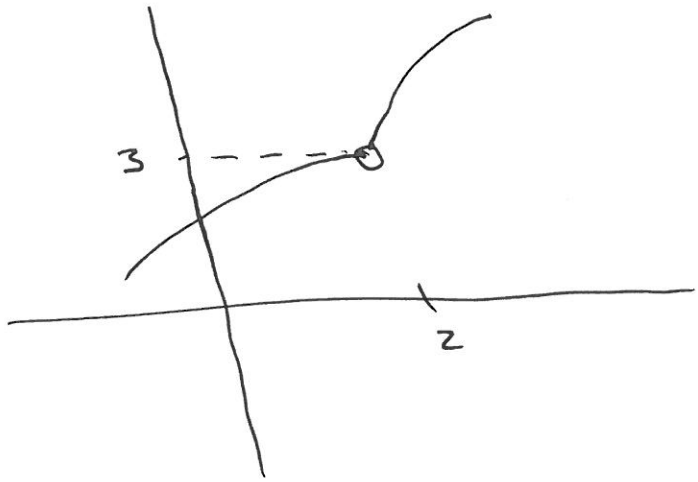
Math 1411
Review of ch 2
limits and continuity

$$\lim_{x \to a} f(x) = L \quad \text{iff} \quad \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L$$

Ex 1

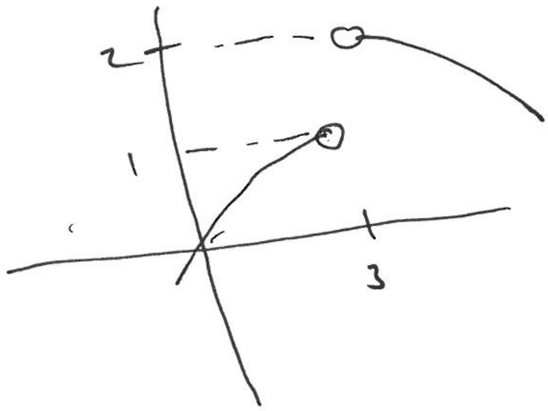


$\lim_{x \to 2^+} f(x) =$

$\lim_{x \to 2^-} f(x) =$

$\lim_{x \to 2} f(x) =$

Ex 2



$\lim_{x \to 3^-} f(x) =$

$\lim_{x \to 3^+} f(x) =$

$\lim_{x \to 3} f(x)$

Ex 3

$$f(x) = \begin{cases} x^2+1, & x \le 2 \\ x+3; & x > 2 \end{cases}$$

$\lim_{x \to 2^-} f(x) = 5$

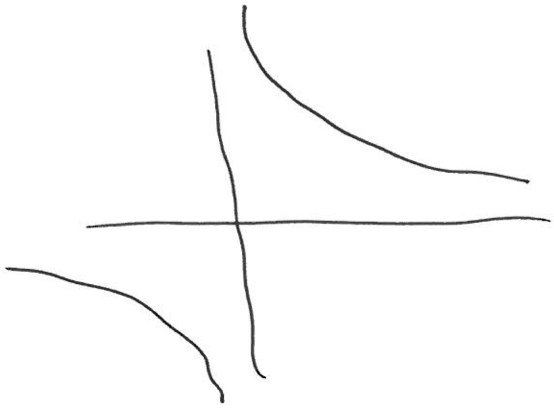
$\lim_{x \to 2^+} f(x) =$

$\lim f(x) =$

Ex 4 $\lim\limits_{x \to 1} \dfrac{x-1}{x+1} = \dfrac{0}{2} = 0$

Ex 5 $\lim\limits_{x \to 1} \dfrac{x^2-1}{x-1} = \lim\limits_{x \to 1} \dfrac{(x-1)(x+1)}{x-1} = 2$

Ex 6 $\lim\limits_{x \to \infty} \dfrac{1}{x}$



Ex 7 $\lim\limits_{x \to 0^+} =$

Ex 8 $\lim\limits_{x \to 1} \dfrac{x^2+x-2}{x^2-x} = \lim\limits_{x \to 1} \dfrac{(x+2)\cancel{(x-1)}}{x\cancel{(x-1)}} = 3$

Ex 9 $\lim\limits_{x \to -1} \dfrac{\sqrt{x^2+8}-3}{x+1} = \lim\limits_{x \to -1} \dfrac{\sqrt{x^2+8}-3}{(x+1)} \dfrac{(\sqrt{x^2+8}+3)}{(\sqrt{x^2+8}+3)}$

$= \lim\limits_{x \to -1} \dfrac{x^2+8-9}{(x+1)(\sqrt{x^2+8}+3)} = \lim\limits_{x \to -1} \dfrac{(x-1)\cancel{(x+1)}}{\cancel{(x+1)}(\sqrt{x^2+8}+3)}$

$= \dfrac{-2}{6} = -\dfrac{1}{3}$

Ex 10

$f(x) = \begin{cases} x+1, & x \le 0 \\ -x, & x > 0 \end{cases}$

$\lim\limits_{x \to 0^-} f(x) = 1$

$\lim\limits_{x \to 0^+} f(x) = 0$

$\lim\limits_{x \to 0} f(x)$ DNE

th Sandwich theorem

If $g(x) \le f(x) \le h(x)$, $\forall x$ around c

then if $\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L$

then $\lim_{x \to c} f(x) = L$

Ex 1 suppose that

$$1 - x^2 \le f(x) \le 1 + x^2$$

then $\lim_{x \to 0} (1 - x^2) \le \lim_{x \to 0} f(x) \le \lim_{x \to 0} (1 + x^2)$

$$L \le \lim_{x \to 0} f(x) \le L$$

$$\Rightarrow \lim_{x \to 0} f(x) = L$$

Ex 2 Find $\lim_{x \to \infty} \frac{\sin x}{x}$

since $\frac{-1}{x} < \frac{\sin x}{x} < \frac{1}{x}$, $x > 0$

and $\lim_{x \to \infty} \frac{1}{x} = 0$

$$\Rightarrow \lim_{x \to \infty} \frac{\sin x}{x} = 0$$

Ex 2 $\quad y = \dfrac{2}{x^2 - 4x + 3} = \dfrac{2}{(x-3)(x-1)}$

Horizontal Asymptote $\lim_{x \to \infty} f(x) = 0$

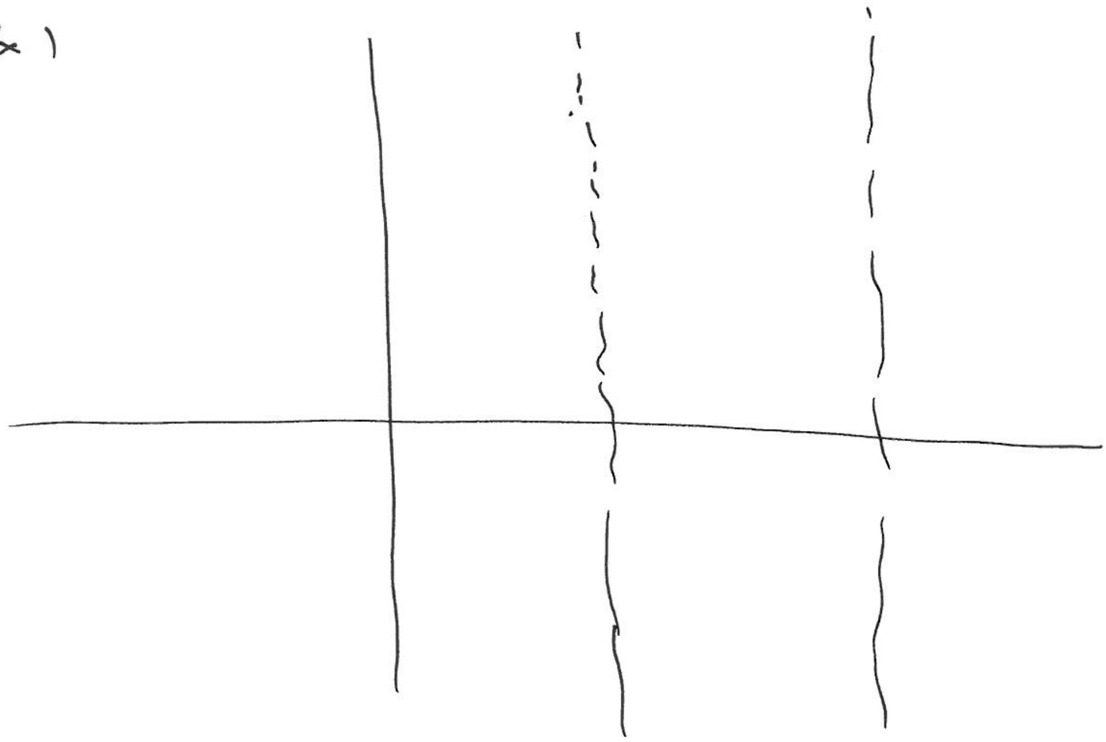
Vertical asymptotes $x = 1, x = 3$

$\lim_{x \to 1^+} f(x) =$

$\lim_{x \to 1^-} f(x) =$

$\lim_{x \to 3^+} f(x)$

$\lim_{x \to 3^-} f(x)$



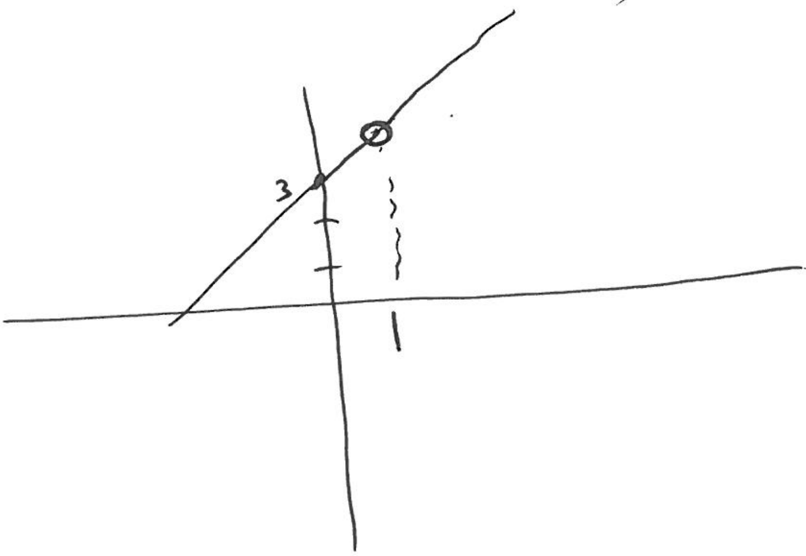
(Ex) Removable discontinuity:

Consider $f(x) = \dfrac{x^2+2x-3}{x-1}$

$f(x)$ is not continuos at $x = 1$

But $\lim\limits_{x\to 1} f(x) = \lim\limits_{x\to 1} \dfrac{x^2+2x-3}{x-1}$

$= \lim\limits_{x\to 1} \dfrac{(x+3)\cancel{(x-1)}}{\cancel{x-1}} = 4$



So if we define

$$g(x) = \begin{cases} \dfrac{x^2+2x-3}{x-1} & , x \neq 1 \\ 4 & , x = 1 \end{cases}$$

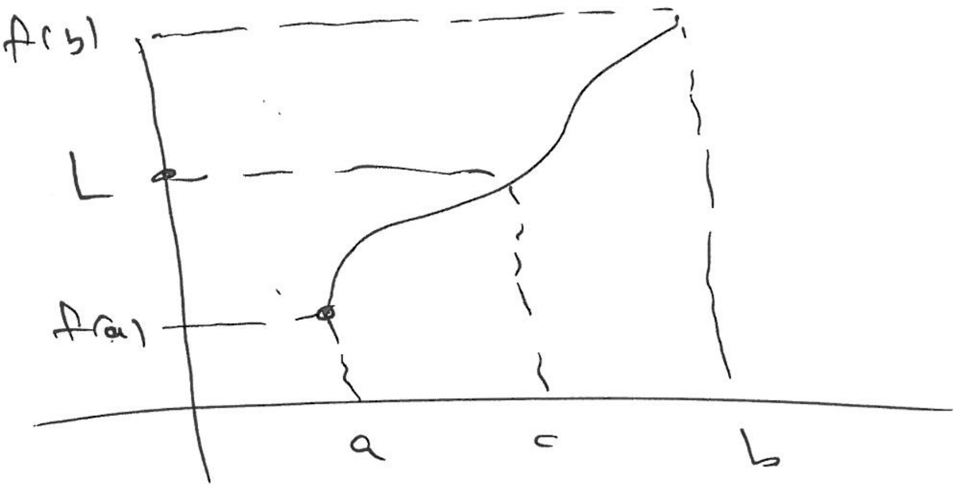
then $g(x)$ is cont. every where.

(Ex2) $f(x) = \dfrac{x^2+2x-3}{x^2-1} = \dfrac{(x+3)(x-1)}{(x+1)(x-1)} = \dfrac{x+3}{x+1}$ is not cont. at $x = 1, -1$

$\lim\limits_{x\to 1} \dfrac{x+3}{x+1} = 2$, If $F(x) = \begin{cases} f(x), & x \neq 1 \\ 2, & x = 1 \end{cases}$ is cont. at $x = 1$

Th The Intermediate value theorem IVT

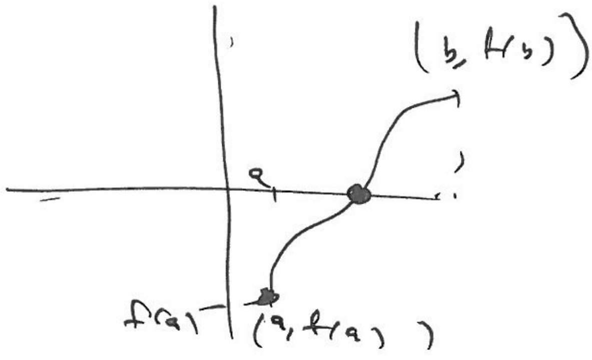
If $f(x)$ is continuous on the closed interval $[a,b]$ and L is any number between $f(a)$ and $f(b)$ then there exists $c \in [a,b]$ such that $f(c) = L$



Th Bolzano, If $f(a) \cdot f(b) < 0$

i.e $L = 0$

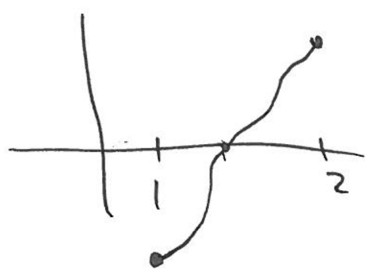
$\Rightarrow \exists\, c \in (a,b)$ s.t $f(c) = 0$



Ex $f(x) = x^3 - x - 1 \quad [1,2]$

$f(1) = -1 \quad , \quad f(2) = 5 \Rightarrow -1 < 0 < 5$

$\Rightarrow \exists\, c \in [1,2]$ s.t $f(c) = 0$



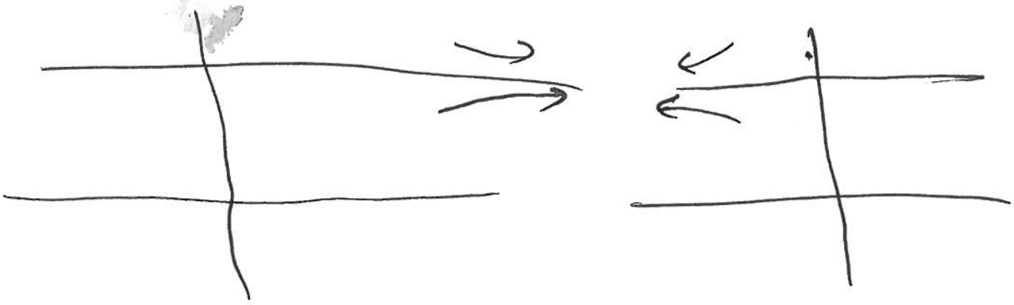
Asymptotes

We consider rational functions

$$f(x) = \frac{\text{Polynom}}{\text{Polynomial}}$$

Our goal to sketch the curve of such functions using limits and asymptotes.

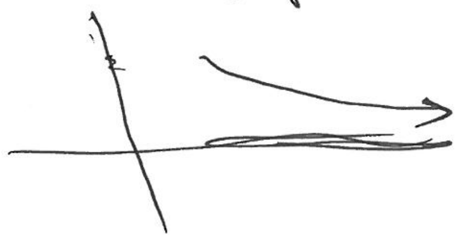
Def: A line $y=b$ is a horizontal asymptote of the graph of the function $y=f(x)$ if either $\lim_{x\to\infty} f(x) = b$ or $\lim_{x\to-\infty} f(x) = b$



Ex 1

$$\lim_{x\to\infty} \frac{x}{x^2+1} = 0$$

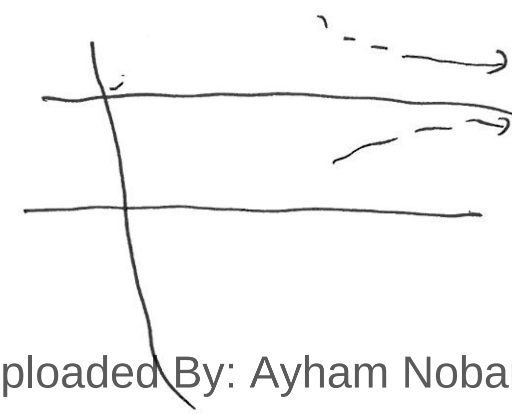
, $y=0$ is a horizontal asymptote



Ex 2

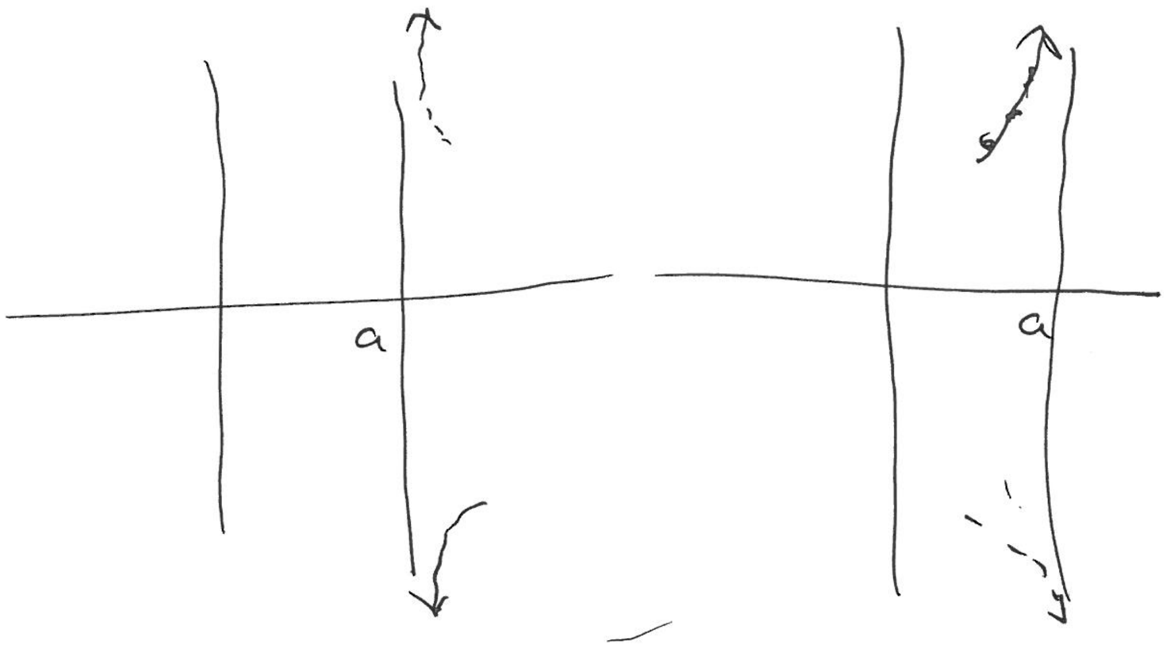
$$\lim_{x\to\infty} \frac{x^2}{x^2+1} = 1$$

$y=1$ is a horizontal asymptote



Def: A line $x = a$ is a vertical asymptote of the graph of the function $y = f(x)$ if

$$\lim_{x \to a^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \to a^-} f(x) = \pm\infty$$



Ex 1 $f(x) = \dfrac{x^2 + 2x - 3}{x^2 - 4} = \dfrac{(x+3)(x-1)}{(x-2)(x+2)}$

vertical asymptotes are $x = 2$, $x = -2$

Ex 2 $f(x) = \dfrac{x^2 + 2x - 3}{x^2 - 1} = \dfrac{(x+3)(x-1)}{(x-1)(x+1)} = \dfrac{x+3}{x+1}$

Only one vertical asymptote, $x = -1$

$x = 1$ is not a vertical asymptote (because the $\lim f(x) = 2 \neq \infty$

Ex 3 $f(x) = \frac{\sin x}{x}$ does not have asymptote at $x = 0$

since $\lim_{x \to 0} \frac{\sin x}{x} = 1$

Ex 1 Sketch the graph of $f(x) = \frac{x+1}{x-1}$

Using Asymptotes & limits.

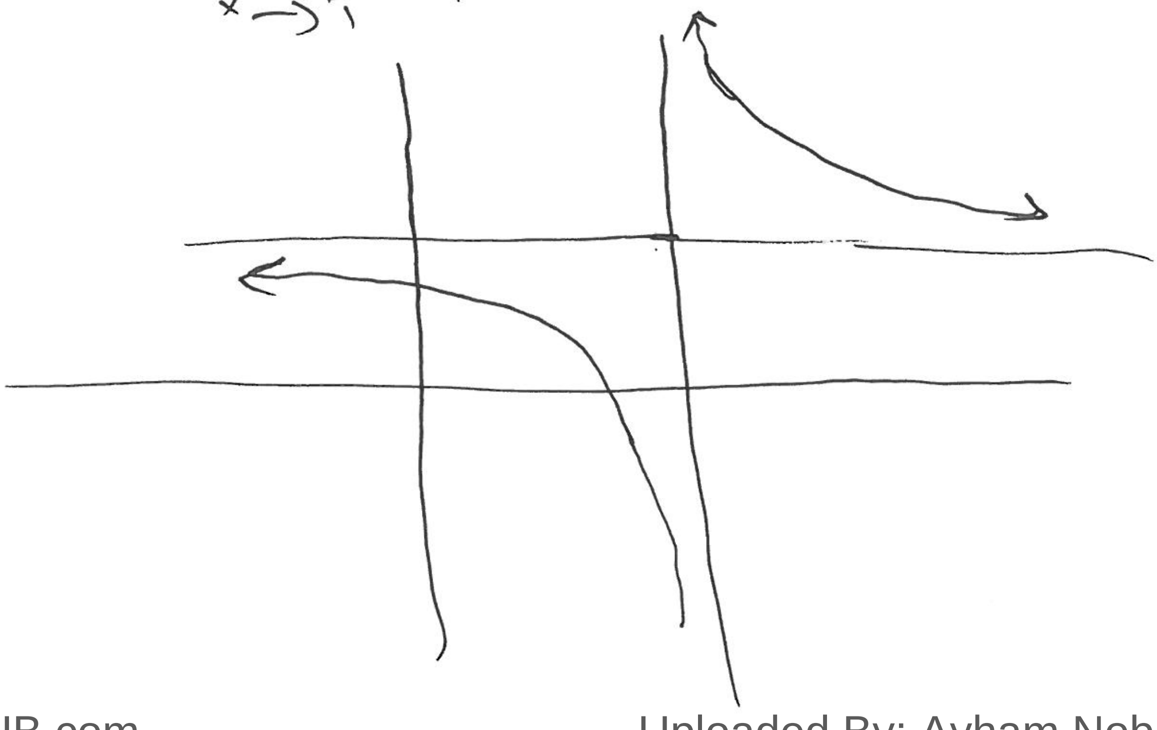
Horizontal asymptote

$\lim_{x \to \infty} f(x) = 1$

Vertical asymptote

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \frac{x+1}{x-1} = +\infty$

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \frac{x+1}{x-1} = -\infty$



3

Continuity!

Def: f is continous at $x=a$ iff

1. $f(a)$ exists
2. $\lim_{x\to a} f(x)$ exists
3. $\lim_{x\to a} f(x) = f(a)$

i.e. $\boxed{\lim_{x\to a} f(x) = f(a)}$

Ex 1 the functions $\sin x$, $\cos x$, $|x|$, e^x, and all polynomials are continous

Ex 2 The rational functions are continous at all points except at the zero's of the denominator.

Ex 1 $f(x) = \dfrac{x^2+x+1}{x^2-1} = \dfrac{x^2+x+1}{(x-1)(x+1)}$

is continous on $(-\infty,\infty)\setminus\{-1,1\}$

Ex 2 $f(x) = \dfrac{x^2+2x-3}{x^2-4} = \dfrac{(x-1)(x+3)}{(x-2)(x+2)}$

is cont. except at $x=2,-2$

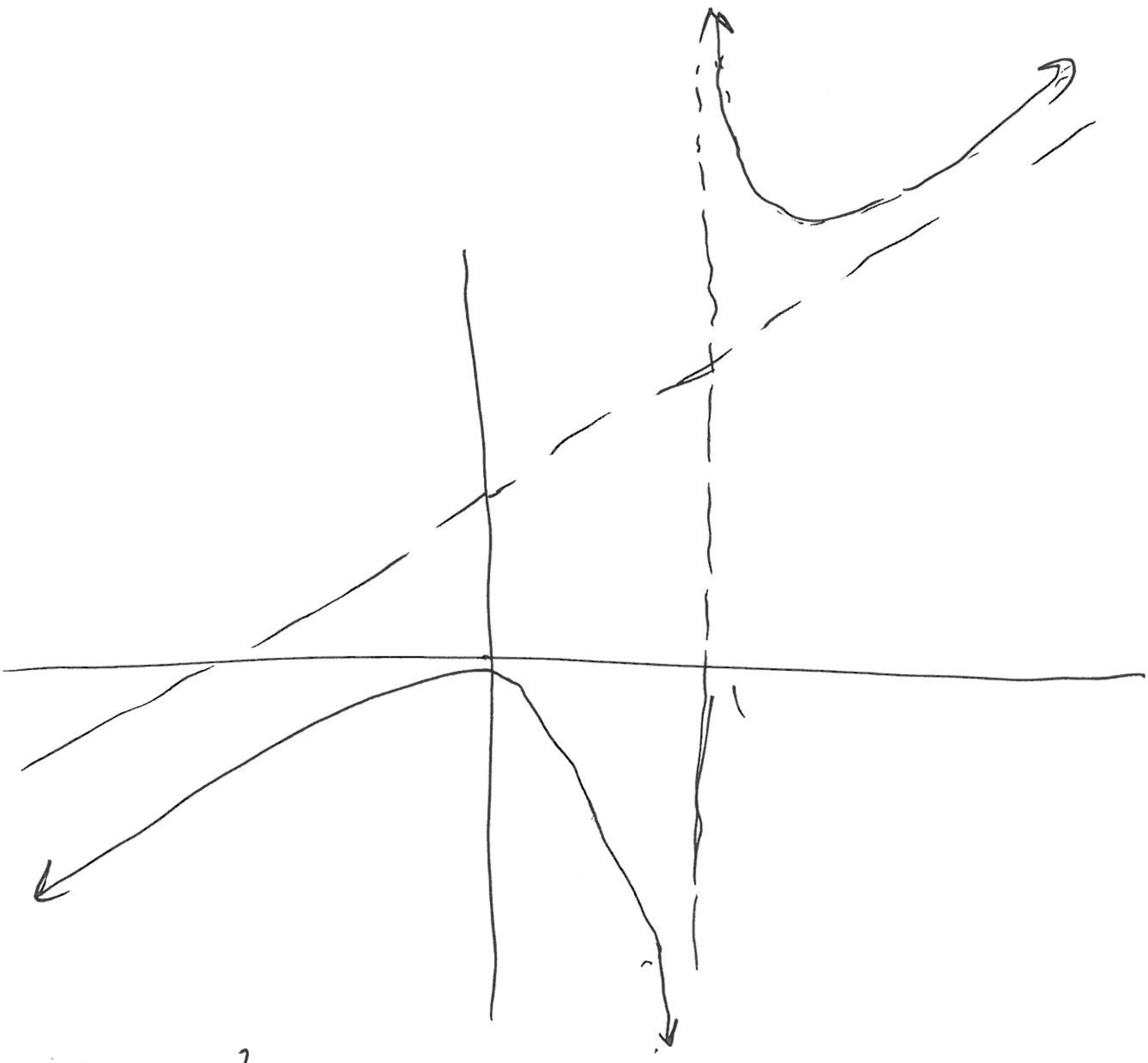
Ex3 $f(x) = \dfrac{x^2}{x-1}$

$$\begin{array}{r|l} & x+1 \\ \hline x-1 & x^2 \\ & \mp x^2 \pm x \\ \hline & x \\ & \mp x \pm 1 \\ \hline & 1 \end{array}$$

$= \underbrace{(x+1)} + \dfrac{1}{x-1}$

The line $y = x+1$ is called oblique asymptote

& $x = 1$ is a vertical asymptote



Ex4 $y = \dfrac{x^2-1}{x-2}$