10.4 (Direct) Comparasion Test

Th 10 "Comparasion Test"

- Let $\sum a_n$, $\sum c_n$, $\sum d_n$ be series with nonnegative terms.
- Suppose that $d_n \le a_n \le c_n$ for all $n > N$.

(a) If $\sum c_n$ converges, then $\sum a_n$ also converges.

(b) If $\sum d_n$ diverges, then $\sum a_n$ also diverges.

Exp Use the Comparasion Test to determine if the following series converges or diverges?

① $\displaystyle\sum_{n=1}^{\infty} \frac{1+\cos n}{n^2} \le 2\sum_{n=1}^{\infty} \frac{1}{n^2}$ → This is p-series with $p=2$ which converges

Thus, $\displaystyle\sum_{n=1}^{\infty} \frac{1+\cos n}{n^2}$ converges by the (Direct) Comparasion Test

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② $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{3n-1}$ diverges because $\lim_{n\to\infty} a_n \ne 0$

$$\lim_{n\to\infty} \frac{2n}{3n-1} = \frac{2}{3} \ne 0$$

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③ $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n+\sqrt{n}} \ge \sum_{n=1}^{\infty} \frac{1}{n}$ → divergent series "harmonic series"

$3n > n+\sqrt{n} \Leftrightarrow$

$n+n+n > n+\sqrt{n}+0 \Leftrightarrow$

$n > \frac{n+\sqrt{n}}{3}$

$\frac{1}{n} < \frac{3}{n+\sqrt{n}}$

Thus $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n+\sqrt{n}}$ diverges by the comparsion Test.

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④ $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{3n+1}\right)^n < \sum_{n=1}^{\infty} \left(\frac{n}{3n}\right)^n = \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$ → geometric series which converges

Thus, $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{3n+1}\right)^n$ converges by the Comparsion Test

The limit Comparsion Test

Th (Limit Comparsion Test)

Suppose that $a_n > 0$ and $b_n > 0$ for all $n \geq N$

1. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then $\sum a_n$ and $\sum b_n$ both converges or both diverge.

2. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges

3. If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

Exp Use the limit Comparasion Test to determine whether the following series converges or diverges:

① $\sum_{n=1}^{\infty} \frac{n-2}{n^3 - n^2 + 3}$ $\qquad a_n = \frac{n-2}{n^3 - n^2 + 3} \qquad b_n = \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n-2}{n^3-n^2+3}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^3 - 2n^2}{n^3 - n^2 + 3} = 1 > 0$$

and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges "p-series with p=2".

Thus, $\sum_{n=1}^{\infty} \frac{n-2}{n^3 - n^2 + 3}$ converges by the limit Comparsion Test.

② $\sum_{n=1}^{\infty} \frac{1}{1 + \ln n}$ $\qquad a_n = \frac{1}{1 + \ln n} \qquad b_n = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{1+\ln n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{1+\ln n} = \lim_{n\to\infty} \frac{1}{\frac{1}{n}} = \infty$$

and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges "harmonic series"

Thus, $\sum_{n=1}^{\infty} \frac{1}{1 + \ln n}$ diverges by the limit Comparsion Test.

(3) $\sum_{n=1}^{\infty} \sin \frac{1}{n}$ $\quad a_n = \sin \frac{1}{n} \quad b_n = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin \frac{1}{n}}{\frac{1}{n}} = \lim_{x\to 0} \frac{\sin x}{x} = 1 > 0$$

and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

Thus, $\sum_{n=1}^{\infty} \sin \frac{1}{n}$ diverges by limit Comparsion Test.

(4) $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^{9/2}}$ $\quad a_n = \frac{(\ln n)^2}{n^{9/2}} \quad b_n = \frac{1}{n^{\frac{3}{2}}}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{(\ln n)^2}{n^{9/2}}}{1/n^{\frac{3}{2}}} = \lim_{n\to\infty} \frac{(\ln n)^2}{n^3} = \lim_{n\to\infty} \frac{\frac{2\ln n}{n}}{3n^2}$$

$$= \lim_{n\to\infty} \frac{2}{3} \frac{\ln n}{n^3} = \frac{2}{3} \lim_{n\to\infty} \frac{\frac{1}{n}}{3n^2} = \frac{2}{9} \lim_{n\to\infty} \frac{1}{n^3}$$

$$= \frac{2}{9}(0) = 0$$

and $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is convergent p-series. Thus, $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^{\frac{9}{2}}}$ converges by the limit Comparsion Test.

Exp
what about $\sum_{n=1}^{\infty} \frac{1-n}{n2^n} = \sum_{n=1}^{\infty} \frac{1}{n2^n} + \sum_{n=1}^{\infty} \frac{-1}{2^n}$

converges by the Direct Comparsion Test since $\frac{1}{n2^n} < \frac{1}{2^n}$

converges since it is a geometric series

Thus, $\sum_{n=1}^{\infty} \frac{1-n}{n2^n}$ converges since it is sum of two convergent series.