Ch. 8.1 Integration by part :-

Evaluate the integral by using integration by part :-

$$\int u \cdot dv = u \cdot v - \int v \cdot du$$

Q6 $\displaystyle\int_1^e x^3 \ln x \, dx$

$u = \ln x \qquad dv = x^3 \, dx$

$du = \frac{1}{x} dx \xleftarrow{\int} \qquad v = \frac{x^4}{4}$

$$I = \frac{x^4}{4} \ln x \Big|_1^e - \int_1^e \frac{x^4}{4} \cdot \frac{1}{x} \, dx$$

$$= \left(\frac{e^4}{4} - \frac{1}{4} \cancel{\ln 1} \right) - \int_1^e \frac{x^3}{4} \, dx$$

$$= \frac{e^4}{4} - \left[\frac{x^4}{16} \right]_1^e$$

$$= \frac{e^4}{4} - \frac{e^4}{16} + \frac{1}{16} = \boxed{\frac{3}{16} e^4 + \frac{1}{16}}$$

Q12 $\displaystyle\int \sin^{-1} y \, dy$

$u = \sin^{-1} y \qquad dv = dy$

$du = \frac{dy}{\sqrt{1-y^2}} \xleftarrow{\int} \qquad v = y$

$$I = y \sin^{-1} y - \int y \cdot \frac{1}{\sqrt{1-y^2}} \, dy$$

let $Z = 1 - y^2$
$dZ = -2y \, dy$

$$= y \sin^{-1} y - \int \frac{\cancel{y}}{\sqrt{Z}} \frac{dZ}{-2\cancel{y}}$$

$$= y \sin^{-1} y + \frac{1}{2} \int Z^{-1/2} \, dZ = \boxed{y \sin^{-1} y + \sqrt{1-y^2} + C}$$

Q16 $\int p^4 e^{-p}\, dp$

Derivative		Integral
p^4	$+$	e^{-p}
$4p^3$	$-$	$-e^{-p}$
$12p^2$	$+$	e^{-p}
$24p$	$-$	$-e^{-p}$
24	$+$	e^{-p}
0		$-e^{-p}$

$$I = -p^4 e^{-p} - 4p^3 e^{-p} - 12p^2 e^{-p} - 24p e^{-p} - 24 e^{-p} + C$$

Q22 $\int e^{-y} \cos y\, dy$

First time

let $u = \cos y$ $\quad dv = e^{-y}\, dy$

$du = -\sin y\, dy$ $\quad v = -e^{-y}$

$$\int e^{-y}\cos y\, dy = -\cos y\, e^{-y} - \int \sin y\, e^{-y}\, dy$$

Second time

let $u = \sin y$ $\quad dv = e^{-y}\, dy$

$du = \cos y\, dy$ $\quad v = -e^{-y}$

$$\int e^{-y}\cos y\, dy = -\cos y\, e^{-y} - \left(-\sin y\, e^{-y} + \int e^{-y}\cos y\, dy\right)$$

$$\int e^{-y}\cos y\, dy = \sin y\, e^{-y} - \cos y\, e^{-y} - \int e^{-y}\cos y\, dy$$

$$\int e^{-y}\cos y\, dy = e^{-y}\left[\sin y - \cos y\right]/2 + C$$

$\boxed{Q_{30}}$ $\displaystyle\int Z(\ln Z)^2 \, dZ$

By substitution:-

let $y = \ln Z \longrightarrow Z = e^y$

$dZ = e^y \, dy$

$$I = \int y^2 e^{2y} \, dy$$

Derivative		Integral
y^2		e^{2y}
$2y$	$+$	$\frac{e^{2y}}{2}$
2	$-$	$\frac{e^{2y}}{4}$
0	$+$	$\frac{e^{2y}}{8}$

$$I = \frac{1}{2} y^2 e^{2y} - \frac{1}{2} y e^{2y} + \frac{1}{4} e^{2y} + C$$

$$= \frac{1}{2} (\ln Z)^2 Z^2 - \frac{1}{2} (\ln Z) Z^2 + \frac{1}{4} Z^2 + C$$

$$= \frac{1}{2} Z^2 (\ln Z)^2 - \frac{1}{2} Z^2 (\ln Z) + \frac{1}{4} Z^2 + C$$

$\boxed{Q_{33}}$ $\displaystyle\int x (\ln x)^2 \, dy$ "Exactly Q_{30}"

$\boxed{Q_{36}}$ $\displaystyle\int \frac{(\ln x)^3}{x} \, dx$

let $y = \ln x$

$dy = \frac{1}{x} \, dx$

$$\int y^3 \, dy$$

$$I = \frac{y^4}{4} + C$$

$$I = \frac{\ln^4 x}{4} + C$$

$\boxed{Q_{39}}$ $\int x^3\sqrt{x^2+1}\,dx$

let $\boxed{y = x^2+1}$ *

$\boxed{dy = 2x\,dx}$

$I = \int x^3 \sqrt{y}\,\frac{dy}{2x}$

$= \frac{1}{2}\int (y-1)\sqrt{y}\,dy$

$= \frac{1}{2}\int \left(y^{\frac{3}{2}} - y^{\frac{1}{2}}\right) dy$

$= \frac{1}{2}\left[\frac{2}{5}y^{5/2} - \frac{2}{3}y^{3/2}\right] + C$

$= \frac{1}{2}\cdot\left[\frac{2}{5}\left(x^2+1\right)^{5/2} - \frac{2}{3}\left(x^2+1\right)^{3/2}\right] + C$

$I = \frac{1}{5}\left(x^2+1\right)^{5/2} - \frac{1}{3}\left(x^2+1\right)^{3/2} + C$

$\boxed{Q\,46}$ $\displaystyle\int \sqrt{x}\, e^{\sqrt{x}}\, dx$

let $y = \sqrt{x} \longrightarrow y^2 = x$

$\boxed{2y\, dy = dx}$

$$I = \int y\, e^{y}\, 2y\, dy$$

$$I = \int 2y^2 e^{y}\, dy$$

Derivative		Integral
$2y^2$	$+$	e^y
$4y$	$-$	e^y
4	$+$	e^y
0		e^y

$$= 2y^2 e^{y} - 4y e^{y} + 4 e^{y} + C$$

$$= 2x e^{\sqrt{x}} - 4\sqrt{x}\, e^{\sqrt{x}} + 4 e^{\sqrt{x}} + C$$

$\boxed{Q_{49}}$ $\displaystyle\int_{2/\sqrt{3}}^{2} t \sec^{-1} t\, dt$

let $u = \sec^{-1} t$ $\qquad dv = t\, dt$

$du = \dfrac{1}{t\sqrt{t^2-1}}\, dt \qquad v = \dfrac{t^2}{2}$

$$\int_{2/\sqrt{3}}^{2} t \sec^{-1} t = \left.\frac{t^2 \sec^{-1} t}{2}\right]_{2/\sqrt{3}}^{2} - \underbrace{\frac{1}{2}\int_{2/\sqrt{3}}^{2} \frac{t}{\sqrt{t^2-1}}\, dt}_{I_1}$$

$$= \frac{1}{2}\left[4 \sec^{-1}(2) - \frac{4}{3}\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)\right] - I_1$$

$$= \frac{1}{2}\left[4 \cos^{-1}\left(\frac{1}{2}\right) - \frac{4}{3}\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)\right] - I_1 = \frac{5\pi}{9} - I_1$$

We need to Find $I_1 = \frac{1}{2}\int_{2/\sqrt{3}}^{2} \frac{t}{\sqrt{t^2-1}}\,dt$

By substitution:

let $y = t^2 - 1$

$dy = 2t\,dt$

$* \quad t = \frac{2}{\sqrt{3}} \longrightarrow y = \frac{1}{3}$

$t = 2 \longrightarrow y = 3$

$$= \frac{1}{2}\int \frac{1}{\sqrt{y}}\,\frac{dy}{2}$$

$$= \frac{1}{4}\int_{1/3}^{3} y^{-1/2}\,dy$$

$$= \frac{1}{4}\cdot 2\, y^{1/2}\Big]_{1/3}^{3}$$

$$= \cancel{2}\,\frac{1}{2}\left[\sqrt{3} - \frac{1}{\sqrt{3}}\right]$$

$$I_1 = \frac{1}{2}\left[\frac{2}{\sqrt{3}}\right] = \frac{1}{\sqrt{3}}$$

Then: $\int_{2/\sqrt{3}}^{2} t\sec^{-1} t\,dt = \frac{5\pi}{9} - \frac{3\sqrt{3}\,1}{3\sqrt{3}\sqrt{3}}$

$$I = \frac{5\pi - 3\sqrt{3}}{9}$$

#.

$\boxed{8.2}$ Trigonometric Integral :-

$\boxed{6}$ $\int \cos^3 4x \, dx$

$\int \cos 4x \left(\cos^2 4x\right) dx$

$= \int \cos 4x \left(1 - \sin^2 4x\right) dx$

let $u = \sin 4x$

$du = 4 \cos 4x \, dx$

$= \frac{1}{4} \int \left(1 - u^2\right) du$

$= \frac{1}{4} \left[u - \frac{u^3}{3}\right] + C$

$I = \frac{1}{4} \left[\sin(4x) - \frac{\sin^3(4x)}{3}\right] + C$

$= \frac{\sin(4x)}{4} - \frac{\sin^3(4x)}{12} + C$

$\boxed{11}$ $\int \sin^3 x \cos^3 x \, dx$

$\int \sin^3 x \left(1 - \sin^2 x\right) \cos x \, dx$

let $U = \sin x$
$dU = \cos x \, dx$

$I = \int U^3 \left(1 - U^2\right) dU$

$= \int U^3 - U^5 \, dU$

$= \frac{U^4}{4} - \frac{U^6}{6} + C$

$= \frac{\sin^4 x}{4} - \frac{\sin^6 x}{6} + C$

$\boxed{17}$ $\int_0^{\pi} 8 \sin^4 x \, dx$

$\int_0^{\pi} 8 \left(\sin^2 x\right)^2 dx$

$= \int_0^{\pi} 8 \left(\frac{1 - \cos(2x)}{2}\right)^2 dx$

$= \int_0^{\pi} \frac{8}{4} \left[1 - 2\cos(2x) + \cos^2(2x)\right] dx$

$= 2\int 1 - 2\cos(2x) + \left(\frac{1 + \cos(4x)}{2}\right) dx$

$$I = 2\left[x - \frac{2\sin(2x)}{2} + \frac{1}{2}x + \frac{1}{2}\,\frac{\sin(4x)}{4} \right]_0^{\pi}$$

$$= 2\left[\frac{3x}{2} - \sin(2x) + \frac{1}{8}\sin(4x) \right]_0^{\pi}$$

$$= 3\pi$$

$\boxed{22}$ $$\int_0^{\pi/2} \sin^2 2\theta \; \cos^3 2\theta \; d\theta$$

$$\int_0^{\pi/2} \sin^2 2\theta \; \cos 2\theta \left(1 - \sin^2 2\theta\right) d\theta$$

Let $U = \sin 2\theta$

$dU = 2\cos 2\theta \, d\theta$

$$I = \int_0^0 U^2 (1 - U^2) \, \frac{dU}{2}$$

$*$ $\theta = 0 \longrightarrow U = 0$

$\theta = \frac{\pi}{2} \longrightarrow U = 0$

$=$ Zero

$\boxed{28}$ $$\int_0^{\pi/6} \sqrt{1 + \sin x} \; dx$$

"Eliminating Square root"

$$\int_0^{\pi/6} \frac{\sqrt{1 - \sin^2 x}}{\sqrt{1 - \sin x}} = \int_0^{\pi/6} \frac{|\cos x|}{\sqrt{1 - \sin x}} \, dx$$

$$I = \int_0^{\pi/6} \frac{\cos x}{\sqrt{1-\sin x}}\, dx$$

let $U = 1 - \sin x$

$dU = -\cos x \, dx$

$x = 0 \longrightarrow U = 1$

$x = \frac{\pi}{6} \longrightarrow U = \frac{1}{2}$

$$= \int_1^{1/2} \frac{-1}{\sqrt{U}}\, dU$$

$$= \int_{1/2}^{1} \frac{1}{\sqrt{U}}\, dU$$

$$= 2U^{1/2}\Big]_{1/2}^{1}$$

$$= 2\left[1 - \sqrt{\frac{1}{2}}\right] = 2\left[\frac{\sqrt{2}-1}{\sqrt{2}}\right]$$

$$= 2 - \sqrt{2}$$

$\boxed{34}$ $\int \sec x \tan^2 x \, dx$

$\int \sec x \left(\sec^2 x - 1\right) dx$

By part By ln

$\int \sec^3 x - \sec x \, dx$

let $U = \sec x$ $dV = \sec^2 x \, dx$

$dU = \sec x \tan x$ $\xleftarrow{\int}$ $V = \tan x$

$\int \sec^3 x - \int \sec x \, dx = \sec x \tan x - \int \tan^2 x \sec x \, dx - \int \sec x \, dx$

$\int \sec x \tan^2 x \, dx = \sec x \tan x - \int \tan^2 x \sec x \, dx - \int \sec x \, dx$

$2 \int \sec x \tan^2 x \, dx = \sec x \tan x - \int \sec x \, dx$

$= \sec x \tan x - \ln |\sec x + \tan x| + C$

Then $\int \sec x \tan^2 x \, dx = \frac{1}{2} \sec x \tan x - \frac{1}{2} \ln |\sec x + \tan x| + C$

$\boxed{41}$ $\displaystyle\int \sec^4 x \, dx$

$$\int \sec^2 x \; \sec^2 x \, dx$$

$$\int (1 + \tan^2 x) \sec^2 x \, dx$$

let $U = \tan x$

$dU = \sec^2 x \, dx$

$$\int (1 + U^2) \, dU$$

$$I = U + \frac{U^3}{3} + C$$

$$I = \tan x + \frac{1}{3} \tan^3 x + C$$

$\boxed{52}$ $\displaystyle\int \sin(2x) \cos(3x) \, dx$

Note:

$\sin(mx)\cos(nx) =$

$= \frac{1}{2}\left[\sin(n-m)x + \sin(n+m)x\right]$

$$= \frac{1}{2} \int \sin(-x) + \sin(5x) \, dx$$

$$= \frac{1}{2} \int -\sin x \, dx + \frac{1}{2} \int \sin(5x) \, dx$$

$$= \frac{+1}{2} \cos x + \frac{-1}{2} \frac{\cos(5x)}{5} + C$$

$$= \frac{1}{2} \cos x - \frac{1}{10} \cos(5x) + C$$

$\boxed{64}$ $\int \frac{\sin^3 x}{\cos^4 x} \, dx$

$\int \frac{\sin^3 x}{\cos^3 x \quad \cos x} \, dx$

$= \int \tan^3 x \quad \sec x \; dx$

$= \int \tan x \quad \tan^2 x \sec x \; dx$

$= \int (\sec^2 x - 1) \tan x \sec x \; dx$

let $u = \sec x$
$du = \sec x \tan x \, dx$

$= \int (u^2 - 1) \; du$

$= \frac{u^3}{3} - u + C$

$= \frac{1}{3} \sec^3 x - \sec x + C$