Quiz#4 Solutions.

Exercise#1 [**7.5 marks**].

(a) Use the **sequential criterion for limits** to show that the following limit does not exist:

$$\lim_{x\to 1} \frac{x-1}{|x-1|}.$$

1 Take $x_n = 1 + \frac{1}{n}$, $y_n = 1 - \frac{1}{n}$.

Clearly, $x_n \longrightarrow 1$ and $y_n \longrightarrow 1$ as $n \longrightarrow \infty$.

1 But $f(x_n) = 1$ and $f(y_n) = -1$, $\forall n \in \mathbb{N}$.

1 So, $f(x_n) \longrightarrow 1$ and $f(y_n) \longrightarrow -1$ as $n \to \infty$.

0.5 Thus, $\lim_{x\to 1} f(x)$ does not exist. $\square$

(b) **Use the** $\varepsilon - \delta$ **definition for limits** to prove that

$$\lim_{x\to 1} \frac{x^2 - x + 1}{x + 1} = \frac{1}{2}.$$

Proof. If $|x-1| < 1$, then $|2x-1| < 3$ and $\frac{1}{|x+1|} < 1$,

So that $\left| \frac{x^2 - x + 1}{x+1} - \frac{1}{2} \right| = \frac{|2x-1|\,|x-1|}{2\,|x+1|}$

$\leq \frac{3}{2}|x-1|$.

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So, set $\delta = \min\left\{ 1, \frac{2\varepsilon}{3} \right\}$. If $|x-1| < \delta$,

2 then $\left| \frac{x^2 - x + 1}{x+1} - \frac{1}{2} \right| \leq \frac{3}{2}|x-1| < \frac{3}{2} \cdot \frac{2\varepsilon}{3} = \varepsilon$.

$\square$

Exercise#2 [7.5 marks]. Prove or disprove.

(a) If $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$, then $\lim_{x\to\infty} ((f(x) - g(x)) = 0$.

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False. Take $f(x) = x+1$, $g(x) = x$.

$\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$. But,

1.5

$\lim_{x\to\infty} (f(x) - g(x)) = \lim_{x\to\infty} 1 = 1 \neq 0$. $\square$

(b) Let $f : (a, \infty) \to \mathbb{R}$. If $\lim_{x\to\infty} xf(x) = L$, where $L \in \mathbb{R}$, then $\lim_{x\to\infty} f(x) = 0$.

1 True. Since $\lim_{x\to\infty} xf(x) = L$, $\exists\, \alpha > 0$ s.t

$|xf(x) - L| < 1$, for $x > \alpha$.

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Hence, $|f(x)| < \frac{|L|+1}{x}$, for $x > \alpha$.

Let $\varepsilon > 0$ be given. Set $\alpha = \dfrac{|L|+1}{\varepsilon}$.

If $x > \alpha$, then $|f(x)| < \dfrac{|L|+1}{x} < \dfrac{|L|+1}{\alpha} = \varepsilon$.

Thus, $\lim\limits_{x\to\infty} f(x) = 0$. $\square$

(c) Let f and g be defined on (a,∞). If $\lim\limits_{x\to\infty} f(x) = L$ and $\lim\limits_{x\to\infty} g(x) = \infty$, then $\lim\limits_{x\to\infty} (f\circ g)(x) = L$.

(1 True. Since $\lim\limits_{x\to\infty} f(x) = L$, then given $\varepsilon > 0$,

$\exists\, M \in \mathbb{R}$ s.t $\boxed{|f(x) - L| < \varepsilon, \text{ for } x > M.}$ (i)

(1.5 Since $\lim\limits_{x\to\infty} g(x) = \infty$. then $\exists\, K \in \mathbb{R}$ s.t

$\boxed{g(y) > K, \text{ for } y > K.}$ (ii)

Thus, (i) and (ii) give

$$|(f\circ g)(y) - L| = |f(g(y)) - L| < \varepsilon, \text{ for } y > K.$$

Thus, $\lim_{x\to\infty} (f\circ g)(x) = L.$ $\square$