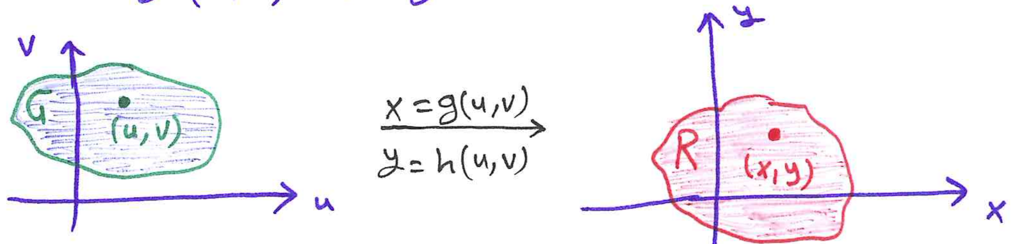
15.8 Substitution in Multiple Integrals

A Substitution in Double Integrals

* Suppose that a region G in the uv-plane is transformed 1-1 into the region R in the xy-plane by the equations

$$x = g(u,v) \quad , \quad y = h(u,v)$$



* R is the image of G and G is the preimage of R
* If $f(x,y)$ is defined on R, then f is also defined on G. That is $f(x,y) = f\big(g(u,v), h(u,v)\big)$.

* If g, h, f have continuous first partial derivatives, then

$$\iint_R f(x,y)\,dx\,dy = \iint_G f\big(g(u,v), h(u,v)\big)\ \text{abs}\big(J(u,v)\big)\,du\,dv$$

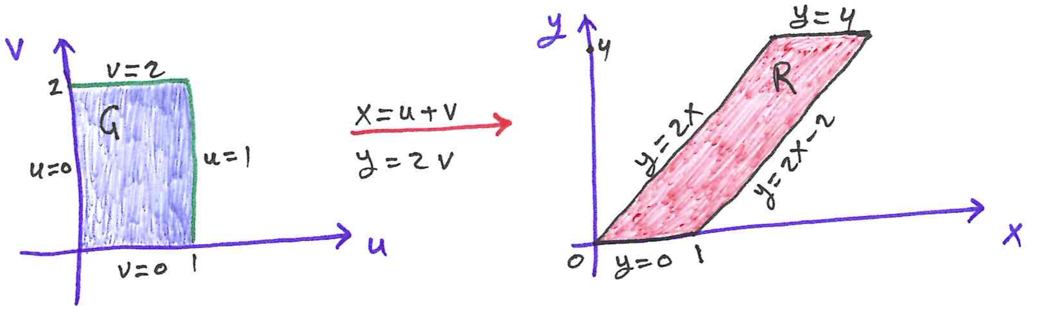
where the Jacobian $J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = x_u y_v - x_v y_u$

Exp Find $\displaystyle\int_0^4 \int_{x=\frac{y}{2}}^{x=\frac{y}{2}+1} \frac{2x-y}{2}\,dx\,dy$ by applying the transformation

$u = \frac{2x-y}{2}$, $v = \frac{y}{2}$ and integrating w.r.t u and v.

- $u = x - \frac{y}{2} = x - v \Rightarrow x = u + v$
  $\Rightarrow y = 2v$ $\Big\} \Rightarrow J(u,v) = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2$

- when $y = 0 \Rightarrow v = 0$ $\qquad x = \frac{y}{2} + 1 = v + 1 \Rightarrow u = 1$
  $y = 4 \Rightarrow v = 2$ $\qquad x = \frac{y}{2} = v \Rightarrow u = 0$



- $\displaystyle\int_0^4 \int_{x=\frac{y}{2}}^{x=\frac{y}{2}+1} \frac{2x-y}{2}\,dx\,dy = \int_{v=0}^{v=2} \int_{u=0}^{u=1} (u)(2)\,du\,dv = \int_0^2 dv = 2$

- Note that $\displaystyle\int_0^4 \int_{x=\frac{y}{2}}^{x=\frac{y}{2}+1} \frac{2x-y}{2}\,dx\,dy = \int_0^4 \frac{dy}{2} = 2$

Exp Evaluate $\displaystyle\int_0^1 \int_0^{1-x} \sqrt{x+y}\,(y-2x)^2\,dy\,dx$

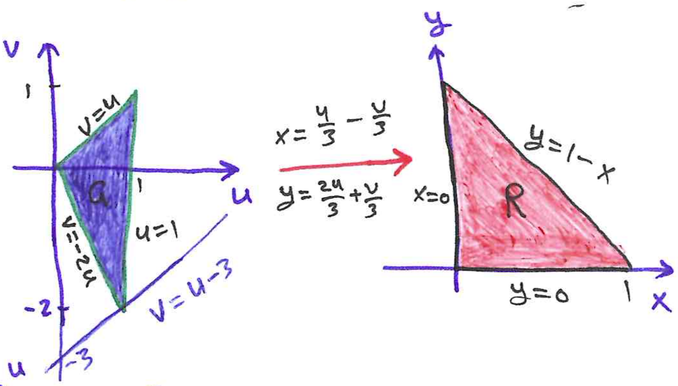
- we suggest $\left.\begin{aligned} u &= x+y \\ v &= y-2x \end{aligned}\right\} \Rightarrow \left.\begin{aligned} x &= \frac{u}{3} - \frac{v}{3} \\ y &= \frac{2u}{3} + \frac{v}{3} \end{aligned}\right\} \Rightarrow J(u,v) = \begin{vmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{vmatrix} = \frac{1}{3}$

- when $x = 0 \Rightarrow u = v$

  $x = 1 \Rightarrow v = u-3$

  $y = 0 \Rightarrow v = -2u$

  $y = 1-x \Rightarrow y+x = 1 \Rightarrow u = 1$



- $\displaystyle\int_0^1 \int_0^{1-x} \sqrt{x+y}\,(y+2x)^2\,dy\,dx = \frac{1}{3} \int_0^1 \int_{-2u}^{u} \sqrt{u}\; v^2\,dv\,du$

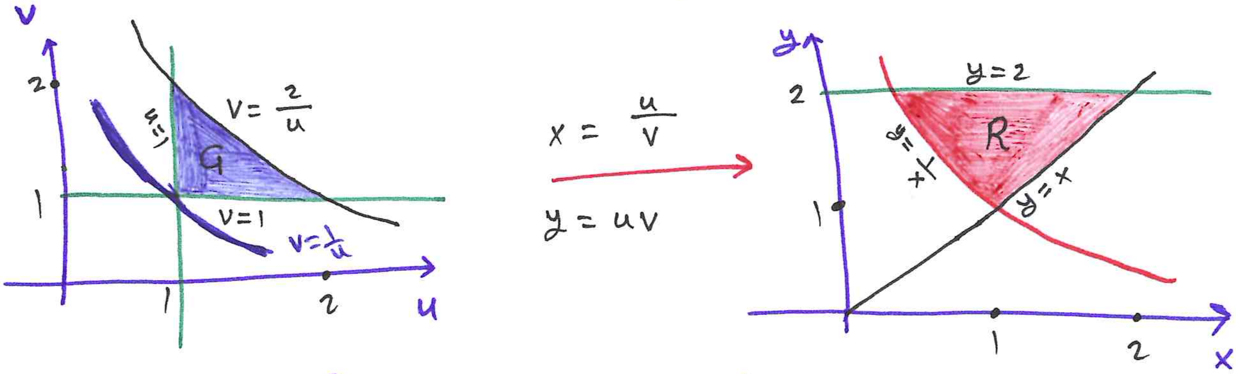
$$= \int_0^1 u^{\frac{7}{2}}\,du = \frac{2}{9}$$

Exp Evaluate the integral

$$\int_1^2 \int_{\frac{1}{y}}^{y} \sqrt{\frac{y}{x}}\; e^{\sqrt{xy}}\,dx\,dy$$

- We suggest $\left.\begin{array}{l} u=\sqrt{xy} \\ v=\sqrt{\frac{y}{x}} \end{array}\right\} \Rightarrow \left.\begin{array}{l} u^2=xy \\ v^2=\frac{y}{x} \end{array}\right\} \Rightarrow \left.\begin{array}{l} x^2=\frac{u^2}{v^2} \\ y^2=u^2v^2 \end{array}\right\} \Rightarrow \begin{array}{l} x=\frac{u}{v} \\ y=uv \end{array}$

$$\Rightarrow J(u,v) = \begin{vmatrix} \frac{1}{v} & \frac{-u}{v^2} \\ v & u \end{vmatrix} = \frac{2u}{v}$$



- when $y=2 \Rightarrow v=\frac{2}{u}$, $y=1 \Rightarrow v=\frac{1}{u}$

  $y=\frac{1}{x} \Rightarrow xy=1 \Rightarrow u=1$

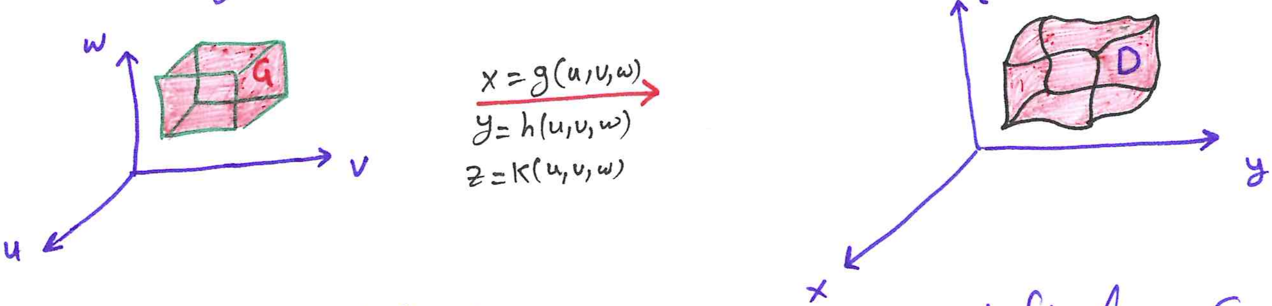
  $y=x \Rightarrow\Rightarrow\Rightarrow v=1$

- $$\int_1^2\int_{\frac{1}{y}}^{y}\sqrt{\frac{y}{x}}\, e^{\sqrt{xy}}\,dx\,dy = \int_1^2\int_1^{\frac{2}{u}} 2u\, e^u\, dv\, du = 2\int_1^2 (2-u)\, e^u\, du$$
  $$= 2e(e-2)$$

$\boxed{B}$ Substitution in Triple Integrals

* Suppose that a region G in uvw-space is transformed 1-1 into region D in xyz-space by the equations:

$$x=g(u,v,w) \quad , \quad y=h(u,v,w) \quad , \quad z=K(u,v,w)$$



* If $F(x,y,z)$ is defined on D, then F is also defined on G. That is $F(x,y,z) = F\big(g(u,v,w), h(u,v,w), K(u,v,w)\big) = H(u,v,w)$

* g, h, k have continuous first partial derivatives, then

$$\iiint_D F(x,y,z)\,dx\,dy\,dz = \iiint_G H(u,v,w)\ \text{abs}\big(J(u,v,w)\big)\,du\,dv\,dw$$

where the Jacobian $J(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$.

Exp Let D be the region in xyz-space defined by the inequalities:

$$1 \le x \le 2, \quad 0 \le xy \le 2, \quad 0 \le z \le 1$$

Evaluate $\iiint_D (x^2y + 3xyz)\,dx\,dy\,dz$ by applying the transformation

- $\left.\begin{array}{l} u = x \\ v = xy \\ w = 3z \end{array}\right\} \Rightarrow \left.\begin{array}{l} x = u \\ y = \frac{v}{u} \\ z = \frac{w}{3} \end{array}\right\} \Rightarrow J(u,v,w) = \begin{vmatrix} 1 & 0 & 0 \\ \frac{-v}{u^2} & \frac{1}{u} & 0 \\ 0 & 0 & \frac{1}{3} \end{vmatrix} = \frac{1}{3u}$

- when $1 \le x \le 2 \Rightarrow 1 \le u \le 2$
  $0 \le xy \le 2 \Rightarrow 0 \le v \le 2$
  $0 \le z \le 1 \Rightarrow 0 \le w \le 3$

- $\iiint_D (x^2y + 3xyz)\,dx\,dy\,dz = \iiint_G (uv - wv)\frac{1}{3u}\,du\,dv\,dw$

$$= \frac{1}{3}\int_0^3\int_0^2\int_1^2 \left(v + \frac{vw}{u}\right) du\,dv\,dw$$

$$= \frac{1}{3}\int_0^3\int_0^2 (v + vw\ln 2)\,dv\,dw = \frac{2}{3}\int_0^2 (1 + w\ln 2)\,dw$$

$$= 2 + 3\ln 2$$
