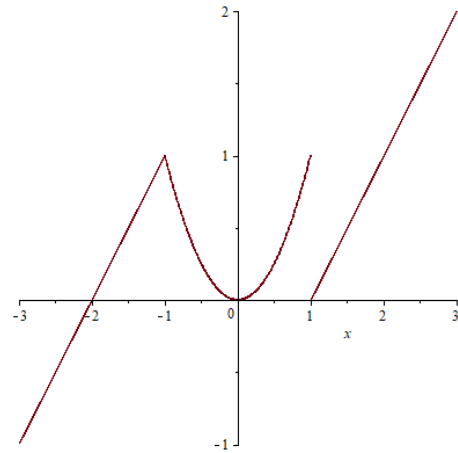
Then,  $\lim_{x \rightarrow 0^+} f(x) = 0$  and  $\lim_{x \rightarrow 0^-} f(x) = 1$ . So,  $\lim_{x \rightarrow 0} f(x)$  does not exist.

We give another example

**Example 2.1.5** Consider the function

$$g(x) = \begin{cases} x + 2 & , \quad x \leq -1 \\ x^2 & , \quad -1 < x \leq 1 \\ x - 1 & , \quad x > 1 \end{cases}$$

Then,  $\lim_{x \rightarrow -1^+} g(x) = \lim_{x \rightarrow -1^-} g(x) = 1$ , so  $\lim_{x \rightarrow -1} g(x) = 1$ . While,  $\lim_{x \rightarrow 1^+} g(x) = 0$ ,  $\lim_{x \rightarrow 1^-} g(x) = 1$ , so  $\lim_{x \rightarrow 1} g(x)$  does not exist.

Figure 2.3: Graph of  $f(x)$ Figure 2.4: Graph of  $g(x)$ 

## 2.2 Continuity

**Definition 2.2.1** A function  $f$  is continuous at a point  $x_0$  if the following conditions are satisfied:

- (a)  $f(x_0)$  exists.
- (b)  $\lim_{x \rightarrow x_0} f(x)$  exists.
- (c)  $\lim_{x \rightarrow x_0} f(x) = f(x_0)$ .

**Example 2.2.1** The functions  $\sin x$ ,  $\cos x$ ,  $|x|$  and all polynomials are continuous on  $(-\infty, \infty)$ .

**Example 2.2.2** The rational functions are continuous at all points except at the zeros of the denominator. For example, the function

$$f(x) = \frac{x^3 + x + 1}{x^2 - 1}$$

is continuous on  $(-\infty, \infty) \setminus \{-1, 1\}$ .

**Example 2.2.3** (a function with removable discontinuity) Consider the function

$$f(x) = \frac{x^2 + 2x - 3}{x^2 - 1}$$

Then

$$\lim_{x \rightarrow 1} \frac{x^2 + 2x - 3}{x^2 - 1} = \lim_{x \rightarrow 1} \frac{(x-1)(x+3)}{(x-1)(x+1)} = \lim_{x \rightarrow 1} \frac{x+3}{x+1} = 2$$

The point  $x = 1$  is called a **removable discontinuity** of the function  $f$  because we can define  $f$  at  $x = 1$  so that we can remove the discontinuity. The following function is called the **continuous extension of  $f$  at  $x = 1$**

$$F(x) = \begin{cases} f(x) & , x \neq 1 \\ 2 & , x = 1 \end{cases}$$

**Theorem 2.2.1** (*The intermediate value theorem*) If  $f$  is a continuous function on a closed interval  $[a, b]$ , and if  $y_0$  is any value between  $f(a)$  and  $f(b)$ , then  $y_0 = f(c)$  for some  $c$  in  $[a, b]$ .

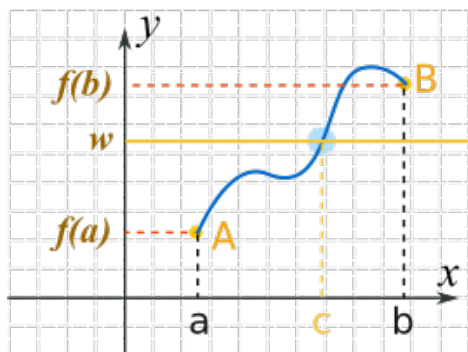


Figure 2.5: Intermediate Value Theorem

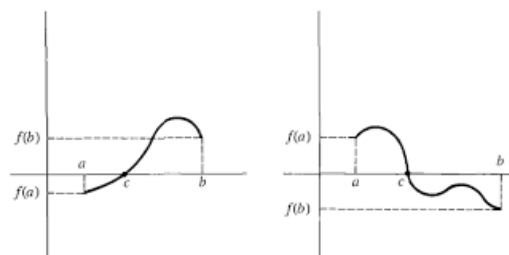


Figure 2.6: Graph of  $g(x)$

Recall that a point  $c$  is called a root of a function  $f$  if  $f(c) = 0$ . We can use the intermediate value theorem to show that a given function has a root in some interval (**Bolzano Theorem**).

**Example 2.2.4** Consider the function  $f(x) = x^3 - x - 1$ . Take  $a = 1$  and  $b = 2$ . Since  $f(1) = -1 < 0$ ,  $f(2) = 5 > 0$  and  $f(1) < 0 < f(2)$  then there exists  $c \in [1, 2]$  such that  $f(c) = 0$ . In fact,  $c \approx 1.324717957$ .

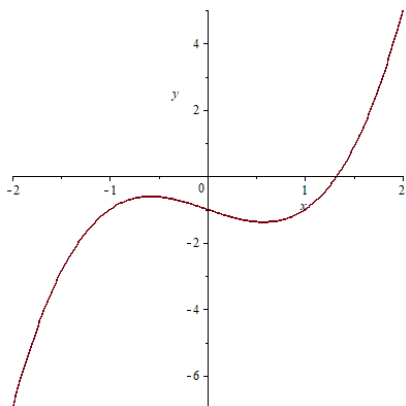


Figure 2.7: Graph of  $y = x^3 - x - 1$

### 2.2.1 Asymptotes

In this section, we are dealing mainly with rational functions. **A rational function is the ratio of two polynomials.** Our objective is to be able to sketch some rational functions using limits and asymptotes. *A method that helps us in finding the limits of a rational function as  $x$  approaches  $+\infty$  or  $-\infty$ , we divide the numerator and denominator by the highest power in the denominator.* Suppose that we want to find the limits of a rational function

$$f(x) = \frac{p(x)}{q(x)}$$

where  $p(x) = a_mx^m + a_{m-1}x^{m-1} + \cdots + a_1x + a_0$  is a polynomial of degree  $m$  and  $q(x) = b_nx^n + b_{n-1}x^{n-1} + \cdots + b_1x + b_0$  is a polynomial of degree  $n$ . Then, we have the following cases:



- (a) if  $m = n$  then  $\lim_{x \rightarrow \pm\infty} f(x) = \frac{a_m}{b_n}$ . For example,  $\lim_{x \rightarrow \pm\infty} \frac{2x^3 - x + 3}{3x^3 + x^2 + x} = \frac{2}{3}$
- (b) if  $m < n$  then  $\lim_{x \rightarrow \pm\infty} f(x) = 0$ . For example,  $\lim_{x \rightarrow \pm\infty} \frac{x^2 + 1}{x^3 + x} = 0$
- (c) if  $m > n$  then  $\lim_{x \rightarrow \pm\infty} f(x) = \pm\infty$ . For example, to find  $\lim_{x \rightarrow \infty} \frac{x^2 + 1}{x + 1}$ , we divide the numerator and denominator by  $x$  to get  $\lim_{x \rightarrow \infty} \frac{x + \frac{1}{x}}{1 + \frac{1}{x}} = +\infty$

**Definition 2.2.2** A line  $y = b$  is a horizontal asymptote of the graph of the function  $y = f(x)$  if either

$$\lim_{x \rightarrow \infty} f(x) = b \quad \text{or} \quad \lim_{x \rightarrow -\infty} f(x) = b$$

**Example 2.2.5** The line  $y = 0$  is a horizontal asymptote for graph of the function  $f(x) = \frac{x}{x^2 + 1}$  since  $\lim_{x \rightarrow +\infty} \frac{x}{x^2 + 1} = \lim_{x \rightarrow -\infty} \frac{x}{x^2 + 1} = 0$ .

**Example 2.2.6** The line  $y = 1$  is a horizontal asymptote for the graph of the function  $f(x) = \frac{x^2}{x^2 + 1}$  since  $\lim_{x \rightarrow +\infty} \frac{x^2}{x^2 + 1} = \lim_{x \rightarrow -\infty} \frac{x^2}{x^2 + 1} = 1$ .

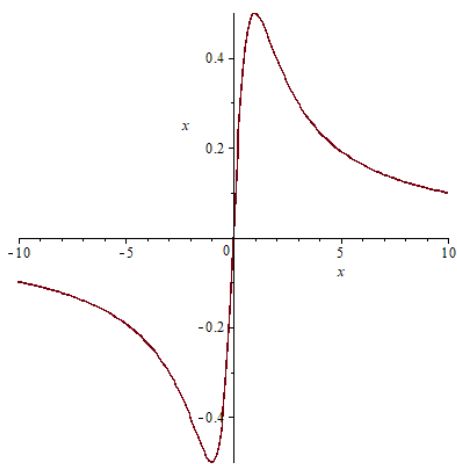


Figure 2.8: Graph of  $f(x) = \frac{x}{x^2 + 1}$

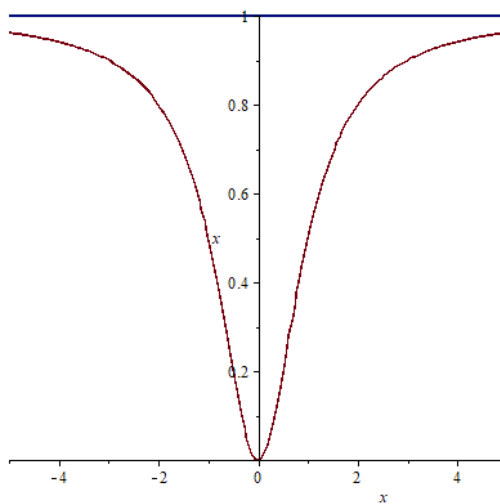


Figure 2.9: Graph of  $f(x) = \frac{x^2}{x^2 + 1}$

**Definition 2.2.3** A line  $x = a$  is a vertical asymptote of the graph of the function  $y = f(x)$  if either

$$\lim_{x \rightarrow a^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \rightarrow a^-} f(x) = \pm\infty$$

**Example 2.2.7** The line  $x = 0$  is a vertical asymptote for  $f(x) = \frac{1}{x}$  since  $\lim_{x \rightarrow 0^+} \frac{1}{x} = +\infty$  and  $\lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty$ .

**Example 2.2.8** Consider the function  $f(x) = \frac{x+1}{x-1}$ . Notice that

$$\lim_{x \rightarrow 1^+} \frac{x+1}{x-1} = +\infty, \quad \lim_{x \rightarrow 1^-} \frac{x+1}{x-1} = -\infty$$

and

$$\lim_{x \rightarrow +\infty} \frac{x+1}{x-1} = \lim_{x \rightarrow -\infty} \frac{x+1}{x-1} = 1$$

Then the line  $x = 1$  is a vertical asymptote and the line  $y = 1$  is a horizontal asymptote.

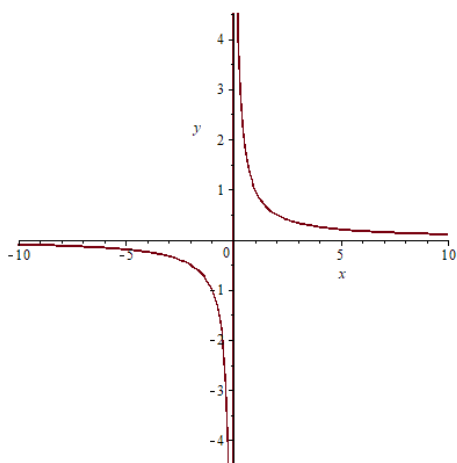


Figure 2.10: Graph of  $f(x) = \frac{1}{x}$

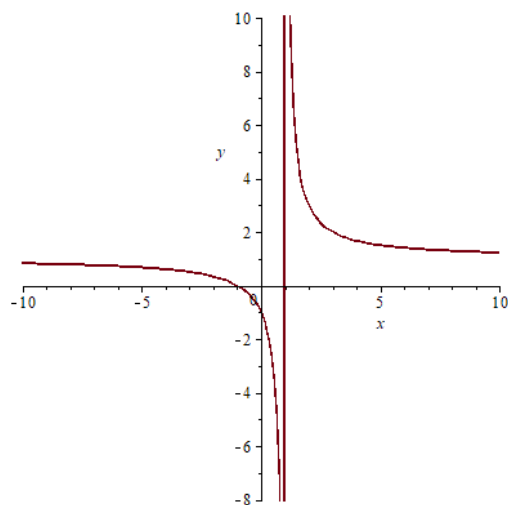


Figure 2.11: Graph of  $f(x) = \frac{x+1}{x-1}$

Consider the following remarks:

**Remark 2.2.1** Suppose that  $f(x)$  is a rational function

- (a) the graph of  $f(x)$  can intersect its horizontal asymptote as in example (2.2.6).
- (b) the graph of  $f(x)$  can have horizontal and vertical asymptotes.
- (c) the graph of  $f$  can have at most one horizontal asymptote.
- (d)  $x = a$  is a vertical asymptote for the graph of  $f$  if  $x = a$  is a root of the denominator of  $f$ . But if  $x = a$  is a root of the denominator of  $f$  then the graph of  $f$  does not have necessarily a vertical asymptote at  $x = a$ . For example, the graph of the function  $f(x) = \frac{x^2+2x-3}{x^2-1}$  does not have a vertical asymptote at  $x = 1$ , see example (2.2.3). Also, the graph of the function  $f(x) = \frac{\sin x}{x}$ , which is not a rational function, does not have a vertical asymptote at  $x = 0$ .

**Example 2.2.9** The function  $f(x) = \frac{\sin x}{x}$  has no vertical asymptote even it is undefined at  $x = 0$  since  $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ .

**Example 2.2.10** Let  $f(x) = \frac{x^2+2x-3}{x^2-1}$ , see example(2.2.3)

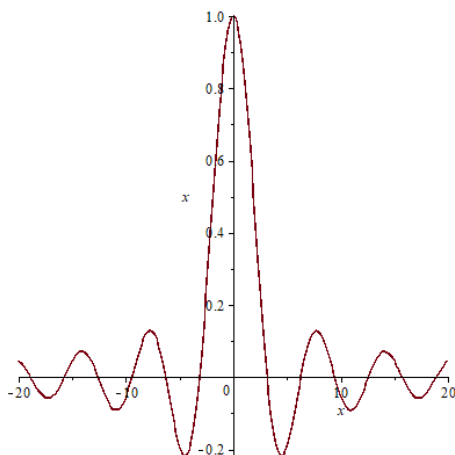
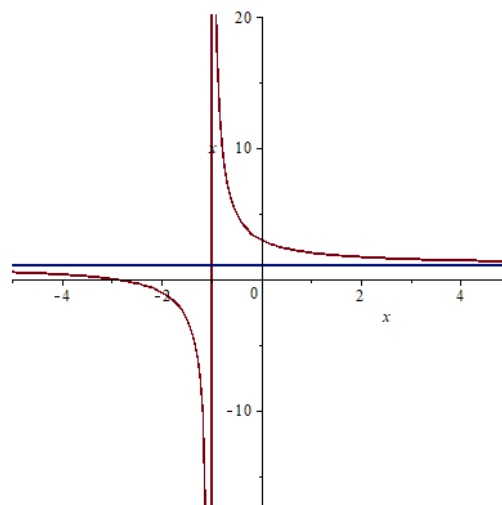
$$\lim_{x \rightarrow 1} \frac{x^2 + 2x - 3}{x^2 - 1} = \lim_{x \rightarrow 1} \frac{(x-1)(x+3)}{(x-1)(x+1)} = \lim_{x \rightarrow 1} \frac{x+3}{x+1} = 2$$

and

$$\lim_{x \rightarrow -1^+} f(x) = \lim_{x \rightarrow -1^+} \frac{x+3}{x+1} = +\infty$$

$$\lim_{x \rightarrow -1^-} f(x) = \lim_{x \rightarrow -1^-} \frac{x+3}{x+1} = -\infty$$

from the previous limits, we conclude that  $x = -1$  is a vertical asymptote but  $x = 1$  is not a vertical asymptote.

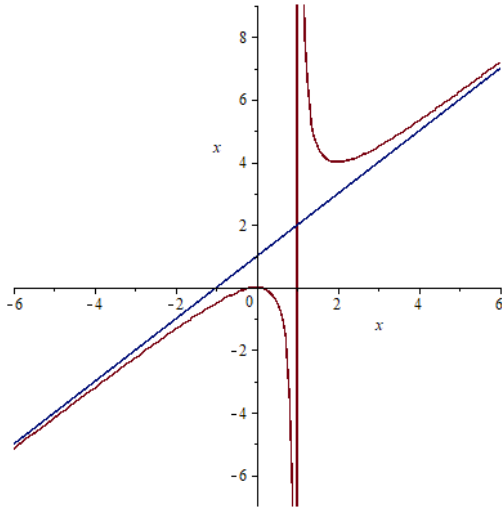
Figure 2.12: Graph of  $f(x) = \frac{\sin x}{x}$ Figure 2.13: Graph of  $f(x) = \frac{x^2+2x-3}{x^2-1}$ 

If the degree of the numerator of a rational function is 1 greater than the degree of the denominator then the graph of  $f$  has an **oblique asymptote**.

**Example 2.2.11** The graph of the function  $f(x) = \frac{x^2}{x-1}$  has an oblique asymptote since the degree of the numerator is 2 and the degree of the denominator is one. Using polynomial division, we can write

$$f(x) = (x + 1) + \frac{1}{x - 1}$$

So, the line  $y = x + 1$  is the oblique asymptote of the graph of  $f$ . Moreover, the line  $x = 1$  is a vertical asymptote for the graph of  $f$  since  $\lim_{x \rightarrow 1^+} f(x) = +\infty$  and  $\lim_{x \rightarrow 1^-} f(x) = -\infty$ . *Note that a rational function cannot have a horizontal and an oblique asymptote at the same time.*

Figure 2.14: Graph of  $y = \frac{x^2}{x-1}$ 

## 2.3 Exercises

1. Find the following limits:

- (a)  $\lim_{t \rightarrow -1} \frac{t^2 + 3t + 2}{t^2 - t - 2}$
- (b)  $\lim_{x \rightarrow 1} \frac{1 - \sqrt{x}}{1 - x}$
- (c)  $\lim_{\theta \rightarrow 1} \frac{\theta^4 - 1}{\theta^3 - 1}$
- (d)  $\lim_{\theta \rightarrow 0} \frac{\sin(2\theta)}{3\theta}$
- (e)  $\lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\sin(2\theta)}$
- (f)  $\lim_{x \rightarrow \infty} \frac{1 + \sqrt{x}}{1 - \sqrt{x}}$
- (g)  $\lim_{x \rightarrow -\infty} \frac{\sqrt{x^2 + 1}}{x + 1}$
- (h)  $\lim_{x \rightarrow -\infty} \frac{\sqrt[3]{x} - \sqrt[5]{x}}{\sqrt[3]{x} + \sqrt[5]{x}}$
- (i)  $\lim_{x \rightarrow \infty} (\sqrt{x^2 + 1} - \sqrt{x^2 - x})$

(j)  $\lim_{t \rightarrow 3^+} \frac{\lfloor t \rfloor}{t}$

(k)  $\lim_{x \rightarrow 0} x \sin\left(\frac{1}{x}\right)$

2. Find the asymptotes of the following functions then sketch their graphs

(a)  $f(x) = \frac{x+1}{x-1}$

(b)  $y = \frac{x^3+1}{x^2}$

(c)  $f(x) = \frac{x^2+1}{x-1}$

(d)  $f(x) = \frac{x^3+1}{x^2-1}$

3. For what values of  $a$  and  $b$  is

$$g(x) = \begin{cases} ax + 2b & , \quad x \leq 0 \\ x^2 + 3a - b & , \quad 0 < x \leq 2 \\ 3x - 5 & , \quad x > 2 \end{cases}$$

continuous at every  $x$ . Then sketch the graph of the function.

4. Find the continuous extension of the function  $h(t) = \frac{t^2+3t-10}{t-2}$ .

5. Use the intermediate value theorem to show that the function  $f(x) = x^3 - 2x^2 + 2$  has a root.

# Chapter 3

## Differentiation

### 3.1 Definition of derivative

**Definition 3.1.1** *The derivative of a function  $f$  at  $x_0$ , denoted  $f'(x_0)$  is defined by*

$$f'(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

*provided this limit exists.*

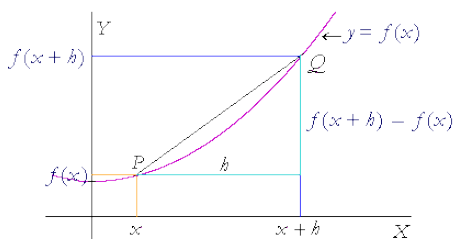


Figure 3.1: Secant line

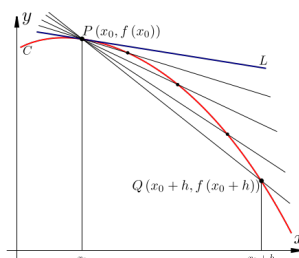


Figure 3.2: Tangent line

Let  $z = x_0 + h$ , then  $h = z - x_0$ . The above limit can be written as

$$f'(x_0) = \lim_{z \rightarrow x_0} \frac{f(z) - f(x_0)}{z - x_0}$$

If  $f'(x_0)$  exists then we say that  $f$  is **differentiable** at  $x_0$ . We say that  $f$  is differentiable on an open interval  $(a, b)$  if it is differentiable at each

point of  $(a, b)$ . We can use the above definition to find the derivative of any differentiable function at any point. The derivative of  $f$  at  $x_0$  gives the rate of change of  $f$  at  $x_0$ . It is also the slope of the tangent line to the graph of  $f$  at  $(x_0, f(x_0))$ .

**Example 3.1.1** Use the definition to find the derivative of the function  $f(x) = \sqrt{x}$ .

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \\ &= \lim_{h \rightarrow 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h} + \sqrt{x})} \\ &= \frac{1}{2\sqrt{x}} \end{aligned}$$

When we say that  $f$  is differentiable on a closed interval  $[a, b]$ , we mean the following

- $f'$  exists at all points in the open interval  $(a, b)$ .
- The **right-hand derivative of  $f$  at  $a$**  exists; that is,

$$f'_+(a) = \lim_{h \rightarrow 0^+} \frac{f(a+h) - f(a)}{h}$$

exists. We denote the right-hand derivative of  $f$  at  $x = a$  by  $f'_+(a)$ .

- The **left-hand derivative of  $f$  at  $b$**  exists; that is,

$$f'_-(b) = \lim_{h \rightarrow 0^-} \frac{f(b+h) - f(b)}{h}$$

exists. We denote the left-hand derivative of  $f$  at  $x = b$  by  $f'_-(b)$ .



**Remark 3.1.1** A function  $f$  is differentiable at  $x = c$  if and only if the right-hand derivative and the left-hand derivative both exist and are equal at  $x = c$ .

If  $f$  is differentiable at  $x = c$  then  $f$  is continuous at  $x = c$ . The converse of this statement is not true, the function  $f(x) = |x|$  is continuous but not differentiable at  $x = 0$ .

**Example 3.1.2** Let  $f(x) = |x|$ . We find the left-hand and right-hand derivatives of  $f$  at  $x = 0$ .

$$f'_+(0) = \lim_{h \rightarrow 0^+} \frac{|0 + h| - |0|}{h} = \lim_{h \rightarrow 0^+} \frac{h}{h} = 1$$

$$f'_-(0) = \lim_{h \rightarrow 0^-} \frac{|0 + h| - |0|}{h} = \lim_{h \rightarrow 0^-} \frac{-h}{h} = -1$$

We conclude that  $f$  is not differentiable at  $x = 0$ .

**Example 3.1.3** Determine whether the following function is differentiable at  $x = 0$

$$f(x) = \begin{cases} x^{2/3} & , x \geq 0 \\ x^{1/3} & , x < 0 \end{cases}$$

Using the definition of the derivative

$$f'_+(0) = \lim_{h \rightarrow 0^+} \frac{f(0 + h) - f(0)}{h} = \lim_{h \rightarrow 0^+} \frac{h^{2/3}}{h} = \lim_{h \rightarrow 0^+} \frac{1}{h^{1/3}} = +\infty$$

$$f'_-(0) = \lim_{h \rightarrow 0^-} \frac{f(0 + h) - f(0)}{h} = \lim_{h \rightarrow 0^-} \frac{h^{1/3}}{h} = \lim_{h \rightarrow 0^-} \frac{1}{h^{2/3}} = +\infty$$

So,  $f$  is not differentiable at  $x = 0$ . The graph of  $f(x)$  has a vertical tangent at  $x = 0$ .

### 3.2 Differentiation rules

**Theorem 3.2.1** Suppose that  $f(x)$  and  $g(x)$  are differentiable at  $x$ ,  $c$  is a constant. Then

$$(1) \frac{d}{dx}(c) = 0$$

$$(2) \frac{d}{dx}x^n = nx^{n-1}, \text{ where } n \text{ is a positive integer.}$$

$$(3) \frac{d}{dx}(cf(x)) = c\frac{df}{dx}$$

$$(4) \frac{d}{dx}(f(x) \pm g(x)) = \frac{df}{dx} \pm \frac{dg}{dx}$$

$$(5) \frac{d}{dx}(f(x)g(x)) = \frac{df}{dx}g(x) + f(x)\frac{dg}{dx}$$

$$(6) \frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{df}{dx} - f(x)\frac{dg}{dx}}{g^2(x)}$$

$$(7) \frac{d}{dx}(f \circ g)(x) = \frac{d}{dx}f(g(x))\frac{dg}{dx}(x) \text{ (Chain Rule).}$$

**Example 3.2.1** Find the derivatives of the functions

$$(1) \frac{d}{dx}(x^5 + 3x^2 + 1) = 5x^4 + 6x$$

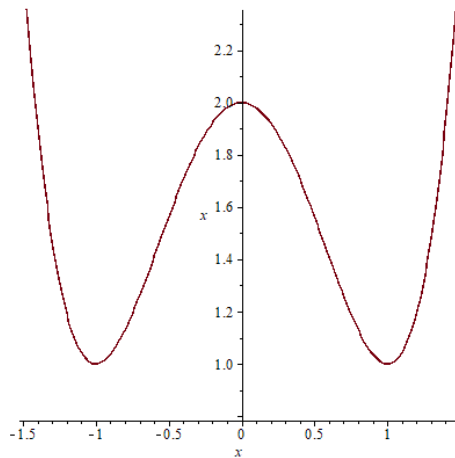
$$(2) \frac{d}{dx}(x^3 + x + 10)(x^4 + x^2 - 20) = (3x^2 + 1)(x^4 + x^2 - 20) + (x^3 + x + 10)(4x^3 + 2x)$$

$$(3) \frac{d}{dx} \frac{x+1}{x^2+1} = \frac{x^2+1-(x+1)(2x)}{(x^2+1)^2} = \frac{1-2x-x^2}{(x^2+1)^2}$$

$$(4) \frac{d}{dx} \frac{1}{x^2+1} = \frac{-2x}{(x^2+1)^2}$$

$$(5) \frac{d}{dx}(x^3 + 2x)^4 = 4(x^3 + 2x)^3(3x^2 + 2)$$

**Example 3.2.2** Where does the graph of  $f(x) = x^4 - 2x^2 + 2$  have horizontal tangent? The curve  $f(x)$  has horizontal tangent if  $f'(x) = 0$ . So,  $f'(x) = 4x^3 - 4x = 0$ , then  $4x(x^2 - 1) = 4x(x - 1)(x + 1) = 0$ . We find that  $f'(x) = 0$  if  $x = 0, 1, -1$ .

Figure 3.3: Graph of  $f(x) = x^4 - 2x^2 + 2$ 

### 3.3 Derivatives of Trigonometric functions

$$(1) \frac{d}{dx}(\sin x) = \cos x.$$

$$(2) \frac{d}{dx}(\cos x) = -\sin x.$$

$$(3) \frac{d}{dx}(\tan x) = \sec^2 x.$$

$$(4) \frac{d}{dx}(\sec x) = \sec x \tan x.$$

$$(5) \frac{d}{dx}(\csc x) = -\csc x \cot x.$$

$$(6) \frac{d}{dx}(\cot x) = -\csc^2 x.$$

To prove (1), we need the following

$$\sin(x + h) = \sin x \cos h + \cos x \sin h$$

$$\sin^2 \theta = \frac{1 - \cos(2\theta)}{2}, \quad \text{let } \theta = \frac{h}{2}, \quad \text{then } \sin^2\left(\frac{h}{2}\right) = \frac{1 - \cos h}{2}$$

So, we get

$$1 - \cos h = 2 \sin^2\left(\frac{h}{2}\right) \quad \Rightarrow \quad \cos h - 1 = -2 \sin^2\left(\frac{h}{2}\right)$$

$$\lim_{h \rightarrow 0} \frac{\cos h - 1}{h} = \lim_{h \rightarrow 0} -2 \frac{\sin^2 \left(\frac{h}{2}\right)}{h} = \lim_{h \rightarrow 0} -2 \frac{\sin \left(\frac{h}{2}\right)}{h} \sin \left(\frac{h}{2}\right) = -1.0 = 0$$

We prove (1)

$$\begin{aligned} \frac{d}{dx} \sin x &= \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin x}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sin x(\cos h - 1)}{h} + \lim_{h \rightarrow 0} \frac{\cos x \sin h}{h} \\ &= \sin x \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} + \cos x \lim_{h \rightarrow 0} \frac{\sin h}{h} \\ &= \sin x \cdot 0 + \cos x \cdot 1 \\ &= \cos x \end{aligned}$$

Similarly, we prove (2)

$$\begin{aligned} \frac{d}{dx} \cos x &= \lim_{h \rightarrow 0} \frac{\cos(x+h) - \cos x}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cos x(\cos h - 1) - \sin x \sin h}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cos x(\cos h - 1)}{h} - \lim_{h \rightarrow 0} \frac{\sin x \sin h}{h} \\ &= \cos x \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} - \sin x \lim_{h \rightarrow 0} \frac{\sin h}{h} \\ &= \cos x \cdot 0 - \sin x \cdot 1 \\ &= -\sin x \end{aligned}$$

Derivative of other trigonometric functions. The derivative of  $y = \tan x$

$$\begin{aligned} \frac{d}{dx} \tan x &= \frac{d \sin x}{dx \cos x} \\ &= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\ &= \frac{1}{\cos^2 x} \\ &= \sec^2 x \end{aligned}$$

The derivative  $y = \cot x$

$$\begin{aligned} \frac{d}{dx} \cot x &= \frac{d \cos x}{dx \sin x} \\ &= \frac{(\sin x)(-\sin x) - (\cos x)(\cos x)}{\sin^2 x} \\ &= -\frac{\sin^2 x + \cos^2 x}{\sin^2 x} \\ &= -\frac{1}{\sin^2 x} \\ &= -\csc^2 x \end{aligned}$$

The derivative of  $y = \sec x$

$$\begin{aligned} \frac{d}{dx} \sec x &= \frac{d}{dx} \frac{1}{\cos x} \\ &= \frac{-(-\sin x)}{\cos^2 x} \\ &= \frac{1}{\cos x} \frac{\sin x}{\cos x} \\ &= \sec x \tan x \end{aligned}$$

Finally, the derivative of  $y = \csc x$

$$\begin{aligned} \frac{d}{dx} \csc x &= \frac{d}{dx} \frac{1}{\sin x} \\ &= \frac{-\cos x}{\sin^2 x} \\ &= -\frac{1}{\sin x} \frac{\cos x}{\sin x} \\ &= -\csc x \cot x \end{aligned}$$

**Example 3.3.1** Find the derivatives of the following functions:

1.  $\frac{d}{dx} \frac{1}{\sin x + \cos x} = -\frac{\cos x - \sin x}{(\sin x + \cos x)^2} = \frac{\sin x - \cos x}{(\sin x + \cos x)^2}$
2.  $\frac{d}{dt} \frac{\tan t}{1 + \sec t} = \frac{(1 + \sec t) \sec^2 t - \tan t (\sec t \tan t)}{(1 + \sec t)^2}$
3.  $\frac{d}{dx} \tan(\sqrt{x}) = (\sec^2 \sqrt{x}) \frac{1}{2\sqrt{x}}$
4.  $\frac{d}{d\theta} \cos(\sin \theta) = -\sin(\sin \theta) \cos \theta$
5.  $\frac{d}{ds} \cot\left(\frac{1}{s}\right) = -\csc^2\left(\frac{1}{s}\right) \left(-\frac{1}{s^2}\right) = \csc^2\left(\frac{1}{s}\right) \left(\frac{1}{s^2}\right)$
6.  $\frac{d}{dx} (\sec x \tan x) = \sec^3 x + \sec x \tan^2 x$

**Example 3.3.2** Find the equation of the tangent line to the curve  $f(x) = \sec x \tan x$  at  $x = \frac{\pi}{4}$ .

From the above example, the slope of the tangent line is  $f'\left(\frac{\pi}{4}\right) = \sec^3\left(\frac{\pi}{4}\right) + \sec\left(\frac{\pi}{4}\right) \tan\left(\frac{\pi}{4}\right) = 3\sqrt{2}$  and  $f\left(\frac{\pi}{4}\right) = \sqrt{2}$ , so the line passes through the point  $\left(\frac{\pi}{4}, \sqrt{2}\right)$ . Then, the equation of the tangent line to the curve  $f(x)$  at the point  $\left(\frac{\pi}{4}, \sqrt{2}\right)$  is

$$y - \sqrt{2} = 3\sqrt{2}\left(x - \frac{\pi}{4}\right)$$

We can find higher order derivatives, for example, if  $y = x^3 + x^2$  then  $y' = 3x^2 + 2x$ ,  $y'' = 6x + 2$ ,  $y''' = 6$ .

### 3.4 Implicit differentiation

In this section, we consider equations that define relation between  $x$  and  $y$ . We will learn how to find  $\frac{dy}{dx}$  using implicit differentiation. Let us consider some examples:

**Example 3.4.1** The equation  $x^2 + y^2 = 1$  defines the unit circle (the circle with center  $(0,0)$  and radius one). To find  $y'$ , we differentiate both sides with respect to  $x$  to get  $2x + 2yy' = 0$ , from which we find that  $y' = -x/y$ .

We can differentiate again to find the second order derivative  $y''$ .

$$y'' = \frac{d^2y}{dx^2} = \frac{-y + xy'}{y^2} = \frac{-y + x\left(\frac{-x}{y}\right)}{y^2} = -\frac{x^2 + y^2}{y^3} = \frac{-1}{y^3}$$

**Example 3.4.2** Consider the implicit equation  $xy = \cot(xy)$ . Differentiate both sides with respect to  $x$ . Then

$$y + x \frac{dy}{dx} = -\csc^2(xy) \left( y + x \frac{dy}{dx} \right) \Rightarrow (x + \csc^2(xy)) \frac{dy}{dx} = -y - y \csc^2(xy)$$

From which we find that

$$\frac{dy}{dx} = \frac{-y - y \csc^2(xy)}{x + x \csc^2(xy)} = \frac{-y(1 + \csc^2(xy))}{x(1 + \csc^2(xy))} = -\frac{y}{x}$$

**Example 3.4.3** The point  $(1, 1)$  lies on the curve  $x^3 + y^3 - 2xy = 0$ . Then find the tangent and normal to the curve there. Differentiating implicitly, we get

$$3x^2 + 3y^2 \frac{dy}{dx} - 2y - 2x \frac{dy}{dx} = 0 \Rightarrow 3x^2 - 2y + (3y^2 - 2x) \frac{dy}{dx} = 0$$

from which we get

$$\frac{dy}{dx} = -\frac{3x^2 - 2y}{3y^2 - 2x}$$

The slope of the tangent line at  $(1, 1)$  equals  $-1$  and the slope of the normal line equals  $1$ . So, the equation of the tangent line and normal line are

$$\text{Tangent line } y - 1 = -(x - 1), \quad \text{normal line } y - 1 = x - 1$$

So, the equation of the tangent line is  $y = 2 - x$  and the equation of the normal line is  $y = x$ .

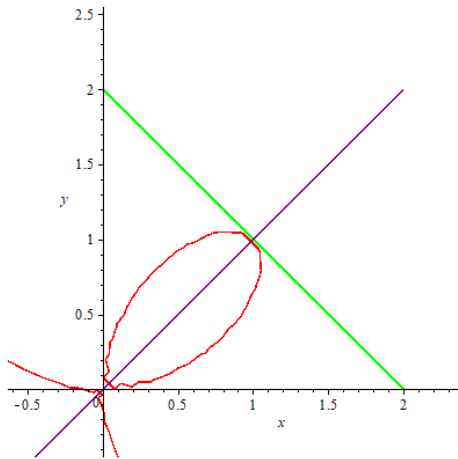


Figure 3.4: Plot of  $x^3 + y^3 - 2xy = 0$  and its tangent and normal lines at  $(1, 1)$

**Example 3.4.4** Find the two points where the curve  $x^2 + xy + y^2 = 7$  crosses the  $x$ -axis and show that the tangents to the curve at these points are parallel. The curve crosses the  $x$ -axis when  $y = 0$ , so we get  $x^2 = 7$  and  $x = \pm\sqrt{7}$ . Then, the curve crosses the  $x$ -axis at  $(\pm\sqrt{7}, 0)$ . Now, we find  $y'$ .

$$2x + y + x \frac{dy}{dx} + 2y \frac{dy}{dx} = 0 \Rightarrow (2x + y) + (x + 2y) \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-2x - y}{x + 2y}$$

when  $y = 0$ , we get

$$\frac{dy}{dx} = \frac{-2x}{x} = -2$$



### 3.5 Linearization and Differentials

Sometimes, we need to approximate a given nonlinear function with a linear function at some point near  $(a, f(a))$ . The best linear function that approximates  $f(x)$  near  $x = a$ , provided that  $f$  is differentiable at  $x = a$ , is its tangent line whose equation is given by

$$L(x) = f(a) + f'(a)(x - a)$$

$L(x)$  is called the **linearization of  $f(x)$  at  $x = a$**  and the approximation  $f(x) \approx L(x)$  is called the **standard linear approximation of  $f$  at  $a$** .

**Example 3.5.1** Find the linearization of the function  $f(x) = \sqrt{1+x}$  at  $x = 0$ . We find that  $f(0) = 1$  and  $f'(x) = \frac{1}{2}(1+x)^{-1/2}$ , so  $f'(0) = \frac{1}{2}$ . The linearization of  $f$  at  $x = 0$  is  $L(x) = 1 + \frac{1}{2}x$ .

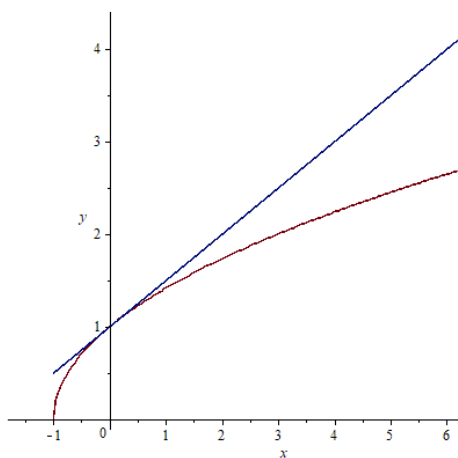


Figure 3.5: Plot of  $f(x) = \sqrt{1+x}$  and its linearization  $L(x) = 1 + \frac{x}{2}$

We can use the linearization to approximate the values of  $f$  near  $x = 0$ . Of course, the closer is  $x$  to 0, the better is the approximation.

$x$	Approximation	True value	$ \text{True value}-\text{Approx.} $
0.2	$\sqrt{1.2} \approx 1 + \frac{0.2}{2} = 1.1$	1.095445	$< 10^{-2}$
0.05	$\sqrt{1.05} \approx 1 + \frac{0.05}{2} = 1.025$	1.024695	$< 10^{-3}$
0.005	$\sqrt{1.005} \approx 1 + \frac{0.005}{2} = 1.00250$	1.002497	$< 10^{-5}$

**Example 3.5.2** Find the linearization of the function  $f(x) = \sqrt{1+x}$  at  $x = 3$ . Note that  $f(3) = 2$ ,  $f'(x) = \frac{1}{2}(1+x)^{-1/2}$ , so  $f'(3) = \frac{1}{4}$ . The linearization of  $f(x)$  at  $x = 3$  is given by

$$L(x) = 2 + \frac{1}{4}(x - 3)$$

We plot the graph of  $f(x)$  with its linearizations at  $x = 0$  and  $x = 3$ .

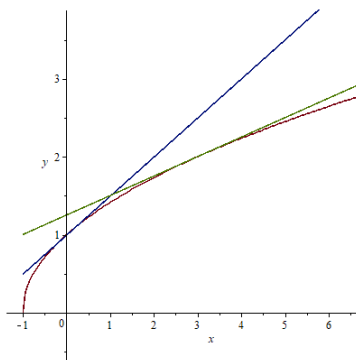


Figure 3.6: The graph of  $f(x)$  with its linearizations

**Example 3.5.3** Find the linearization of the function  $f(x) = \sec x$  at  $x = \frac{\pi}{4}$ . We need to find  $f(\frac{\pi}{4})$  and  $f'(\frac{\pi}{4})$ . Now,  $f'(x) = \sec x \tan x$ , so  $f'(\frac{\pi}{4}) = \sqrt{2}$  and  $f(\frac{\pi}{4}) = \sqrt{2}$ . Then the linearization is

$$L(x) = \sqrt{2} + \sqrt{2}(x - \frac{\pi}{4})$$

Now, suppose that we move from a point  $x = a$  to a nearby point  $a + dx$ . The change in  $f$  is

$$\Delta f = f(a + dx) - f(a)$$

while the change in  $L$  is

$$\begin{aligned}\Delta L &= L(a + dx) - L(a) \\ &= \cancel{f(a)} + f'(a)(a + dx - a) - \cancel{f(a)} \\ &= f'(a)dx\end{aligned}$$

Now, near  $x = a$ , we have

$$f \approx L \quad \text{then} \quad \Delta f \approx \Delta L = f'(a)dx$$

Therefore,  $f'(a)dx$  gives an approximation for  $\Delta f$ . The quantity  $f'(a)dx$  is called the **differential of  $f$  at  $x = a$** . So, we get

$$\Delta f \approx df$$

**Example 3.5.4** Find the differentials of the following functions

(1)  $f(x) = \tan^2 x$ , then  $df(x) = 2 \tan x \sec^2 x \, dx$

(2)  $g(x) = \frac{1}{x}$  then  $df(x) = -\frac{dx}{x^2}$

**Example 3.5.5** The radius  $r$  of a circle increases from 10 to 10.1 m. Use  $dA$  to estimate the increase in the circle's area  $A$ . Estimate the area of the enlarged circle and compare your estimate to the true area found by direct calculations.

**Solution:** The area of the circle is  $A(r) = \pi r^2$ . Then  $dA = 2\pi r dr$ . The estimated increase in the area of the circle is

$$dA = 2\pi(10)0.1 = 2\pi$$

The exact change in the area of the circle is

$$\Delta A = A(10.1) - A(10) = 102.01\pi - 100\pi = 2.02\pi$$

The estimate area of the enlarged circle is

$$A(10.1) \approx A(10) + dA = 100\pi + 2\pi = 102\pi$$

The exact value of the area of the enlarged circle is  $A(10.1) = \pi(10.1)^2 = 102.01\pi$ . The error in this estimation is  $|102.01\pi - 102\pi| = 0.01\pi$ .

### 3.6 Exercises

1. Find the derivatives of the following functions:

(a)  $f(s) = \frac{\sqrt{s-1}}{\sqrt{s+1}}$

(b)  $f(x) = \left(\frac{1}{x} - x\right)(x^2 + 1)$

(c)  $g(x) = \sec(2x + 1) \cot(x^2)$

(d)  $s(t) = \frac{1 + \csc t}{1 - \csc t}$

(e)  $f(x) = x^3 \sin x \cos x$ .

(f)  $x^{1/2} + y^{1/2} = 1$ .

2. Find  $\frac{dy}{dx}$  for the following:

(i)  $y = \cot^2 x$

(ii)  $x^2 + y^2 = x$ .

(iii)  $y = \frac{\sin x}{1 - \cos x}$ .

3. Find the points on the curve  $y = 2x^3 - 3x^2 - 12x + 20$  where the tangent is parallel to the  $x$ -axis.

4. For what values of the constant  $a$ , if any, is

$$f(x) = \begin{cases} \sin(2x) & , \quad x \leq 0 \\ ax & , \quad x > 0 \end{cases}$$

(i) continuous at  $x = 0$ ?

(ii) Differentiable at  $x = 0$ .

5. Find the normals to the curve  $xy + 2x - y = 0$  that are parallel to the line  $2x + y = 0$ .
6. Find the linearization of the following functions at the given points
  - (a)  $f(x) = \tan x$ ,  $x = \pi/4$ .
  - (b)  $g(x) = \frac{1}{x}$ ,  $x = 1$ .
  - (c)  $h(x) = \frac{x^2}{x^2+1}$ ,  $x = 0$ .
  - (d)  $f(x) = 1 + \cos \theta$ ,  $\theta = \frac{\pi}{3}$ .
7. The radius of a circle is increased from 2 to 2.02 m.
  - (a) Estimate the resulting change in area.
  - (b) Express the estimate as a percentage of the circle's original area.



## Chapter 4

# Applications of derivatives

In this chapter, we show how can we use derivatives to find the periods in which a given function  $f(x)$  is increasing or decreasing and the periods in which  $f$  is concave up or concave down. Moreover, we use derivatives to find the extreme values of  $f(x)$ .

### 4.1 Increasing and decreasing functions

**Definition 4.1.1** Let  $f(x)$  be a function defined on an interval  $I$ . Then,

- (a)  $f$  is increasing on  $I$  if whenever  $x_2 > x_1$  then  $f(x_2) > f(x_1)$ , for all  $x_1, x_2$  in  $I$ .
- (b)  $f$  is decreasing on  $I$  if whenever  $x_2 > x_1$  then  $f(x_2) < f(x_1)$ , for all  $x_1, x_2$  in  $I$ .

For example, the functions  $x, x^3, \sqrt{x}$  are increasing functions, while the functions  $1 - x, -x^3$  and  $\frac{1}{x}, x > 0$  are all decreasing. In general, it may be not easy to find the intervals over a given function is increasing or decreasing. We use the first derivative to find these intervals as in the following theorem

**Theorem 4.1.1** Suppose that  $f$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$  then

(a) If  $f'(x) > 0$ , for all  $x \in (a, b)$  then  $f$  is increasing on  $[a, b]$ .

(b) If  $f'(x) < 0$ , for all  $x \in (a, b)$ , then  $f$  is decreasing on  $[a, b]$ .

**Example 4.1.1** Let  $f(x) = x^3 - 12x - 5$ . Then

$$f'(x) = 3x^2 - 12 = 3(x - 2)(x + 2)$$

Note that  $f'(x) > 0$  for all  $x \in (-\infty, -2) \cup (2, \infty)$  and  $f'(x) < 0$  for all  $x \in (-2, 2)$ . So,  $f$  is increasing on  $(-\infty, -2] \cup [2, \infty)$  and decreasing on  $[-2, 2]$ .

**Example 4.1.2** Let  $g(x) = x^3 + x^2 - x + 1$  then

$$g'(x) = 3x^2 + 2x - 1 = (3x - 1)(x + 1)$$

Then,  $g'(x) > 0$  for all  $x \in (-\infty, -1) \cup (\frac{1}{3}, \infty)$  and  $g'(x) < 0$  for all  $x \in (-1, \frac{1}{3})$ . So,  $g$  is increasing on  $(-\infty, -1] \cup [\frac{1}{3}, \infty)$  and is decreasing on  $[-1, \frac{1}{3}]$ .

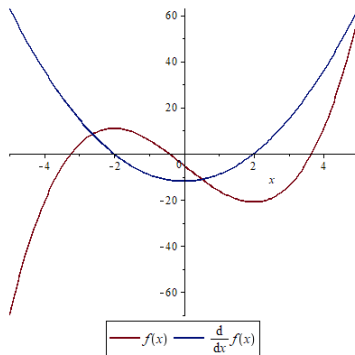


Figure 4.1: Limit of a function

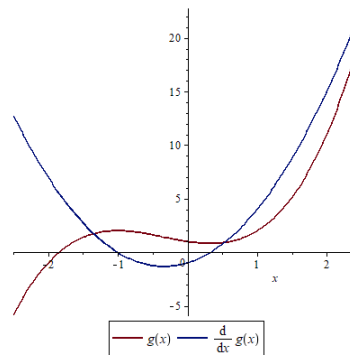


Figure 4.2: Example of limits



## 4.2 Extreme values of functions

**Definition 4.2.1** Let  $f$  be a function with domain  $D$ . Then,

(a)  $f$  has an **absolute maximum** value on  $D$  at a point  $c$  if  $f(x) \leq f(c)$ , for all  $x \in D$ .

(b)  $f$  has an **absolute minimum** value on  $D$  at a point  $c$  if  $f(x) \geq f(c)$ , for all  $x \in D$ .

$f(c)$  is called local maximum (resp. local minimum) if the inequality in (a) (resp. (b)) holds in a small interval around  $x = c$ .

**Example 4.2.1** The function  $f(x) = x^3$ ,  $D = [-1, 1]$  has absolute minimum value  $f(-1) = -1$  and absolute maximum value  $f(1) = 1$ . Similarly, the function  $f(x) = x^2$  on  $[-1, 1]$  has absolute maximum at  $x = \pm 1$  and absolute minimum at  $x = 0$ . But if we consider the functions  $x^2$  and  $x^3$  over the open interval  $(-1, 1)$  then  $x^3$  has neither maximum nor minimum on  $(-1, 1)$  and  $x^2$  has absolute minimum at  $x = 0$ .

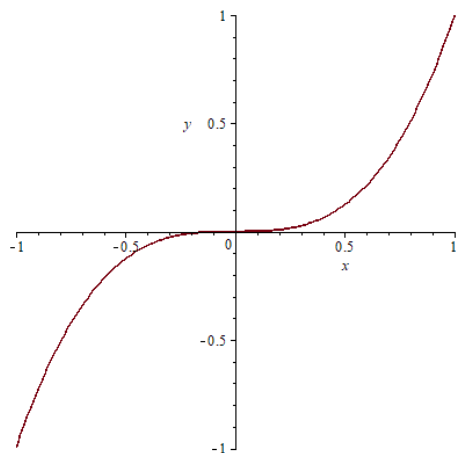


Figure 4.3: The graph of  $f(x) = x^3$  on  $[-1, 1]$

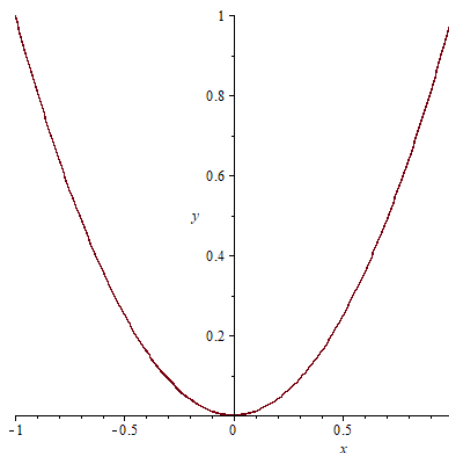


Figure 4.4: The graph of  $f(x) = x^2$  on  $[-1, 1]$

**Theorem 4.2.1** *If  $f$  is continuous on a closed interval  $[a, b]$  then  $f$  has both an absolute maximum value and an absolute minimum value.*

To find the extreme values of a function  $f$  on a closed interval, we look for these values at the endpoints of the interval and at the interior points where  $f' = 0$  or undefined (**critical points**).

**Definition 4.2.2** *An interior point where  $f'$  equals zero or undefined is called a critical point of  $f$ .*

**Example 4.2.2** Let  $f(x) = x\sqrt{1-x^2}$ . The domain of this function is  $D = [-1, 1]$  and  $f$  is differentiable on  $(-1, 1)$  with derivative

$$f'(x) = \sqrt{1-x^2} + x \frac{-2x}{2\sqrt{1-x^2}} = \frac{1-2x^2}{\sqrt{1-x^2}}$$

Then,  $f'(x) = 0$  when  $1 - 2x^2 = 0$  and  $f$  has two critical points  $x = \pm \frac{1}{\sqrt{2}}$ .

**Example 4.2.3** Let  $f(x) = x^{2/3}$ ,  $D = [-1, 8]$ . The derivative of  $f$  is  $f'(x) = \frac{2}{3x^{1/3}}$ . Then  $f'(0)$  is undefined. To find the extreme values of  $f$ , we evaluate  $f$  at the endpoints  $x = -1, x = 8$  and at the critical point  $x = 0$ . Since  $f(-1) = 1, f(0) = 0, f(8) = 4$ , then  $f(0) = 0$  is an absolute minimum and  $f(8) = 4$  is an absolute maximum.

**Theorem 4.2.2** *If  $f$  is differentiable and has an extreme value at an interior point  $c$  then  $f'(c) = 0$ .*

If  $f'(c) = 0$ , this does not mean that  $f$  has an extreme value (maximum or minimum) at  $x = c$ . For example,  $x = 0$  is a critical point of  $f(x) = x^3$  but  $f(0)$  is neither maximum nor minimum for  $y = x^3$ .

To classify the critical as maximum or minimum, we can use either the first derivative test or the second derivative test which we state now.

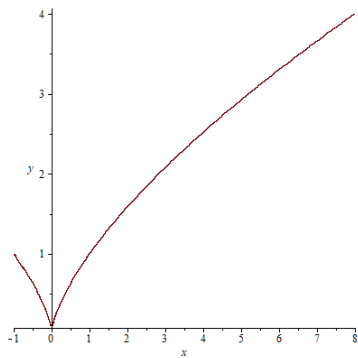


Figure 4.5: Graph of  $f(x) = x^{2/3}$

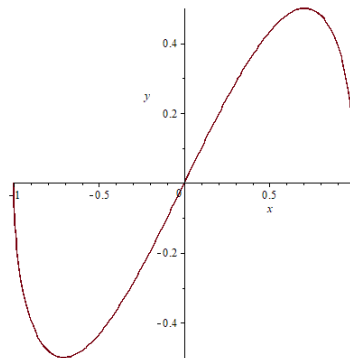


Figure 4.6: Graph of  $f(x) = x\sqrt{1-x^2}$

**Theorem 4.2.3 (First derivative test)** Suppose that  $f$  has a critical point at  $x = c$  and that  $f'(x)$  exists in an open interval containing  $x = c$ . Then

- (a) If  $f'$  changes sign from positive to negative at  $x = c$  then  $f(c)$  is a local maximum.
- (b) If  $f'$  changes sign from negative to positive at  $x = c$  then  $f(c)$  is a local minimum.
- (c) If  $f'$  does not change sign at  $x = c$  then  $f$  does not have an extreme value at  $x = c$ .

**Example 4.2.4** Consider the function  $f(x) = x\sqrt{1-x^2}$  from example (4.2.2) whose derivative is

$$f'(x) = \frac{1 - 2x^2}{\sqrt{1 - x^2}}$$

$f$  has two critical point  $x = \pm \frac{1}{\sqrt{2}}$ , the sign of  $f'$  is

$$- \quad - \quad - \quad - \quad - \quad \frac{-1}{\sqrt{2}} \quad + \quad + \quad + \quad + \quad \frac{1}{\sqrt{2}} \quad - \quad - \quad - \quad - \quad -$$

So,  $f$  has a local minimum at  $x = -\frac{1}{\sqrt{2}}$  and local maximum at  $x = \frac{1}{\sqrt{2}}$ . Its maximum value is  $f\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{2}$  and its minimum value is  $f\left(-\frac{1}{\sqrt{2}}\right) = -\frac{1}{2}$ . In fact, as it is clear from figure (4.6) these extreme values are absolute.

**Theorem 4.2.4 (Second derivative test)** Suppose that  $f'(c) = 0$  and that  $f''$  is continuous in an open interval containing  $c$ . Then

(a) If  $f''(c) < 0$  then  $f(c)$  is a local maximum.

(b) If  $f''(c) > 0$  then  $f(c)$  is a local minimum.

(c) If  $f''(c) = 0$  then the test fails.

If  $f''(x) \geq 0$  for all  $x$  in an interval  $I$  then  $f$  is concave up on  $I$ . If  $f''(x) \leq 0$  for all  $x$  in an interval  $I$  then  $f$  is concave down on  $I$ .

**Definition 4.2.3** A point where  $f$  has tangent line and changes concavity is called **an inflection point** of  $f$ .

**Example 4.2.5** Find the intervals at which the function

$$f(x) = x^4 - 4x^3 + 10$$

is increasing, decreasing, concave up and concave down. Then, find the extreme values of  $f$ .

**Solution:** The first and second derivatives of  $f$  are given by

$$f'(x) = 4x^2(x - 3) \quad \text{and} \quad f''(x) = 12x(x - 2)$$

We find that  $f'(x) = 0$  at  $x = 0$  and  $x = 3$ ,  $f''(x) = 0$  at  $x = 0$  and  $x = 2$ , so  $f$  has two critical points  $x = 0$  and  $x = 3$ . The signs of  $f'$  and  $f''$  are found to be as

$$f' \quad - - - - - 0 - - - - - 3 + + + + +$$

$$f'' \quad + + + + + + + 0 - - - - - 2 + + + + + + +$$

Hence,  $f'(x) < 0$  for all  $x \in (-\infty, 0) \cup (0, 3)$  and  $f'(x) > 0$  for all  $x \in (3, \infty)$ . We conclude that  $f$  is decreasing on  $(-\infty, 3]$  and  $f$  is increasing on  $[3, \infty)$ . It follows that  $f(3) = -17$  is an absolute minimum.

Moreover,  $f''(x) > 0$  for all  $x \in (-\infty, 0) \cup (2, \infty)$  and  $f''(x) < 0$  for all  $x \in (0, 2)$ . We conclude that  $f$  is concave up on  $(-\infty, 0] \cup [2, \infty)$  and  $f$  is concave down on  $[0, 2]$ . Finally,  $f$  has inflection points at  $(0, 10)$  and  $(2, -6)$ .

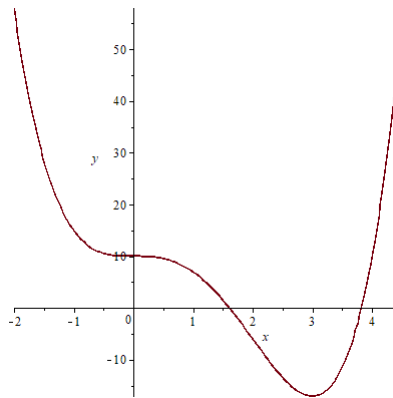


Figure 4.7: Graph of  $y = x^4 - 4x^3 + 10$

**Example 4.2.6** Consider the function

$$f(x) = \frac{x^2}{x+1} = x - 1 + \frac{1}{x+1}$$

Then,

$$f'(x) = \frac{(x+1)2x - x^2}{(x+1)^2} = \frac{x^2 + 2x}{(x+1)^2} = \frac{x(x+2)}{(x+1)^2}$$

and

$$\begin{aligned} f''(x) &= \frac{(x+1)^2(2x+2) - (x^2+2x)(2)(x+1)}{(x+1)^4} \\ &= \frac{2(x+1)^2 - 2(x^2+2x)}{(x+1)^3} \\ &= \frac{2x^2 + 4x + 2 - 2x^2 - 4x}{(x+1)^3} \\ &= \frac{2}{(x+1)^3} \end{aligned}$$

(1) Domain of  $f$ :  $(-\infty, \infty) \setminus \{-1\}$

(2)  $\lim_{x \rightarrow +\infty} \frac{x^2}{x+1} = \lim_{x \rightarrow +\infty} \frac{x}{1+\frac{1}{x}} = +\infty$

(3)  $\lim_{x \rightarrow -\infty} \frac{x^2}{x+1} = \lim_{x \rightarrow -\infty} \frac{x}{1+\frac{1}{x}} = -\infty$

(4) Horizontal asymptotes: None

(5)  $\lim_{x \rightarrow -1^+} \frac{x^2}{x+1} = +\infty$

(6)  $\lim_{x \rightarrow -1^-} \frac{x^2}{x+1} = -\infty$

(7) Vertical asymptote:  $x = -1$

(8) Oblique asymptote  $y = x - 1$

(9) Critical points  $x = 0, -2$  since  $f'(x) = 0$  at  $x = 0, x = -2$

- (10)  $f'$     + + + + + (-2) - - - (-1) - - - 0 + + + + +, so  $f$  is increasing on  $(-\infty, -2] \cup [0, \infty)$  and decreasing on  $[-2, -1) \cup (-1, 0]$
- (11)  $f(-2) = -4$  is a local maximum.
- (12)  $f(0) = 0$  is a local minimum.
- (13)  $f''$     - - - - - (-1) + + + + +, so  $f$  is concave down on  $(-\infty, -1)$  and concave up on  $(-1, \infty)$
- (14) Absolute maximum and absolute minimum values: None.
- (15) Inflection points: None.
- (16) Range of  $f$ :  $(-\infty, -4] \cup [0, \infty)$

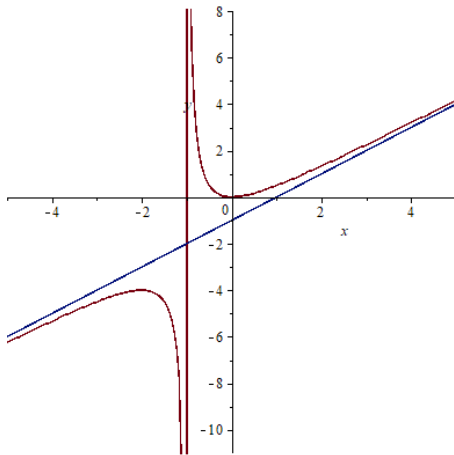


Figure 4.8: Graph of  $f(x) = \frac{x^2}{x+1}$  and its asymptotes

**Example 4.2.7** Consider the function

$$f(x) = \frac{x^2}{x^2 - 1}$$

Then

$$f'(x) = \frac{(x^2 - 1)(2x) - x^2(2x)}{(x^2 - 1)^2} = \frac{2x^3 - 2x - 2x^3}{(x^2 - 1)^2} = \frac{-2x}{(x^2 - 1)^2}$$

and

$$\begin{aligned} f''(x) &= \frac{(x^2 - 1)^2(-2) + 2x(2)(2x)(x^2 - 1)}{(x^2 - 1)^4} \\ &= \frac{-2(x^2 - 1) + 8x^2}{(x^2 - 1)^3} \\ &= \frac{6x^2 + 2}{(x^2 - 1)^3} \end{aligned}$$

- (1) Domain  $(-\infty, \infty) \setminus \{\pm 1\}$
- (2)  $\lim_{x \rightarrow \pm\infty} \frac{x^2}{x^2 - 1} = 1$
- (3) Horizontal asymptote  $y = 1$
- (4)  $\lim_{x \rightarrow 1^+} \frac{x^2}{x^2 - 1} = +\infty$
- (5)  $\lim_{x \rightarrow 1^-} \frac{x^2}{x^2 - 1} = -\infty$
- (6)  $\lim_{x \rightarrow -1^+} \frac{x^2}{x^2 - 1} = -\infty$
- (7)  $\lim_{x \rightarrow -1^-} \frac{x^2}{x^2 - 1} = +\infty$
- (8) Vertical asymptotes:  $x = 1$  and  $x = -1$
- (9) Critical point  $x = 0$  since  $f'(0) = 0$



(10)  $f'$  +++++(-1)+++++0-----1-----, so  $f$  is increasing on  $(-\infty, -1) \cup (-1, 0]$  and  $f$  is decreasing on  $[0, 1) \cup (1, \infty)$

(11)  $f(0) = 0$  is a local maximum.

(12) Local minimum: None

(13) Absolute maximum and absolute minimum: None

(14)  $f''$  +++++(-1)-----1+++++, so  $f$  is concave up on  $(-\infty, -1) \cup (1, \infty)$  and concave down on  $(-1, 1)$ .

(15) Inflection points: None.

(16) Range of  $f$ :  $(-\infty, 0] \cup (1, \infty)$

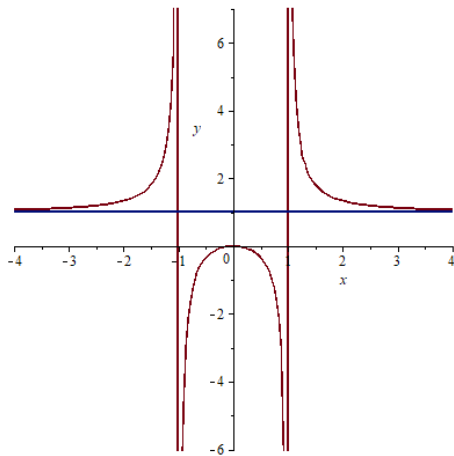


Figure 4.9: Graph of  $f(x) = \frac{x^2}{x^2-1}$  and its asymptotes

**Example 4.2.8** Consider the function

$$f(x) = \frac{x}{x^2 + 1}$$

Then

$$f'(x) = \frac{(x^2 + 1) - x(2x)}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2}$$

$$\begin{aligned} f''(x) &= \frac{(x^2 + 1)^2(-2x) - (1 - x^2)(2)(1 + x^2)(2x)}{(x^2 + 1)^2} \\ &= \frac{-2x(x^2 + 1) - 4x(1 - x^2)}{(x^2 + 1)^3} \\ &= \frac{2x^3 - 6x}{(x^2 + 1)^3} \\ &= \frac{2x(x^2 - 3)}{(x^2 + 1)^3} \end{aligned}$$

- (1) Domain:  $(-\infty, \infty)$
- (2)  $\lim_{x \rightarrow \pm\infty} \frac{x}{x^2+1} = 0$
- (3) Horizontal asymptote:  $y = 0$
- (4) Vertical asymptote: None
- (5) Oblique asymptote: None
- (6) Critical points:  $x = 1$  and  $x = -1$  since  $f'(\pm 1) = 0$
- (7)  $f'$   $- - - - - (-1) + + + + + 1 - - - - -$ , so  $f$  is increasing on  $[-1, 1]$  and  $f$  is decreasing on  $(-\infty, -1] \cup [1, \infty)$
- (8) Local maximum  $f(1) = \frac{1}{2}$
- (9) Local minimum  $f(-1) = -\frac{1}{2}$

(10) Absolute maximum  $f(1) = \frac{1}{2}$

(11) Absolute minimum  $f(-1) = -\frac{1}{2}$

(12)  $f''$   $- - - - - (-\sqrt{3}) + + + + + 0 - - - - - \sqrt{3} + + + + +$ , so  $f$  is concave up on  $[-\sqrt{3}, 0] \cup [\sqrt{3}, \infty)$  and  $f$  is concave down on  $(-\infty, -\sqrt{3}] \cup [0, \sqrt{3}]$

(13) Inflection points  $(-\sqrt{3}, \frac{-\sqrt{3}}{4})$ ,  $(0, 0)$ ,  $(\sqrt{3}, \frac{\sqrt{3}}{4})$

(14) Range of  $f$ :  $[-\frac{1}{2}, \frac{1}{2}]$

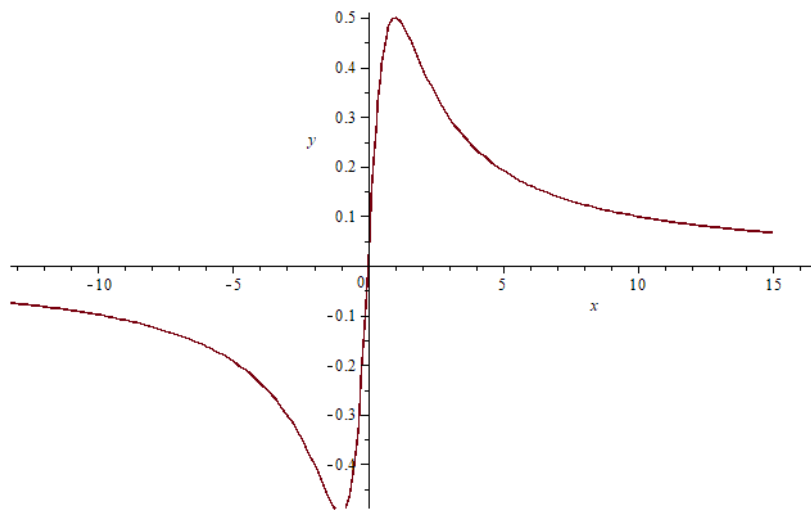


Figure 4.10: Graph of  $f(x) = \frac{x}{x^2+1}$

### 4.3 The Mean Value Theorem

**Theorem 4.3.1 Rolle's Theorem** If  $y = f(x)$  is continuous on the closed interval  $[a, b]$  and differentiable on  $(a, b)$  and  $f(a) = f(b)$ , then there is at least one point  $c$  in  $(a, b)$  such that  $f'(c) = 0$ .

**Theorem 4.3.2 The Mean Values Theorem** If  $y = f(x)$  is continuous on the closed interval  $[a, b]$  and differentiable on  $(a, b)$ , then there is at least one point  $c$  in  $(a, b)$  such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

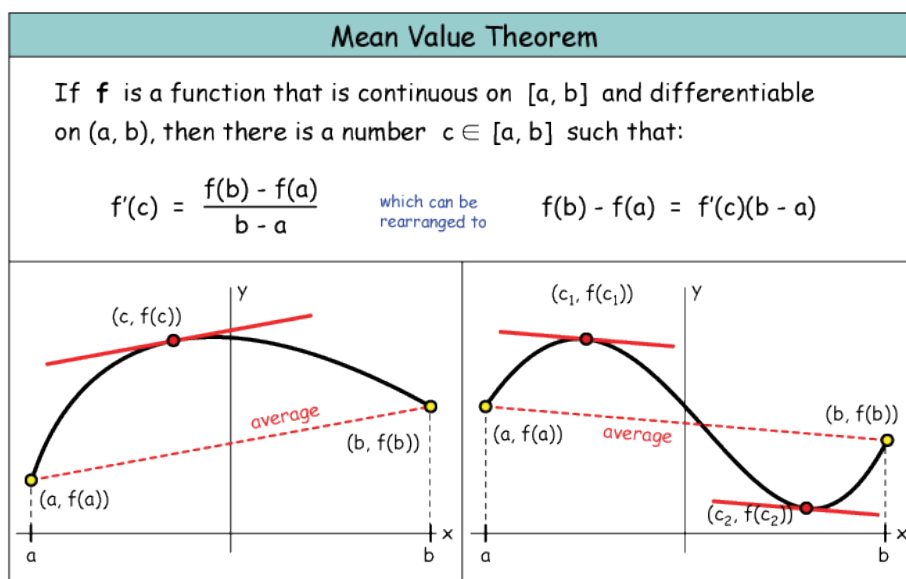


Figure 4.11: Graph of  $f(x) = \frac{x^2}{x^2-1}$  and its asymptotes

The mean value theorem means that, at some point  $c$  in the interval  $[a, b]$ , the slope of the tangent line at  $(c, f(c))$  equals the slope of the secant line through the points  $(a, f(a))$  and  $(b, f(b))$ .

**Example 4.3.1** Let  $f(x) = x^2$ ,  $x \in [1, 4]$ . Find the point  $c$  in the conclusion of the mean value theorem. Note that  $f$  is continuous on

$[1, 4]$  and differentiable on  $(1, 4)$ . Then,

$$\frac{f(4) - f(1)}{4 - 1} = \frac{16 - 1}{4 - 1} = \frac{15}{3} = 5, \quad f'(c) = 2c \Rightarrow 5 = 2c \Rightarrow c = \frac{5}{2}$$

#### 4.4 Exercises

1. Find the intervals in which the following functions are increasing, decreasing, concave up and concave down. Then, find the extreme values and inflection points and sketch their graphs:

(a)  $y = 1 - (x + 1)^3$

(b)  $y = \frac{x^2+1}{x}$

(c)  $y = x^4 - 2x^2$

(d)  $y = \frac{x^2-3}{x-2}$

(e)  $y = \sqrt[3]{x^3 + 1}$

(f)  $y = \frac{x}{x^2-1}$

(g)  $y = x\sqrt{8 - x^2}$

2. Find the value of  $c$  in the conclusion of the mean value theorem for the function  $f(x) = \sqrt{x}$  on the interval  $[a, b]$ ,  $a > 0$ .

3. For what values of  $a, m$  and  $b$  does the function

$$f(x) = \begin{cases} 3 & , \quad x = 0 \\ -x^2 + 3x + a & , \quad 0 < x < 1 \\ mx + b & , \quad 1 \leq x \leq 2 \end{cases}$$

satisfy the hypotheses of the mean value theorem on the interval  $[0, 2]$ .



## Chapter 5

# Integration

### 5.1 Antiderivative and integration

**Definition 5.1.1** A function  $F$  is called an **antiderivative** of a function  $f$  on an interval  $I$  if  $F'(x) = f(x)$ , for all  $x$  in  $I$ . The set of all antiderivatives of  $f$  is called the **indefinite integral** of  $f$  and is denoted by  $\int f(x)dx$ .

**Example 5.1.1** An antiderivative of the function  $f(x) = 2x$  is  $F(x) = x^2$  since  $F'(x) = 2x = f(x)$ . All antiderivatives of  $f(x) = 2x$  are given by  $F(x) = x^2 + C$ , for any constant  $C$ .

**Example 5.1.2** In this example, we give the indefinite integrals of some important functions

(a)  $\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$

(b)  $\int \sin x dx = -\cos x + C$

(c)  $\int \cos x dx = \sin x + C$

(d)  $\int \sec^2 x dx = \tan x + C$

(e)  $\int \sec x \tan x dx = \sec x + C$

$$(f) \int \csc x \cot x dx = -\csc x + C$$

$$(g) \int \csc^2 x dx = -\cot x + C$$

**Example 5.1.3** Consider the following examples:

$$(a) \int (x^{-2} - x^2 + 1) dx = -\frac{1}{x} - \frac{1}{3}x^3 + x + C$$

$$(b) \int \cos^2 \theta d\theta = \int \frac{1+\cos(2\theta)}{2} d\theta = \frac{1}{2} \int (1 + \cos(2\theta)) d\theta = \frac{1}{2}(\theta + \frac{\sin(2\theta)}{2}) + C$$

$$(c) \int \sin^2 x dx = \int \frac{1-\cos(2x)}{2} dx = \frac{1}{2} \int (1 - \cos(2x)) dx = \frac{1}{2}(x - \frac{\sin(2x)}{2}) + C$$

$$(d) \int \cot^2 x dx = \int (\csc^2 x - 1) dx = -\cot x - x + C$$

## 5.2 Definite integrals and areas

Sometimes, we evaluate integrals on given intervals. Such integrals are called definite integrals and take the form

$$\int_a^b f(x) dx$$

We can solve definite integrals using the fundamental theorem of calculus:

### Theorem 5.2.1 Fundamental Theorem of Calculus

(I) Suppose that  $f$  is continuous on  $[a, b]$  and  $F$  is an antiderivative of  $f$  on  $[a, b]$  then

$$\int_a^b f(x) dx = F(b) - F(a)$$

(II) Suppose that  $f$  is continuous on  $[a, b]$  and  $F(x) = \int_a^x f(t) dt$  then  $F$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$  and  $F'(x) = f(x)$ .



If  $f(x) \geq 0$  is an integrable function on  $[a, b]$  then  $\int_a^b f(x)dx$  is the area enclosed between the curve  $f(x)$  and the  $x$ -axis.

**Example 5.2.1** Find the derivatives of the following functions

$$(a) \frac{d}{dx} \int_0^x \sin t dt = \sin x.$$

$$(b) \frac{d}{dx} \int_1^{x^2} \frac{dt}{1+t^2} = \frac{2x}{1+x^4}$$

$$(c) \frac{d}{dx} \int_{\sin x}^1 \frac{dt}{t} = \frac{d}{dx} \left( - \int_1^{\sin x} \frac{dt}{t} \right) = -\frac{\cos x}{\sin x} = -\cot x$$

$$(d) \frac{d}{dx} \int_{x^2}^{x^3} \sin t dt = \sin(x^3)(3x^2) - \sin(x^2)(2x)$$

**Example 5.2.2** Find the area enclosed between the following curves and the  $x$ -axis in the given intervals

(a)  $f(x) = 2x\sqrt{x^2 + 1}$ ,  $x \in [0, 1]$ . The area is given by the following integral

$$A = \int_0^1 2x\sqrt{x^2 + 1} dx$$

using substitution  $u = x^2 + 1$ ,  $du = 2x dx$ . The integral can be written as

$$A = \int_1^2 u^{1/2} du = \frac{2}{3} u^{3/2} \Big|_1^2 = \frac{2}{3} (2\sqrt{2} - 1)$$

We can find the area enclosed between two functions  $f(x)$  and  $g(x)$  in some interval  $[a, b]$  where  $f(x) \geq g(x)$ , using the formula

$$A = \int_a^b (f(x) - g(x)) dx$$

Sometimes, the functions are expressed in terms of  $y$  in some interval  $[c, d]$ , so the area in this case is

$$A = \int_c^d (f(y) - g(y)) dy$$

The next examples explain both cases.

**Example 5.2.3** Find the area enclosed between the curves  $f(x) = 2 - x^2$  and  $y = -x$ .

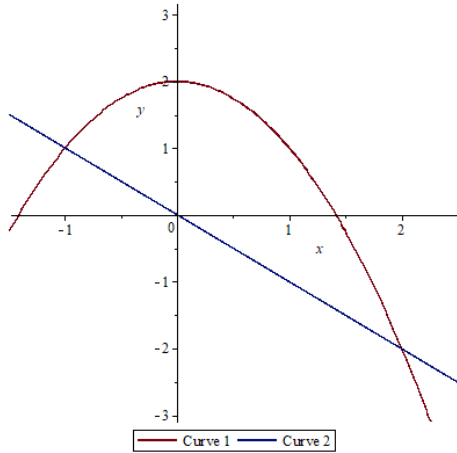


Figure 5.1: Plot of  $f(x) = 2 - x^2$ ,  $g(x) = -x$

**Solution** We first find the points at which the two curves intersect by equating the functions

$$-x = 2 - x^2 \quad \text{which is equivalent to} \quad x^2 - x - 2 = 0$$

The last equation can be factorized as  $(x + 1)(x - 2) = 0$ . Thus, the two curves intersect at  $x = -1$  and  $x = 2$ . So, the area is given by

$$\begin{aligned} A &= \int_{-1}^2 (2 - x^2 + x) dx \\ &= \left( 2x - \frac{x^3}{3} + \frac{x^2}{2} \right) \Big|_{-1}^2 \\ &= 4 - \frac{8}{3} + 2 + 2 - \frac{1}{3} - \frac{1}{2} \\ &= \frac{9}{2} \end{aligned}$$

**Example 5.2.4** Find the area enclosed between the curves  $y = \sqrt{x}$ , the  $x$ -axis and the line  $y = x - 2$ . It is easier to write  $x$  as a function of  $y$  and to integrate with respect to  $y$ . In this case, we have  $x = y^2$  and  $x = y + 2$ . The two curves intersect at the point  $y = 2$ . The area is given by the integral

$$\begin{aligned} A &= \int_0^2 (y + 2 - y^2) dy \\ &= \left( \frac{y^2}{2} + 2y - \frac{y^3}{3} \right) \Big|_0^2 \\ &= 2 + 4 - \frac{8}{3} \\ &= \frac{10}{3} \end{aligned}$$

integrating with respect to  $x$ ,

$$A = \int_0^2 \sqrt{x} dx + \int_2^4 (\sqrt{x} - (x - 2)) dx = \frac{10}{3} \quad (\text{check!!!})$$

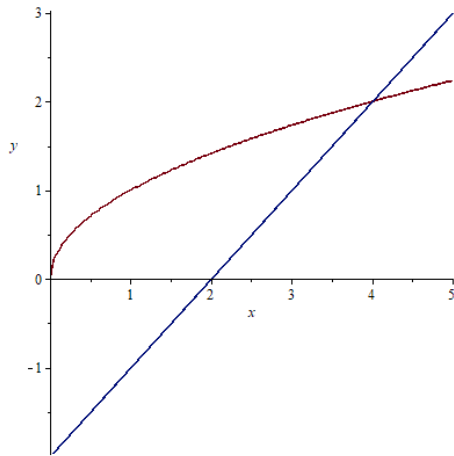


Figure 5.2: Plot of  $y = \sqrt{x}$  and  $y = x - 2$

### 5.3 Additional Examples

**Example 5.3.1** Solve  $\int \sqrt{\frac{x^4}{x^3-1}} dx$

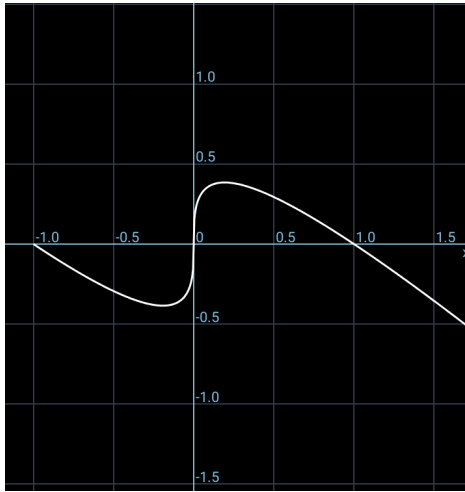
$$\int \sqrt{\frac{x^4}{x^3-1}} dx = \int \frac{x^2}{\sqrt{x^3-1}} dx$$

using the substitution  $u = x^3 - 1$ ,  $du = 3x^2 dx$ , the integral becomes

$$\frac{1}{3} \int \frac{du}{u^{1/2}} = \frac{1}{3} \int u^{-1/2} du = \frac{2}{3} \sqrt{u} = \frac{2}{3} \sqrt{x^3-1} + C$$

**Example 5.3.2** Find the area enclosed between the curve  $f(x) = x^{1/3} - x$  and the  $x$ -axis in the interval  $[-1, 8]$ . Notice that  $f(x) = 0$  at  $x = -1, 0, 1$ , and its graph lies below the  $x$ -axis in the intervals  $[-1, 0]$ ,  $[1, 8]$  and above the  $x$ -axis in the interval  $[0, 1]$ . So,

$$\begin{aligned} A &= \left| \int_{-1}^0 (x^{1/3} - x) dx \right| + \int_0^1 (x^{1/3} - x) dx + \left| \int_1^8 (x^{1/3} - x) dx \right| \\ &= \left| \left. \frac{3}{4} x^{4/3} - \frac{x^2}{2} \right|_{-1}^0 \right| + \left( \left. \frac{3}{4} x^{4/3} - \frac{x^2}{2} \right|_0^1 + \left| \left. \frac{3}{4} x^{4/3} - \frac{x^2}{2} \right|_1^8 \right) \\ &= \left| -\frac{3}{4} + \frac{1}{2} \right| + \left( \frac{3}{4} - \frac{1}{2} \right) + \left| 12 - 32 - \frac{3}{4} + \frac{1}{2} \right| \\ &= \frac{1}{4} + \frac{1}{4} + \frac{81}{4} \\ &= \frac{83}{4} \end{aligned}$$

Figure 5.3: Plot of  $f(x) = x^{1/3} - x$ 

## 5.4 Exercises

1. Solve the following integrals:

- (a)  $\int \sin(5x) dx$
- (b)  $\int \tan^2 x dx$
- (c)  $\int (1 + \cot^2 \theta) d\theta$ .
- (d)  $\int \frac{\csc \theta d\theta}{\csc \theta - \sin \theta}$

2. Find the derivatives of the following functions

- (a)  $y = \int_1^x \frac{dt}{t}$
- (b)  $y = \int_0^{\sqrt{x}} \cos t dt$
- (c)  $y = \int_{\tan x}^0 \frac{dt}{1+t^2}$

3. Find the linearization of  $g(x) = 3 + \int_1^{x^2} \sec(t-1) dt$  at  $x = -1$

4. Solve the following definite integrals

- (a)  $\int_1^{\sqrt{2}} \frac{s^2 + \sqrt{s}}{s^2} ds$

$$(b) \int_0^{\pi/6} (\sec x + \tan x)^2 dx$$

$$(c) \int_0^{\pi} (\cos x + |\cos x|) dx$$

5. Use substitution to solve the following integrals:

$$(a) \int \frac{dx}{\sqrt{x}(1+\sqrt{x})^2}$$

$$(b) \int \frac{\sec z \tan z}{\sqrt{\sec z}} dz$$

$$(c) \int \sqrt{\frac{x-1}{x^5}} dx$$

$$(d) \int x^3 \sqrt{x^2 + 1} dx$$

6. Find the area enclosed between the given functions:

$$(a) y = x^2 - 2x, y = x$$

$$(b) y = x^2, y = -x^2 + 4x$$

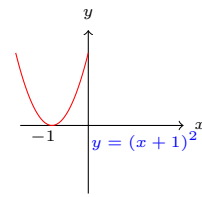
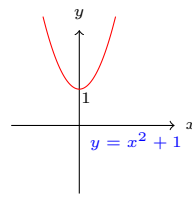
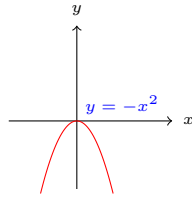
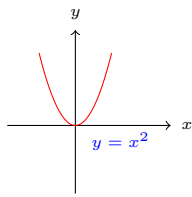
$$(c) x = y^2, x = 3 - 2y^2$$

$$(d) x = y^3 - y^2, x = 2y$$

# Basics

(Self Study)

- **Functions:** are maps in which every  $x$  value has only one image  $f(x) = y$
- **$y$ -intercept:** Where  $f$  crosses  $y$ -axis  $\rightarrow$  Let  $x = 0$ , then find  $y = f(0)$
- **$x$ -intercept (zero or root):** Where  $f$  crosses  $x$ -axis  $\rightarrow$  Let  $y = 0$ , then find  $x$
- **Shifting and reflections:** Given a function  $y = f(x)$  and a constant  $c > 0$ , then
  - 1)  $y = f(x) + c$  : Shift the graph of  $f(x)$   $c$  units **upward**.
  - 2)  $y = f(x) - c$  : Shift the graph of  $f(x)$   $c$  units **downward**.
  - 3)  $y = f(x + c)$  : Shift the graph of  $f(x)$   $c$  units **leftward**.
  - 4)  $y = f(x - c)$  : Shift the graph of  $f(x)$   $c$  units **rightward**.
  - 5)  $y = -f(x)$  : Reflect the graph of  $f(x)$  **about  $x$ -axis**.
  - 6)  $y = f(-x)$  : Reflect the graph of  $f(x)$  **about  $y$ -axis**



- 
- **Linear functions (Lines):**
  - **General Form:**  $y = f(x) = mx + b$ , where  $m = \frac{\Delta y}{\Delta x} = y'$  is the slope of the line.
  - **$(y - y_0) = m(x - x_0)$ :** Gives the equation of the line with slope  $m$  and passes through  $(x_0, y_0)$
  - **Horizontal line:**  $y = c \rightarrow$  Slope = 0
  - **Vertical line:**  $x = c \rightarrow$  Slope undefined
  - If  $L_1$  and  $L_2$  are two lines with slopes  $m_1$  and  $m_2$  respectively, then
    - 1)  $L_1$  and  $L_2$  are **parallel** if  $m_1 = m_2$
    - 2)  $L_1$  and  $L_2$  are **perpendicular (normal)** if  $m_1 = -\frac{1}{m_2}$

- 
- **Solving Equations and inequalities with absolute value:**

- $|x| = a \rightarrow x = \pm a$
- $|x| \leq a \rightarrow -a \leq x \leq a$
- $|x| \geq a \rightarrow x \leq -a$  or  $x \geq a$

- 
- **Special Factorizations:**

- $x^2 - a^2 = (x - a)(x + a)$
- $x^3 - a^3 = (x - a)(x^2 + ax + a^2)$
- $x^3 + a^3 = (x + a)(x^2 - ax + a^2)$

- **Quadratic functions (Parabolas):**
- **General Form:**  $y = f(x) = ax^2 + bx + c$ ;  $a \neq 0$
- **Vertex:** is the point  $(\frac{-b}{2a}, f(\frac{-b}{2a}))$
- **Discriminant**  $= b^2 - 4ac$ 
  - 1) If discriminant  $> 0$ , then  $f(x)$  has two real roots.
  - 2) If discriminant  $= 0$ , then  $f(x)$  has one real root.
  - 3) If discriminant  $< 0$ , then  $f(x)$  has no real roots.
- **Quadratic formula:**  $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

If  $a > 0$  then the parabola is open upward (concave up)

If  $a < 0$  then the parabola is open downward (concave down)

- **Square Completion:** Given  $x^2 + bx + c$ , (notice that  $a = 1$ ), add  $\pm(\frac{b}{2})^2$   
 $\rightarrow x^2 + bx + c = x^2 + bx + (\frac{b}{2})^2 - (\frac{b}{2})^2 + c = (x - |\frac{b}{2}|)^2 - (\frac{b}{2})^2 + c$   
 Ex:  $x^2 - 6x + 11 = x^2 - 6x + 9 - 9 + 11 = (x - 3)^2 + 2$

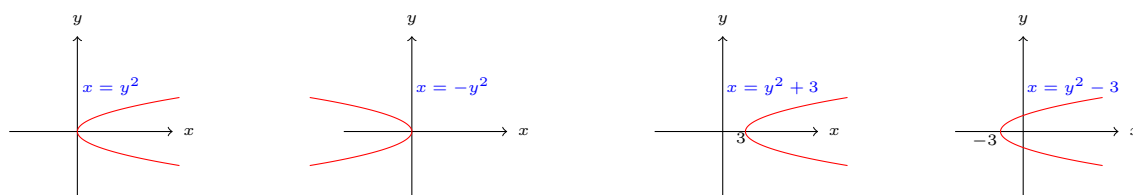
- **Special Quadratic Curves in  $y$ :**  $x = y^2$  and  $x = -y^2$

$x = y^2$ : a parabola open to the right with vertex  $(0, 0)$

$x = -y^2$ : a parabola open to the left with vertex  $(0, 0)$

Examples of shifts on  $x = y^2$ :

- 1)  $x = y^2 + 3$ : Shift the graph of  $x = y^2$  three units to the right
- 2)  $x = y^2 - 3$ : Shift the graph of  $x = y^2$  three units to the left
- 3)  $x = (y + 3)^2$ : Shift the graph of  $x = y^2$  three units downward
- 4)  $x = (y - 3)^2$ : Shift the graph of  $x = y^2$  three units upward



- **Circles:**

$(x - a)^2 + (y - b)^2 = r^2$ : a circle with center  $(a, b)$  and radius  $r$

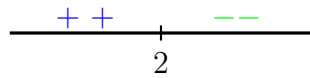
- **Unit circle:**  $x^2 + y^2 = 1$ : center =  $(0, 0)$  and radius = 1



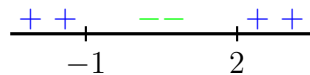
- **Determine the sign of  $y = f(x)$ :** Sometimes we need to know when  $y$  is positive (above  $x$ -axis) and when  $y$  is negative (below  $x$ -axis)

1) **Polynomials: Find the zeros, if any, then substitute values**

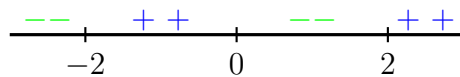
Ex:  $f(x) = 4 - 2x \rightarrow 4 - 2x = 0 \rightarrow x = 2$  (take  $f(0) = 4 > 0$  but  $f(3) = -2 < 0$ )



Ex:  $f(x) = x^2 - x - 2 \rightarrow x^2 - x - 2 = 0 \rightarrow x = -1, 2$   
 $(f(-2) = 4 > 0, f(0) = -2 < 0, f(3) = 4 > 0)$



Ex:  $f(x) = x^3 - 4x \rightarrow x^3 - 4x = 0 \rightarrow x = -2, 0, 2$



Ex:  $f(x) = x^2 + 3$  has no zeros, so substitute any value  $f(1) = 4 > 0$

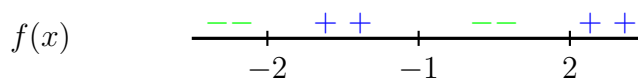
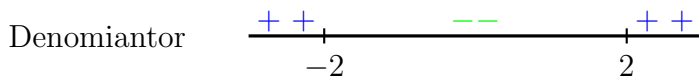
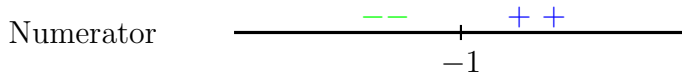


2) **Rational functions =  $\frac{\text{polynomial}}{\text{polynomial}}$ :** Determine sign of numerator, then denominator, then divide

Ex:  $f(x) = \frac{x^3+1}{x^2-4}$

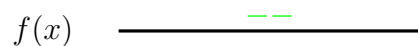
Numerator:  $x^3 + 1 = 0 \rightarrow x = -1$

Denominator:  $x^2 - 4 = 0 \rightarrow x = -2, 2$

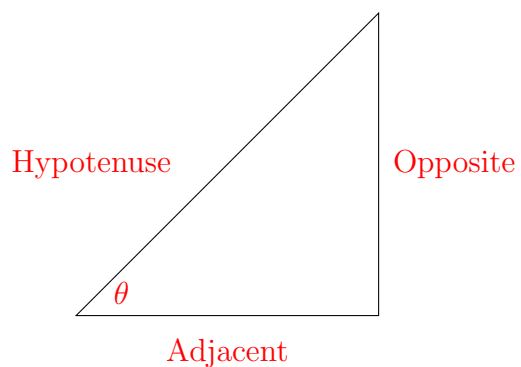


Ex:  $f(x) = \frac{-2}{x^2+1}$

'The numerator is always negative and the denominator is always positive, so  $f$  is always negative.



- Trigonometric functions



$$\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\sec \theta = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{1}{\cos \theta}$$

$$\tan \theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{\sin \theta}{\cos \theta} = \frac{1}{\cot \theta}$$

$$\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\csc \theta = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{1}{\sin \theta}$$

$$\cot \theta = \frac{\text{adjacent}}{\text{opposite}} = \frac{\cos \theta}{\sin \theta} = \frac{1}{\tan \theta}$$

$\theta$	$\sin \theta$	$\cos \theta$
0	0	1
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{\pi}{2}$	1	0
$\pi$	0	-1
$\frac{3\pi}{2}$	-1	0
$2\pi$	0	1

• Unit Circle and trigonometric functions:

Recall: Unit Circle:  $x^2 + y^2 = 1$  and  $\cos^2 \theta + \sin^2 \theta = 1$

→ For any point on this circle:  $(x, y) = (\cos \theta, \sin \theta)$ , where  $\theta$ : is the angle (counterclockwise) between the positive  $x$ -axis and the line segment from origin to point  $(x, y)$

Ex:  $(\frac{\sqrt{3}}{2}, \frac{1}{2}) = (\cos(\frac{\pi}{6}), \sin(\frac{\pi}{6}))$ ,  $(0, 1) = (\cos(\frac{\pi}{2}), \sin(\frac{\pi}{2}))$ ,  $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = (\cos(\frac{3\pi}{4}), \sin(\frac{3\pi}{4}))$

