Chapter 3 - Vectors

I. Definition

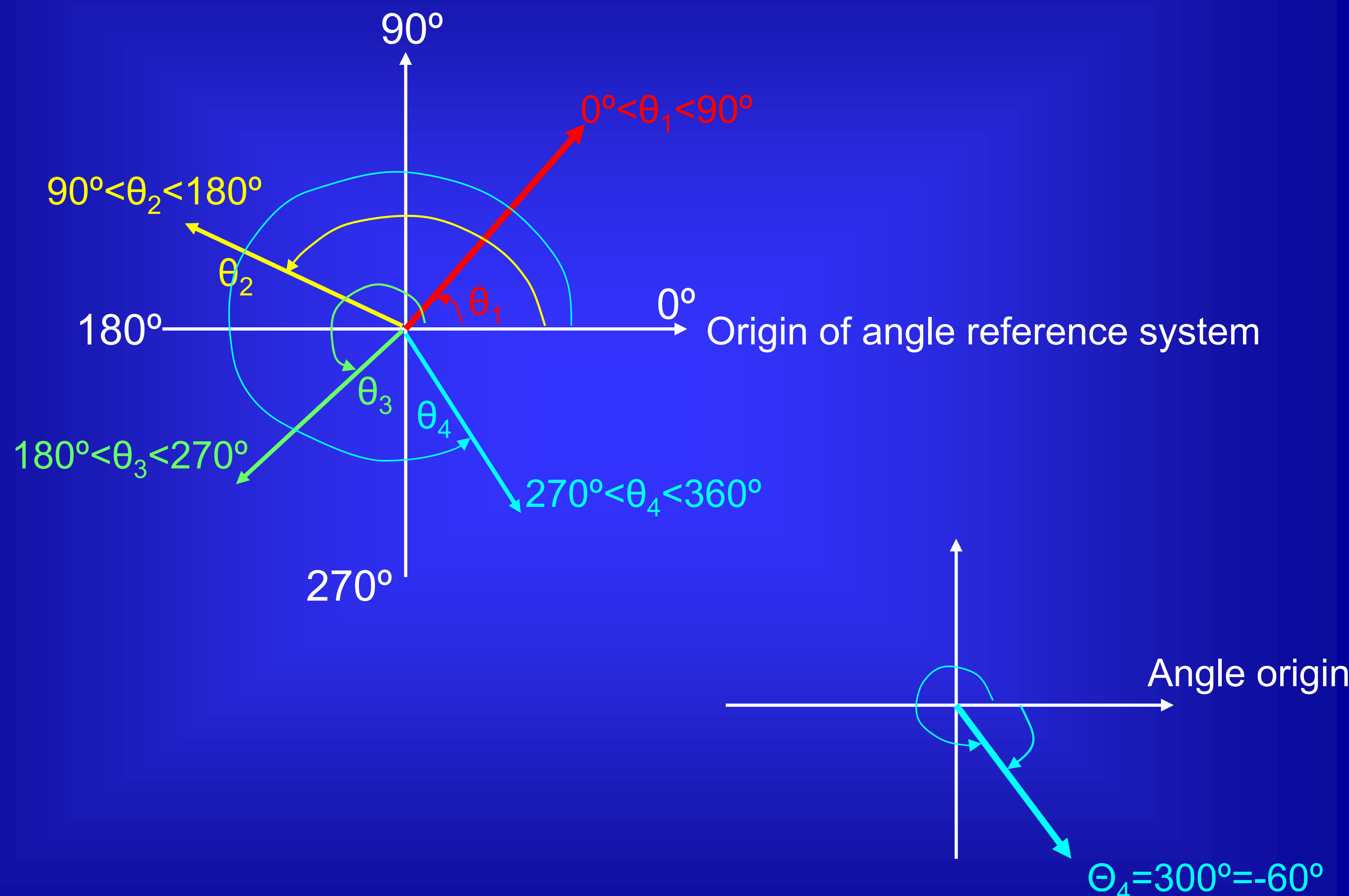
II. Arithmetic operations involving vectors

A) Addition and subtraction

- Graphical method
- Analytical method → Vector components

B) Multiplication

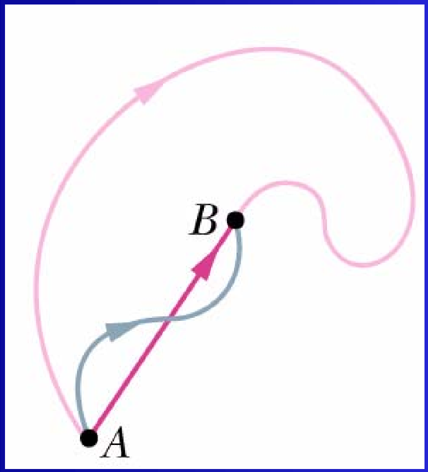
Review of angle reference system

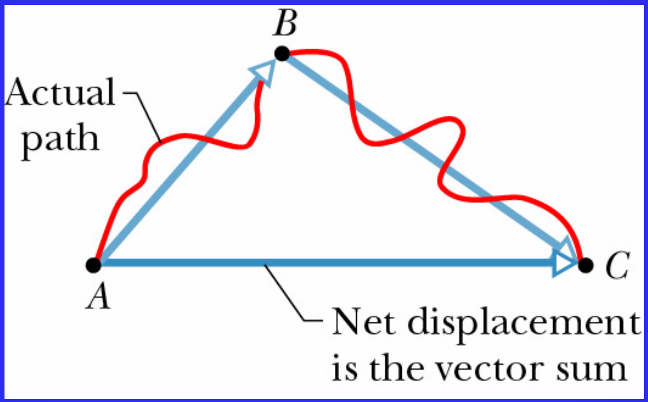


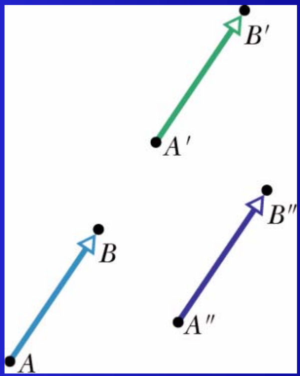
I. Definition

Vector quantity: quantity with a magnitude and a direction. It can be represented by a vector.

Examples: displacement, velocity, acceleration.







Same displacement

Displacement → does not describe the object's path.

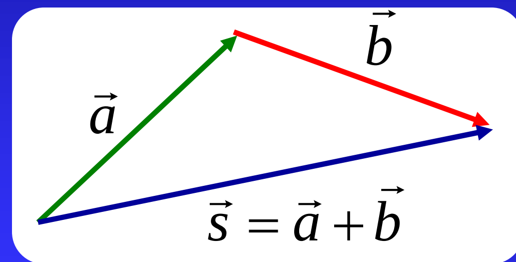
Scalar quantity: quantity with magnitude, no direction.

Examples: temperature, pressure

II. Arithmetic operations involving vectors

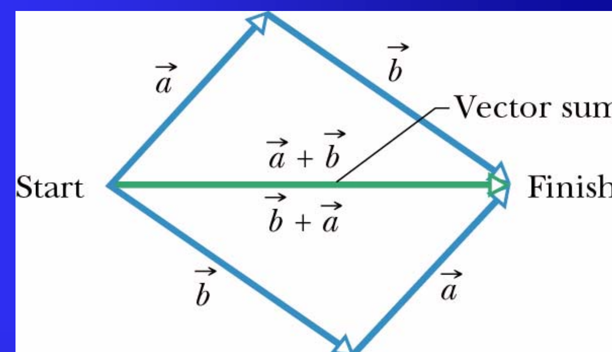
Vector addition: $\vec{s} = \vec{a} + \vec{b}$

- <u>**Geometrical method**</u>

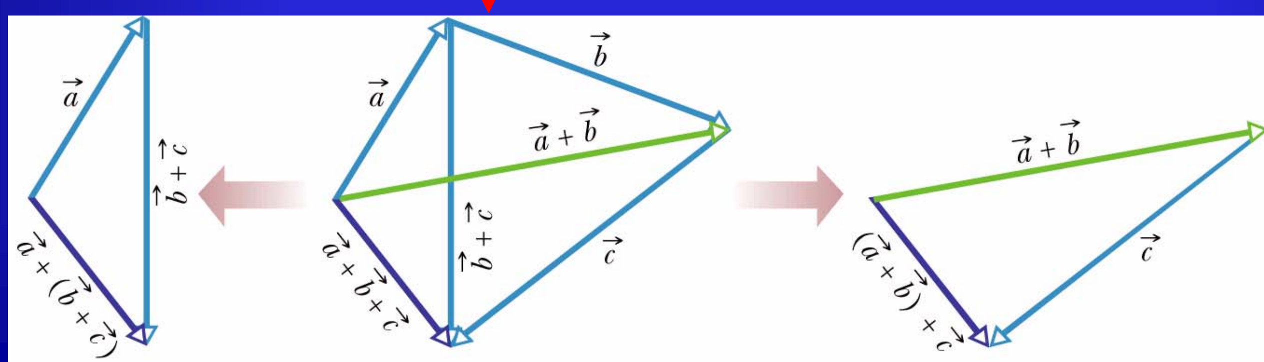


Rules:

$$\vec{a} + \vec{b} = \vec{b} + \vec{a} \qquad (commutative\ law) \qquad (3.1)$$

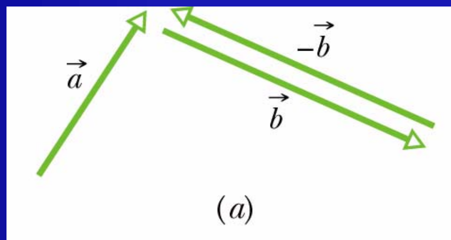


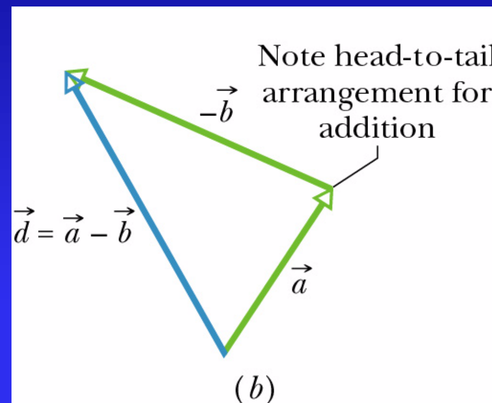
$$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c}) \qquad (associative\ law) \qquad (3.2)$$



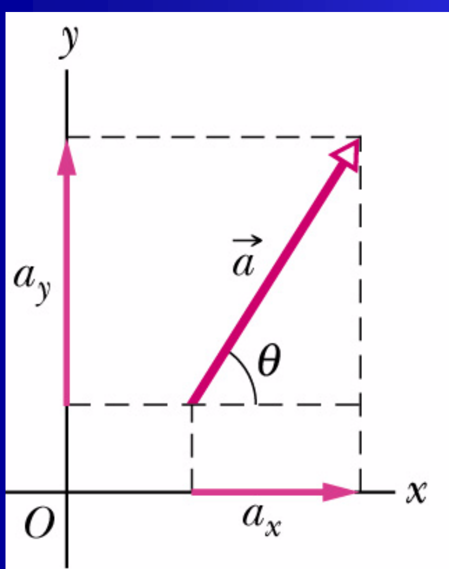
Vector subtraction:

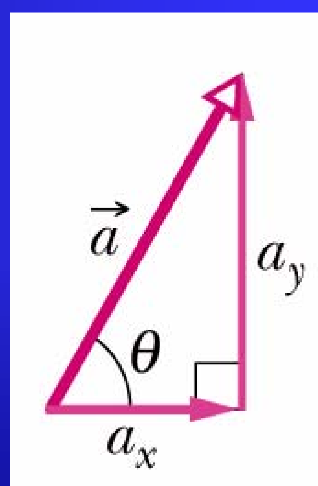
$$\vec{d} = \vec{a} - \vec{b} = \vec{a} + (-\vec{b}) \qquad (3.3)$$





Vector component: projection of the vector on an axis.





$$a_x = a\cos\theta$$
$$a_y = a\sin\theta \qquad (3.4)$$

→ *Scalar components of* $\vec{a}$

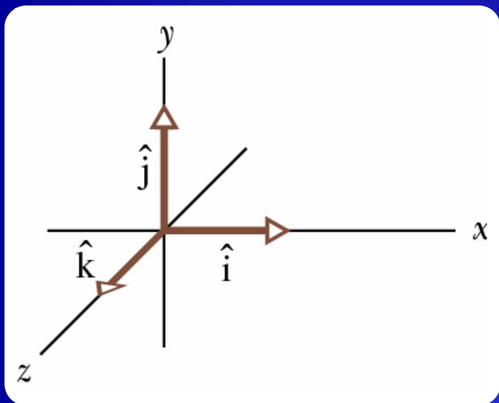
$$a = \sqrt{a_x^2 + a_y^2} \qquad (3.5)$$

Vector magnitude

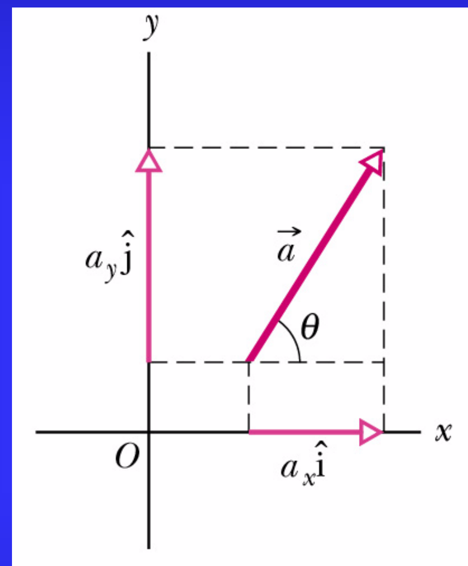
$$\tan\theta = \frac{a_y}{a_x}$$

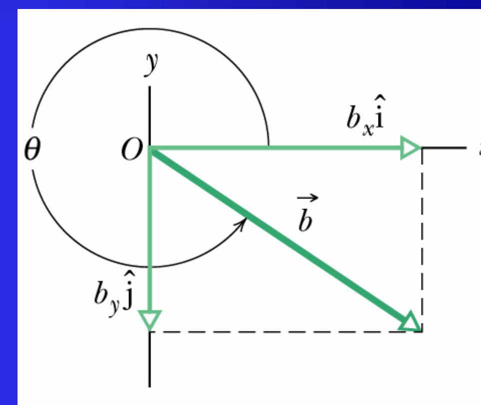
Vector direction

Unit vector: Vector with magnitude 1.
No dimensions, no units.



$\hat{i}, \hat{j}, \hat{k} \rightarrow$ *unit vectors in positive direction of x, y, z axes*





$$\vec{a} = a_x\hat{i} + a_y\hat{j} \qquad (3.6)$$

Vector component

Vector addition:

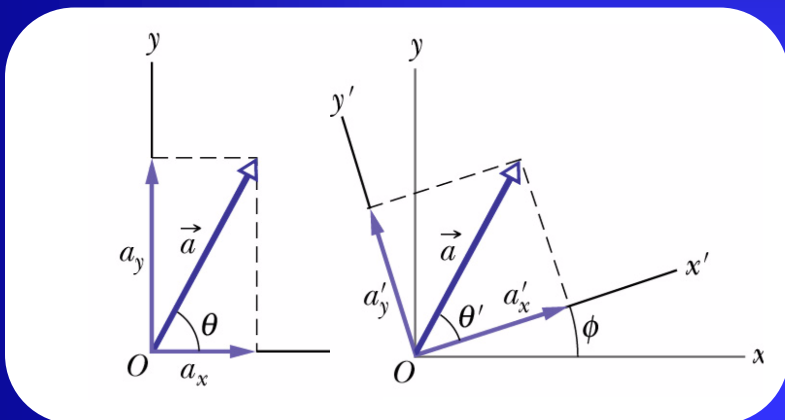
- **Analytical method:** adding vectors by components.

$$\vec{r} = \vec{a} + \vec{b} = (a_x + b_x)\hat{i} + (a_y + b_y)\hat{j} \qquad (3.7)$$

Vectors & Physics:

-The relationships among vectors do not depend on the location of the origin of the coordinate system or on the orientation of the axes.

- The laws of physics are independent of the choice of coordinate system.



$$a = \sqrt{a_x^2 + a_y^2} = \sqrt{a'^2_x + a'^2_y} \qquad (3.8)$$

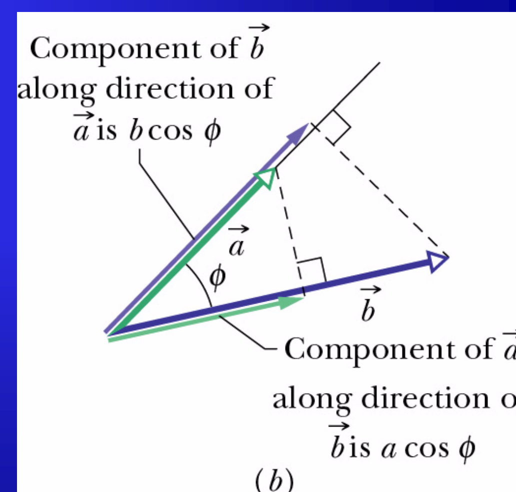
$$\theta = \theta' + \phi$$

Multiplying vectors:

- Vector by a scalar: $\vec{f} = s \cdot \vec{a}$
- Vector by a vector:

Scalar product = scalar quantity

(dot product)



$$\vec{a} \cdot \vec{b} = ab\cos\phi = a_x b_x + a_y b_y + a_z b_z \qquad (3.9)$$

Rule: $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}$ (3.10)

$\vec{a}\cdot\vec{b}=ab \leftarrow \cos\phi=1 \ (\phi=0^\circ)$

$\vec{a}\cdot\vec{b}=0 \leftarrow \cos\phi=0 \ (\phi=90^\circ)$

$$\hat{\imath}.\hat{\imath} = \hat{\jmath}.\hat{\jmath} = \hat{k}.\hat{k} = (1)(1)\cos 0^\circ = 1$$

$$\hat{\imath}.\hat{\jmath} = \hat{\jmath}.\hat{k} = \hat{k}.\hat{\imath} = \hat{\jmath}.\hat{\imath} = \hat{k}.\hat{\jmath} = \hat{\imath}.\hat{k} = (1)(1)\cos 90^\circ = 0$$

Angle between two vectors:

$$\cos\varphi = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|}$$

Multiplying vectors:

- Vector by a vector

<u>Vector product</u> = vector (cross product)

$$\vec{a}\times\vec{b}=\vec{c}=(a_y b_z - b_y a_z)\hat{i}-(b_z a_x - a_z b_x)\hat{j}+(a_x b_y - b_x a_y)\hat{k}$$

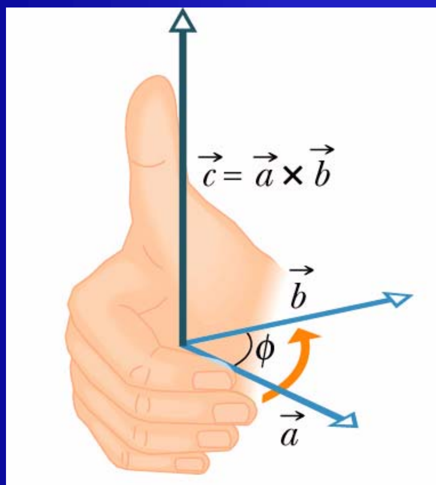
$c = ab\sin\phi$ **Magnitude**

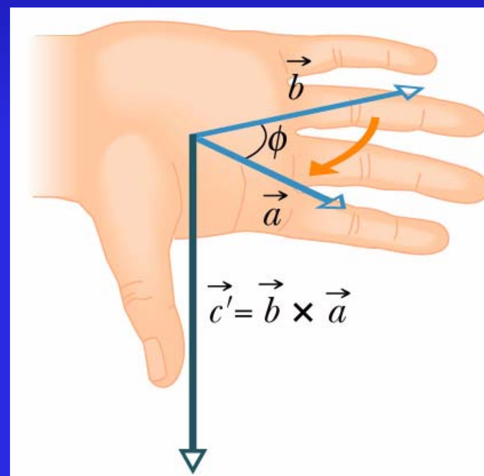
$\vec{a}\times\vec{b}=0 \quad \leftarrow \sin\phi = 0 \ \ (\phi = 0^\circ)$

$\left|\vec{a}\times\vec{b}\right| = ab \quad \leftarrow \sin\phi = 1 \ \ (\phi = 90^\circ)$

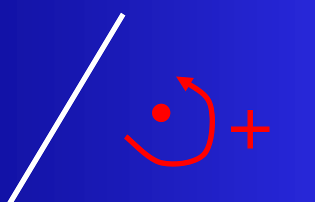
Vector product

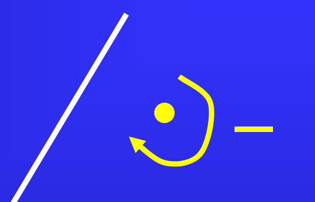
Direction → right hand rule





Rule: $\vec{b}\times\vec{a} = -(\vec{a}\times\vec{b}) \qquad (3.12)$





$\vec{c}$ *perpendicular to plane containing* $\vec{a},\vec{b}$

1) Place $\vec{a}$ and $\vec{b}$ tail to tail without altering their orientations.
2) $\vec{c}$ will be along a line perpendicular to the plane that contains $\vec{a}$ and $\vec{b}$ where they meet.
3) Sweep $\vec{a}$ into $\vec{b}$ through the smallest angle between them.

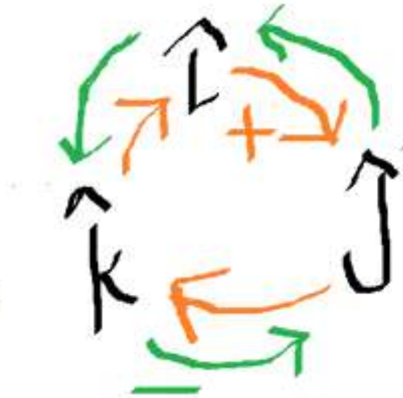
Products of Vectors

Let $\hat{\mathrm{i}}$, $\hat{\mathrm{j}}$, and $\hat{\mathrm{k}}$ be unit vectors in the x, y, and z directions. Then

$$\hat{\mathrm{i}}\cdot\hat{\mathrm{i}} = \hat{\mathrm{j}}\cdot\hat{\mathrm{j}} = \hat{\mathrm{k}}\cdot\hat{\mathrm{k}} = 1, \quad \hat{\mathrm{i}}\cdot\hat{\mathrm{j}} = \hat{\mathrm{j}}\cdot\hat{\mathrm{k}} = \hat{\mathrm{k}}\cdot\hat{\mathrm{i}} = 0,$$

$$\hat{\mathrm{i}}\times\hat{\mathrm{i}} = \hat{\mathrm{j}}\times\hat{\mathrm{j}} = \hat{\mathrm{k}}\times\hat{\mathrm{k}} = 0,$$

$$\hat{\mathrm{i}}\times\hat{\mathrm{j}} = \hat{\mathrm{k}}, \quad \hat{\mathrm{j}}\times\hat{\mathrm{k}} = \hat{\mathrm{i}}, \quad \hat{\mathrm{k}}\times\hat{\mathrm{i}} = \hat{\mathrm{j}}$$



Any vector $\vec{a}$ with components a_x, a_y, and a_z along the x, y, and z axes can be written as

$$\vec{a} = a_x\hat{\mathrm{i}} + a_y\hat{\mathrm{j}} + a_z\hat{\mathrm{k}}.$$

Let $\vec{a}$, $\vec{b}$, and $\vec{c}$ be arbitrary vectors with magnitudes a, b, and c. Then

$$\vec{a}\times(\vec{b}+\vec{c}) = (\vec{a}\times\vec{b}) + (\vec{a}\times\vec{c})$$

$$(s\vec{a})\times\vec{b} = \vec{a}\times(s\vec{b}) = s(\vec{a}\times\vec{b}) \qquad (s = \text{a scalar}).$$

Let θ be the smaller of the two angles between $\vec{a}$ and $\vec{b}$. Then

$$\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a} = a_xb_x + a_yb_y + a_zb_z = ab\cos\theta$$

$$\vec{a}\times\vec{b} = -\vec{b}\times\vec{a} = \begin{vmatrix} \hat{\mathrm{i}} & \hat{\mathrm{j}} & \hat{\mathrm{k}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$$= \hat{\mathrm{i}}\begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} - \hat{\mathrm{j}}\begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} + \hat{\mathrm{k}}\begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix}$$

$$= (a_yb_z - b_ya_z)\hat{\mathrm{i}} + (a_zb_x - b_za_x)\hat{\mathrm{j}} + (a_xb_y - b_xa_y)\hat{\mathrm{k}}$$

$$|\vec{a}\times\vec{b}| = ab\sin\theta$$

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{b}\cdot(\vec{c}\times\vec{a}) = \vec{c}\cdot(\vec{a}\times\vec{b})$$

$$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$$

P2

$\vec{d}_1 = 4\hat{i} + 5\hat{j} - 6\hat{k}$

$\vec{d}_2 = -\hat{i} + 2\hat{j} + 3\hat{k}$

$\vec{d}_3 = 4\hat{i} + 3\hat{j} + 2\hat{k}$

(a) $\vec{r} = \vec{d}_1 - \vec{d}_2 + \vec{d}_3$?

(b) Angle between $\vec{r}$ and $+z$?

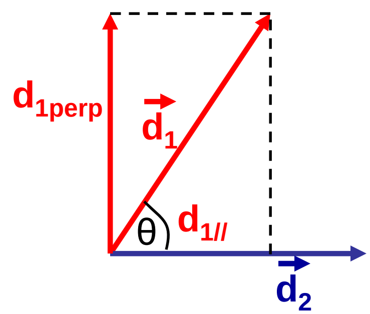
(c) Component of $\vec{d}_1$ along $\vec{d}_2$?

(d) Component of $\vec{d}_1$ perpendicular to $\vec{d}_2$ and in plane of $\vec{d}_1, \vec{d}_2$?

(a) $\vec{r} = \vec{d}_1 - \vec{d}_2 + \vec{d}_3 = (4\hat{i} + 5\hat{j} - 6\hat{k}) - (-\hat{i} + 2\hat{j} + 3\hat{k}) + (4\hat{i} + 3\hat{j} + 2\hat{k}) = 9\hat{i} + 6\hat{j} - 7\hat{k}$

(b) $\vec{r} \cdot \hat{k} = r \cdot 1 \cdot \cos\theta = -7 \rightarrow \theta = \cos^{-1}\left(\frac{-7}{12.88}\right) = 123^\circ$

$r = \sqrt{9^2 + 6^2 + 7^2} = 12.88m$



(c) $\vec{d}_1 \cdot \vec{d}_2 = -4 + 10 - 18 = -12 = d_1 d_2 \cos\theta \rightarrow \cos\theta = \frac{\vec{d}_1 \cdot \vec{d}_2}{d_1 d_2}$

$d_{1//} = d_1 \cos\theta = d_1 \frac{\vec{d}_1 \cdot \vec{d}_2}{d_1 d_2} = \frac{-12}{3.74} = -3.2m$

$d_2 = \sqrt{1^2 + 2^2 + 3^2} = 3.74m$

(d) $d_1 = \sqrt{d_{1//}^2 + d_{1perp}^2} \rightarrow d_{1perp} = \sqrt{8.77^2 - 3.2^2} = 8.16m$

$d_1 = \sqrt{4^2 + 5^2 + 6^2} = 8.77m$

P3

If $\vec{d}_1 = 3\hat{i} - 2\hat{j} + 4\hat{k}$

$\vec{d}_2 = -5\hat{i} + 2\hat{j} - \hat{k}$

$(\vec{d}_1 + \vec{d}_2) \cdot (\vec{d}_1 \times 4\vec{d}_2)$?

$(\vec{d}_1 + \vec{d}_2) = \vec{a} \rightarrow$ contained in $(\vec{d}_1, \vec{d}_2)$ plane

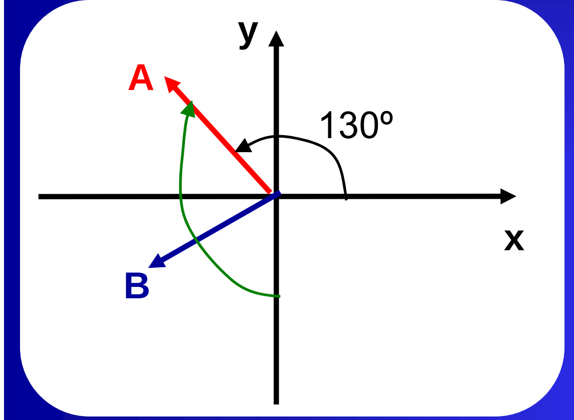
$(\vec{d}_1 \times 4\vec{d}_2) = 4(\vec{d}_1 \times \vec{d}_2) = 4\vec{b} \rightarrow$ perpendicular to $(\vec{d}_1, \vec{d}_2)$ plane

$\vec{a}$ perpendicular to $\vec{b} \rightarrow \cos 90^\circ = 0 \rightarrow 4\vec{a} \cdot \vec{b} = 0$

Tip: Think before calculate !!!

P4:

Vectors $\vec{A}$ and $\vec{B}$ lie in an xy plane. $\vec{A}$ has a magnitude 8.00 and angle 130º; $\vec{B}$ has components B_x= -7.72, B_y= -9.20. What are the angles between the negative direction of the y axis and (a) the direction of $\vec{A}$, (b) the direction of $\vec{A}\times\vec{B}$, (c) the direction of $\vec{A}\times(\vec{B}+3\hat{k})$?



$$(a)\ \text{Angle between } -y \text{ and } \vec{A} = 90^\circ + 50^\circ = 140^\circ$$

$$(b)\ \text{Angle } -y\ ,(\vec{A}\times\vec{B}) = \vec{C} \rightarrow \text{angle } -\hat{j}\ ,\hat{k} \text{ because } \vec{C} \text{ perpendicular}$$

$$\text{plane } (\vec{A},\vec{B}) = (xy) \rightarrow 90^\circ$$

$$(c)\ \text{Direction } \vec{A}\times(\vec{B}+3\hat{k}) = \vec{D}$$

$$\vec{E} = \vec{B}+3\hat{k} = -7.72\hat{i} - 9.2\hat{j} + 3\hat{k}$$

$$\vec{D} = \vec{A}\times\vec{E} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -5.14 & 6.13 & 0 \\ -7.72 & -9.20 & 3 \end{vmatrix} = 18.39\hat{i} + 15.42\hat{j} + 94.61\hat{k}$$

$$|D| = \sqrt{18.39^2 + 15.42^2 + 94.61^2} = 97.61$$

$$-\hat{j}\cdot\vec{D} = -\hat{j}\cdot(18.39\hat{i} + 15.42\hat{j} + 94.61\hat{k}) = -15.42$$

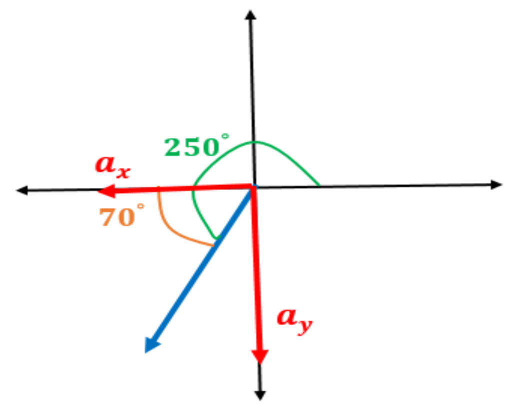
$$\cos\theta = \left(\frac{-\hat{j}\cdot\vec{D}}{1\cdot|D|}\right) = \left(\frac{-15.42}{97.61}\right) \rightarrow \theta = 99^\circ$$

•1 SSM What are (a) the x component and (b) the y component of a vector $\vec{a}$ in the xy plane if its direction is 250° counterclockwise from the positive direction of the x axis and its magnitude is 7.3 m?

$$\vec{a} = a_x\,\hat{\imath} + a_y\,\hat{\jmath}$$

$$a_x = a\cos\theta = 7.3\,m\cos(250^\circ) = -2.5\,m$$

$$a_y = a\sin\theta = 7.3\,m\sin(250^\circ) = -6.9\,m$$



$$a_x = a\cos\theta = -7.3\,m\cos(70^\circ) = -2.5\,m$$

$$a_y = a\sin\theta = -7.3\,m\sin(70^\circ) = -6.9\,m$$

•11 SSM (a) In unit-vector notation, what is the sum $\vec{a} + \vec{b}$ if $\vec{a} = (4.0\text{ m})\hat{\imath} + (3.0\text{ m})\hat{\jmath}$ and $\vec{b} = (-13.0\text{ m})\hat{\imath} + (7.0\text{ m})\hat{\jmath}$? What are the (b) magnitude and (c) direction of $\vec{a} + \vec{b}$?

$$\vec{a} = (4.0\,m)\hat{\imath} + (3.0\,m)\hat{\jmath}$$

$$\vec{b} = (-13.0\,m)\hat{\imath} + (7.0\,m)\hat{\jmath}$$

a) $\vec{a} + \vec{b} = (4.0\,m)\hat{\imath} + (3.0\,m)\hat{\jmath} + (-13.0\,m)\hat{\imath} + (7.0\,m)\hat{\jmath}$

$\vec{a} + \vec{b} = (4.0 - 13.0)m\hat{\imath} + (3.0 + 7.0)m\hat{\jmath}$

$\vec{r} = \vec{a} + \vec{b} = (-9.0\,m)\hat{\imath} + (10.0\,m)\hat{\jmath}$

b) $\vec{r} = (-9.0\,m)\hat{\imath} + (10.0\,m)\hat{\jmath}$

$$r = \sqrt{r_x^2 + r_y^2} = \sqrt{(-9.0)^2 + (10.0)^2} = 13.5\,m$$

c) $\theta = tan^{-1}\left(\frac{r_y}{r_x}\right) = tan^{-1}\left(\frac{10.0}{-9.0}\right) = -48^\circ$

The vector has negative x component and positive y component so it lies in the second quadrant.

$\vec{a} + \vec{b}$ has a magnitude of 13.45 m with 132° counterclockwise the positive x-axis. $(42^\circ\, west\ from\ north)$

•15 SSM ILW WWW The two vectors $\vec{a}$ and $\vec{b}$ in Fig. 3-28 have equal magnitudes of 10.0 m and the angles are $\theta_1 = 30°$ and $\theta_2 = 105°$. Find the (a) x and (b) y components of their vector sum $\vec{r}$, (c) the magnitude of $\vec{r}$, and (d) the angle $\vec{r}$ makes with the positive direction of the x axis.

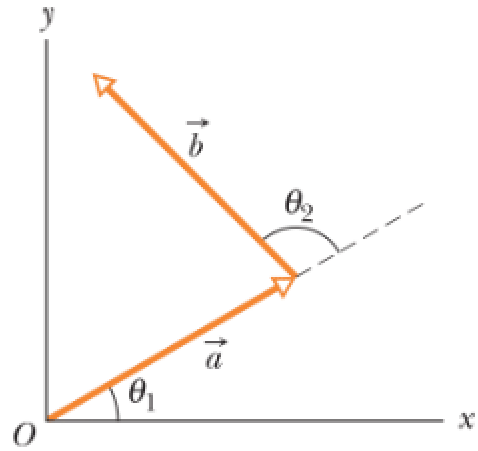


Figure 3-28 Problem 15.

$$\vec{a} = (a\cos\theta_1)\,\hat{\imath} + (a\ \sin\theta_1)\,\hat{\jmath}$$

$$\vec{a} = (10.0\cos\ 30^\circ)\,m\,\hat{\imath} + (10.0\ \sin\ 30^\circ)\,m\,\hat{\jmath} = 8.66\,m\,\hat{\imath} + \ 5.0\,m\,\hat{\jmath}$$

$$\vec{b} = (b\cos\theta)\,\hat{\imath} + (b\ \sin\theta)\,\hat{\jmath}\,,\ \theta = 105^\circ + 30^\circ = 135^\circ$$

$$\vec{b} = (10.0\cos\ 135^\circ)\,m\,\hat{\imath} + (10.0\ \sin\ 135^\circ)\,m\,\hat{\jmath} = (-7.07\,m)\,\hat{\imath} + (7.07\,m)\,\hat{\jmath}$$

$$\vec{r} = \ \vec{a} + \vec{b} = (8.66\,m\,\hat{\imath} + \ 5.0\,m\,\hat{\jmath}) + ((-7.07\,m)\hat{\imath} + (7.07\,m)\hat{\jmath})$$

$$\vec{r} = (1.59\,m)\,\hat{\imath} + (12.07\,m)\,\hat{\jmath}$$

a) $r_x = 1.59\,m$

b) $r_y = 12.07\,m$

c) $r = \sqrt{r_x^2 + r_y^2} = \sqrt{(1.59)^2 + (12.07)^2} = 12.17\,m$

d) $\theta = tan^{-1}\left(\frac{r_y}{r_x}\right) = tan^{-1}\left(\frac{12.07}{1.59}\right) = \ 82.5^\circ$

••30 GO Here are two vectors:

$$\vec{a} = (4.0\text{ m})\hat{i} - (3.0\text{ m})\hat{j} \quad \text{and} \quad \vec{b} = (6.0\text{ m})\hat{i} + (8.0\text{ m})\hat{j}.$$

What are (a) the magnitude and (b) the angle (relative to $\hat{i}$) of $\vec{a}$? What are (c) the magnitude and (d) the angle of $\vec{b}$? What are (e) the magnitude and (f) the angle of $\vec{a} + \vec{b}$; (g) the magnitude and (h) the angle of $\vec{b} - \vec{a}$; and (i) the magnitude and (j) the angle of $\vec{a} - \vec{b}$? (k) What is the angle between the directions of $\vec{b} - \vec{a}$ and $\vec{a} - \vec{b}$?

$\vec{a} = (4.0\text{m})\hat{i} - (3.0\text{m})\hat{j}$

$\vec{b} = (6.0\text{m})\hat{i} + (8.0\text{m})\hat{j}$

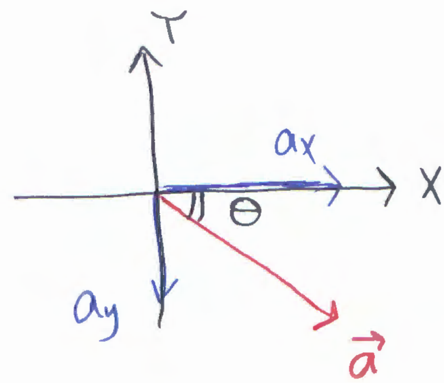
a) The magnitude of vector $\vec{a}$

$$a = \sqrt{a_x^2 + a_y^2} = \sqrt{(4.0)^2 + (-3.0)^2} = 5.0\text{ m}$$

b) The angle of vector $\vec{a}$

$$\tan\theta = \frac{a_y}{a_x} \Rightarrow \theta = \tan^{-1}\frac{3}{4}$$

$$\boxed{\theta = -36.9^\circ}$$



$$\Rightarrow \vec{a}\cdot\hat{i} = |\vec{a}|\,|\hat{i}|\cos\theta = \left[(4.0\text{m})\hat{i} - (3.0\text{m})\hat{j}\right]\cdot\hat{i}$$

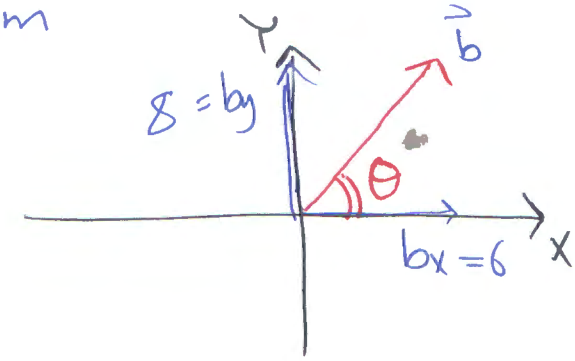
$$(5)(1)\cos\theta = 4$$

$$\theta = \cos^{-1}\frac{4}{5} = 36.9^\circ \Rightarrow \theta = -36.9^\circ$$

c) The magnitude of vector $\vec{b}$

$$b = \sqrt{(6.0)^2 + (8.0)^2} = 10.0\,m$$

d) $\tan\theta = \dfrac{b_y}{b_x}$



$$\theta = \tan^{-1}\left(\frac{8}{6}\right) = 53.1^\circ$$

$$\Rightarrow \vec{b}\cdot\hat{i} = |\vec{b}|\,|\hat{i}|\cos\theta = \vec{b}\cdot\hat{i}$$

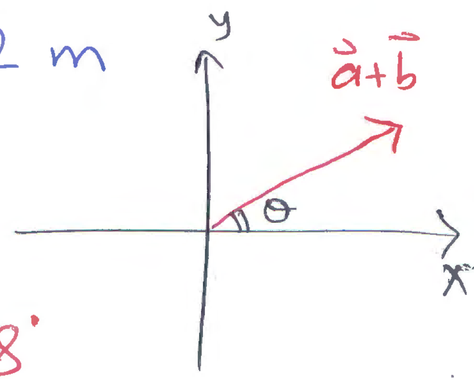
$$(10)(1)\cos\theta = \left[(6.0m)\hat{i} + (8.0m)\hat{j}\right]\cdot\hat{i} = 6$$

$$\boxed{\theta = \cos^{-1}(0.6) = 53.1^\circ}$$

e) $\vec{a} + \vec{b} = (4.0m)\hat{i} - (3.0m)\hat{j} + (6.0m)\hat{i} + (8.0m)\hat{j}$

$$\vec{a} + \vec{b} = (10.0\,m)\hat{i} + (5.0\,m)\hat{j}$$

$$|\vec{a} + \vec{b}| = \sqrt{(10.0)^2 + (5.0)^2} = 11.2\,m$$



f) $(\vec{a} + \vec{b})\cdot\hat{i} = (11.2)(1)\cos\theta = 10$

$$\theta = \cos^{-1}\frac{(\vec{a}+\vec{b})\cdot\hat{i}}{|\vec{a}+\vec{b}|\,|\hat{i}|} = 26.8^\circ$$

••38 GO For the following three vectors, what is $3\vec{C}\cdot(2\vec{A}\times\vec{B})$?

$$\vec{A} = 2.00\hat{i} + 3.00\hat{j} - 4.00\hat{k}$$

$$\vec{B} = -3.00\hat{i} + 4.00\hat{j} + 2.00\hat{k} \qquad \vec{C} = 7.00\hat{i} - 8.00\hat{j}$$

$\Rightarrow 2\vec{A} = 2(2.00\hat{i} + 3.00\hat{j} - 4.00\hat{k})$

$2\vec{A} = 4.00\hat{i} + 6.00\hat{j} - 8.00\hat{k}$

$$\Rightarrow 2\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4.00 & 6.00 & -8.00 \\ -3.00 & 4.00 & 2.00 \end{vmatrix}$$

$2\vec{A}\times\vec{B} = 44.00\hat{i} + 16.00\hat{j} + 34.00\hat{k}$

$\Rightarrow 3\vec{C} = 3(7.00\hat{i} - 8.00\hat{j}) = 21.00\hat{i} - 24.00\hat{j}$

$\Rightarrow 3\vec{C}\cdot(2\vec{A}\times\vec{B})$

$(21.00\hat{i} - 24.00\hat{j})\cdot(44.00\hat{i} + 16.00\hat{j} + 34.00\hat{k})$

$924 - 384 = 540$

••39 Vector $\vec{A}$ has a magnitude of 6.00 units, vector $\vec{B}$ has a magnitude of 7.00 units, and $\vec{A} \cdot \vec{B}$ has a value of 14.0. What is the angle between the directions of $\vec{A}$ and $\vec{B}$?

$A = 6.00$ units

$B = 7.00$ units

$\vec{A} \cdot \vec{B} = 14.0$

The angle between $\vec{A}$ and $\vec{B}$

By using the Dot Product

$$\vec{A} \cdot \vec{B} = |\vec{A}|\,|\vec{B}| \cos\theta = 14.0$$

$$\cos\theta = \frac{\vec{A} \cdot \vec{B}}{|\vec{A}|\,|\vec{B}|} = \frac{14.0}{(6.0)(7.0)}$$

$$\theta = 70.5^\circ$$