3.5 Change of Basis

Definition Let V be a vector space and let $E = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be an ordered basis for V. If $\mathbf{v}$ is any element of V, then $\mathbf{v}$ can be written in the form

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$$

where $c_1, c_2, \ldots, c_n$ are scalars. Thus, we can associate with each vector $\mathbf{v}$ a unique vector $\mathbf{c} = (c_1, c_2, \ldots, c_n)^T$ in $\mathbb{R}^n$. The vector $\mathbf{c}$ defined in this way is called the **coordinate vector** of $\mathbf{v}$ with respect to the ordered basis E and is denoted $[\mathbf{v}]_E$. The c_i's are called the **coordinates** of $\mathbf{v}$ relative to E.

EXAMPLE 1 Let $\mathbf{y} = (2, 1)^T$ and $\mathbf{z} = (1, 4)^T$. The vectors $\mathbf{y}$ and $\mathbf{z}$ are linearly independent and hence form a basis for $\mathbb{R}^2$. The vector $\mathbf{x} = (7, 7)^T$ can be written as a linear combination

$$\mathbf{x} = 3\mathbf{y} + \mathbf{z}$$

Thus, the coordinate vector of $\mathbf{x}$ with respect to $[\mathbf{y}, \mathbf{z}]$ is $(3, 1)^T$.

The Change-of-Basis Problem:

Let V be a finite-dimensional vector space and B be a basis for V. Let $\mathbf{v} \in V$ and $[\mathbf{v}]_B$ is the coordinate vector of $\mathbf{v}$ relative to B.

If we change the basis for V from a basis B to a basis B', how are the coordinate vectors $[\mathbf{v}]_B$ and $[\mathbf{v}]_{B'}$ related?

Solution of the Change-of-Basis Problem:

If we change the basis for a vector space V from an old basis $B = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ to a new basis $B' = \{\mathbf{u}'_1, \mathbf{u}'_2, \dots, \mathbf{u}'_n\}$, then for each vector $\mathbf{v}$ in V, the old coordinate vector $[\mathbf{v}]_B$ is related to the new coordinate vector $[\mathbf{v}]_{B'}$ by the equation

$$[\mathbf{v}]_B = P_{B' \to B} [\mathbf{v}]_{B'}$$

where the columns of the $n \times n$ matrix $P_{B' \to B}$ are the coordinate vectors of the new basis vectors relative to the old basis; that is,

$$P_{B' \to B} = \left[\; [\mathbf{u}'_1]_B \;|\; [\mathbf{u}'_2]_B \;|\; \cdots \;|\; [\mathbf{u}'_n]_B \; \right]$$

Transition Matrices:

The transition matrix from B' to B:

$$P_{B'\to B} = \left[\ [\mathbf{u}'_1]_B \mid [\mathbf{u}'_2]_B \mid \cdots \mid [\mathbf{u}'_n]_B\ \right]$$

The transition matrix from B to B':

$$P_{B\to B'} = \left[\ [\mathbf{u}_1]_{B'} \mid [\mathbf{u}_2]_{B'} \mid \cdots \mid [\mathbf{u}_n]_{B'}\ \right]$$

Remark

$$(P_{B'\to B})^{-1} = P_{B\to B'}$$

Remark

$$[\mathbf{v}]_B = P_{B'\to B}[\mathbf{v}]_{B'}$$

$$[\mathbf{v}]_{B'} = P_{B\to B'}[\mathbf{v}]_B$$

A Procedure for Computing $P_{B \to B'}$: (In $\mathbb{R}^n$)

1. B is the matrix whose columns are the old bases vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$.
2. B' is the matrix whose columns are the new bases vectors $\mathbf{u}'_1, \mathbf{u}'_2, \dots, \mathbf{u}'_n$.
3. Form the matrix $[\ B' \mid B\ ]$.
4. Use elementary row operations to reduce the matrix in Step 1 to reduced row echelon form.
5. The resulting matrix will be $[\ I \mid P_{B \to B'}\ ]$. i.e. $P_{B \to B'} = (B')^{-1} B$

Remark:

We can also compute $P_{B' \to B}$ by using elementary row operations to reduce the matrix

$$[\ B \mid B'\ ]$$

to

$$[\ I \mid P_{B' \to B}\ ]$$

i.e. $P_{B' \to B} = B^{-1} B'$

Example.

Consider the bases $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $B' = \{\mathbf{u}_1', \mathbf{u}_2'\}$ for $\mathbb{R}^2$, where

$$\mathbf{u}_1 = (1,0), \quad \mathbf{u}_2 = (0,1), \quad \mathbf{u}_1' = (1,1), \quad \mathbf{u}_2' = (2,1),$$

a) Find the transition matrix $P_{B \to B'}$ from B to B'.

b) Find the transition matrix $P_{B' \to B}$ from B' to B.

c) Find $[\mathbf{v}]_{B'}$ given that

$$[\mathbf{v}]_B = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$$

a)

$$[B' \; B] = \left[\begin{array}{cccc|cccc} 1 & 2 & 1 & 0 & 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & -1 & -1 & 1 \end{array}\right]$$

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 1 & 0 & 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 & 0 & 1 & 1 & -1 \end{array}\right]$$

$$\therefore \quad P_{B \to B'} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

b)

$$[B \; B'] = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\therefore \quad P_{B' \to B} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

c)

$$[v]_{B'} = P_{B \to B'} [v]_B = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 7 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$$

EXAMPLE 5 If

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix}$$

and

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \quad \mathbf{u}_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

then $E = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $F = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ are ordered bases for $\mathbb{R}^3$. Let

$$\mathbf{x} = 3\mathbf{v}_1 + 2\mathbf{v}_2 - \mathbf{v}_3 \qquad \text{and} \qquad \mathbf{y} = \mathbf{v}_1 - 3\mathbf{v}_2 + 2\mathbf{v}_3$$

Find the transition matrix from E to F and use it to find the coordinates of $\mathbf{x}$ and $\mathbf{y}$ with respect to the ordered basis F.

Solution

the transition matrix is given by

$$U^{-1}V = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 5 \\ 1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -3 \\ -1 & -1 & 0 \\ 1 & 2 & 4 \end{bmatrix}$$

The coordinate vectors of **x** and **y** with respect to the ordered basis F are given by

$$[\mathbf{x}]_F = \begin{bmatrix} 1 & 1 & -3 \\ -1 & -1 & 0 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 8 \\ -5 \\ 3 \end{bmatrix}$$

and

$$[\mathbf{y}]_F = \begin{bmatrix} 1 & 1 & -3 \\ -1 & -1 & 0 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -8 \\ 2 \\ 3 \end{bmatrix}$$

The reader may verify that

$$8\mathbf{u}_1 - 5\mathbf{u}_2 + 3\mathbf{u}_3 = 3\mathbf{v}_1 + 2\mathbf{v}_2 - \mathbf{v}_3$$
$$-8\mathbf{u}_1 + 2\mathbf{u}_2 + 3\mathbf{u}_3 = \mathbf{v}_1 - 3\mathbf{v}_2 + 2\mathbf{v}_3$$

EXAMPLE 6 Suppose that in P_3 we want to change from the ordered basis $[1, x, x^2]$ to the ordered basis $[1, 2x, 4x^2 - 2]$. Because $[1, x, x^2]$ is the standard basis for P_3, it is easier to find the transition matrix from $[1, 2x, 4x^2 - 2]$ to $[1, x, x^2]$. Since

$$\begin{aligned} 1 &= 1 \cdot 1 + 0x + 0x^2 \\ 2x &= 0 \cdot 1 + 2x + 0x^2 \\ 4x^2 - 2 &= -2 \cdot 1 + 0x + 4x^2 \end{aligned}$$

the transition matrix is

$$S = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

The inverse of S will be the transition matrix from $[1, x, x^2]$ to $[1, 2x, 4x^2 - 2]$:

$$S^{-1} = \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix}$$

Given any $p(x) = a + bx + cx^2$ in P_3, to find the coordinates of $p(x)$ with respect to $[1, 2x, 4x^2 - 2]$, we simply multiply

$$\begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a + \frac{1}{2}c \\ \frac{1}{2}b \\ \frac{1}{4}c \end{pmatrix}$$

Thus,

$$p(x) = (a + \tfrac{1}{2}c) \cdot 1 + (\tfrac{1}{2}b) \cdot 2x + \tfrac{1}{4}c \cdot (4x^2 - 2) \qquad \blacksquare$$

SECTION 3.5 EXERCISES

1. For each of the following, find the transition matrix corresponding to the change of basis from $\{\mathbf{u}_1, \mathbf{u}_2\}$ to $\{\mathbf{e}_1, \mathbf{e}_2\}$.

(a) $\mathbf{u}_1 = (1, 1)^T, \quad \mathbf{u}_2 = (-1, 1)^T$

(b) $\mathbf{u}_1 = (1, 2)^T, \quad \mathbf{u}_2 = (2, 5)^T$

(c) $\mathbf{u}_1 = (0, 1)^T, \quad \mathbf{u}_2 = (1, 0)^T$

2. For each of the ordered bases $\{\mathbf{u}_1, \mathbf{u}_2\}$ in Exercise 1, find the transition matrix corresponding to the change of basis from $\{\mathbf{e}_1, \mathbf{e}_2\}$ to $\{\mathbf{u}_1, \mathbf{u}_2\}$.

3. Let $\mathbf{v}_1 = (3, 2)^T$ and $\mathbf{v}_2 = (4, 3)^T$. For each ordered basis $\{\mathbf{u}_1, \mathbf{u}_2\}$ given in Exercise 1, find the transition matrix from $\{\mathbf{v}_1, \mathbf{v}_2\}$ to $\{\mathbf{u}_1, \mathbf{u}_2\}$.

4. Let $E = [(5, 3)^T, (3, 2)^T]$ and let $\mathbf{x} = (1, 1)^T$, $\mathbf{y} = (1, -1)^T$, and $\mathbf{z} = (10, 7)^T$. Determine the values of $[\mathbf{x}]_E$, $[\mathbf{y}]_E$, and $[\mathbf{z}]_E$.

9. Let $[x, 1]$ and $[2x - 1, 2x + 1]$ be ordered bases for P_2.

(a) Find the transition matrix representing the change in coordinates from $[2x - 1, 2x + 1]$ to $[x, 1]$.

(b) Find the transition matrix representing the change in coordinates from $[x, 1]$ to $[2x - 1, 2x + 1]$.