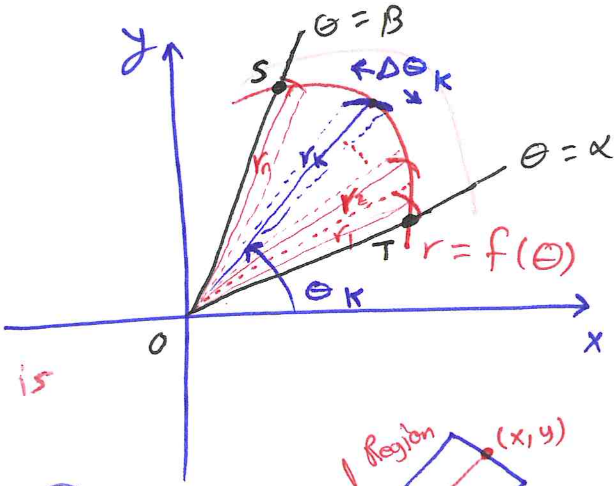
11.5 Areas and lengths in Polar Coordinates

- The region OTS is bounded by the rays $\theta = \alpha$ and $\theta = \beta$ and the curve $r = f(\theta)$.



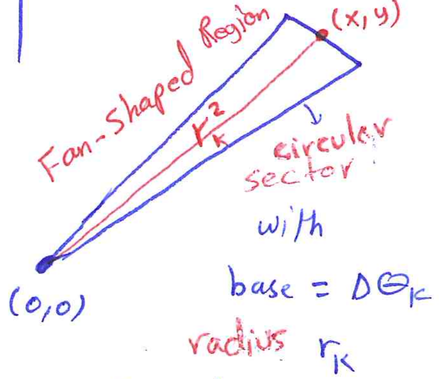
- Area of the circular sector k is

$$A_k = \frac{1}{2} r_k^2 \, \Delta\theta_k$$

$$= \frac{1}{2}\left[f(\theta_k)\right]^2 \Delta\theta_k$$

$A = \pi r^2 = \pi r^2 \frac{\theta}{2\pi}$

$= \frac{r^2 \theta}{2}$



- The area of the region can be approximated by

$$\tilde{A} = \sum_{k=1}^{n} A_k = \sum_{k=1}^{n} \frac{1}{2}\left[f(\theta_k)\right]^2 \Delta\theta_k$$

- The approximation is improved as $\|P\| \to 0$ "$n \to \infty$"
- $$\boxed{A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta} = \lim_{\|P\| \to 0} \tilde{A}$$

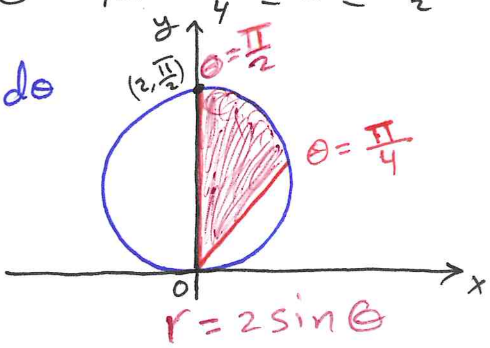
Exp Find the areas of the region

$\equiv x^2 + (y-1)^2 = 1$

1. bounded by the circle $r = 2\sin\theta$ for $\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$

$$A = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{2}\left[2\sin\theta\right]^2 d\theta = 2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\frac{1-\cos 2\theta}{2}\right) d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (1 - \cos 2\theta)\, d\theta = \frac{\pi}{4} + \frac{1}{2}$$

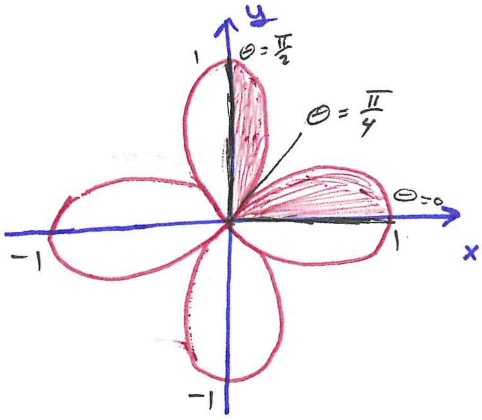


[2] Inside one leaf of the four-leaved rose $r = \cos 2\theta$

$$A = 2\int_0^{\frac{\pi}{4}} \frac{1}{2}\left[\cos 2\theta\right]^2 d\theta$$

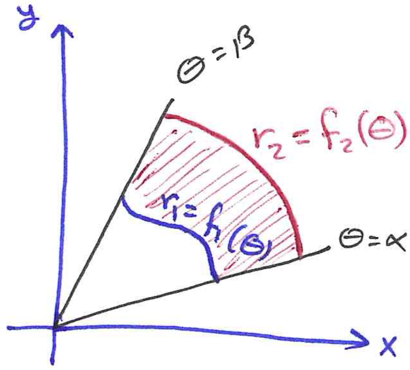
$$= \int_0^{\frac{\pi}{4}} \frac{1+\cos 4\theta}{2}\, d\theta = \frac{\pi}{8}$$

$$= 2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{2}\left[\cos 2\theta\right]^2 d\theta$$



$$A = \int_\alpha^\beta \frac{1}{2} r_2^2\, d\theta - \int_\alpha^\beta \frac{1}{2} r_1^2\, d\theta$$

$$= \int_\alpha^\beta \frac{1}{2}\left(r_2^2 - r_1^2\right) d\theta$$



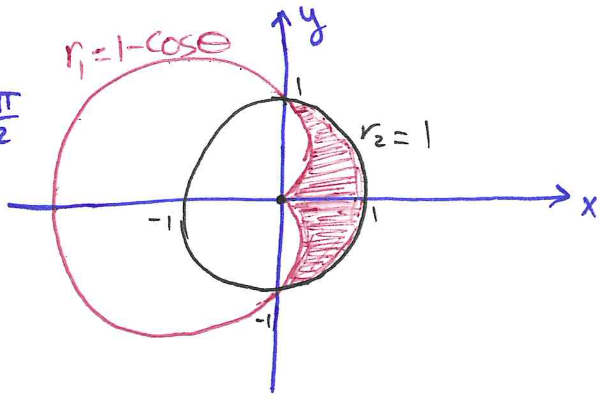
Exp Find the area that lies inside the circle $r=1$ and outside the cardioid $r = 1-\cos\theta$

$r = 1 = 1-\cos\theta \iff \cos\theta = 0 \iff \theta = \frac{\pi}{2}, -\frac{\pi}{2}$

$$A = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}\left(r_2^2 - r_1^2\right) d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}\left[1^2 - (1-\cos\theta)^2\right] d\theta = 2 - \frac{\pi}{4}$$

$$= 2\int_0^{\frac{\pi}{2}} \frac{1}{2}\left[1^2 - (1-\cos\theta)^2\right] d\theta = \int_0^{\frac{\pi}{2}} \left[1 - (1-\cos\theta)^2\right] d\theta = 2 - \frac{\pi}{4}$$



Length of a Polar Curve

- Recall that $x = r\cos\theta = f(\theta)\cos\theta$
  $y = r\sin\theta = f(\theta)\sin\theta$ $\quad \alpha \le \theta \le \beta$

are parametrization for the curve $r = f(\theta)$.

- The length of r is $L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$

$$\frac{dx}{d\theta} = -r\sin\theta + \cos\theta \frac{dr}{d\theta}$$

$$\frac{dy}{d\theta} = r\cos\theta + \sin\theta \frac{dr}{d\theta}$$

$$L = \int_{\alpha}^{\beta} \sqrt{r^2\sin^2\theta + \cos^2\theta\left(\frac{dr}{d\theta}\right)^2 - \cancel{2r\sin\theta\cos\theta\frac{dr}{d\theta}} + r^2\cos^2\theta + \sin^2\theta\left(\frac{dr}{d\theta}\right)^2 + \cancel{2r\sin\theta\cos\theta\frac{dr}{d\theta}}}$$

$$\boxed{L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta}$$

Exp Find the length of the curve given by the spiral

$r = \frac{e^{\theta}}{\sqrt{2}}$, $0 \le \theta \le \pi$

$$L = \int_0^{\pi} \sqrt{\frac{e^{2\theta}}{2} + \frac{e^{2\theta}}{2}}\, d\theta = \int_0^{\pi} \sqrt{e^{2\theta}}\, d\theta = \int_0^{\pi} e^{\theta}\, d\theta$$

$$= e^{\pi} - 1$$