Chapter 8: External Direct Products.

Def : External Direct product.

let $G_1, G_2, \dots G_n$ be a finite collection of groups. The external direct product of $G_1, G_2, \dots G_n$ written as $G_1 \oplus G_2 \oplus \dots G_n$, is the set of all n-tuples for which the ith component is an element of G_i and the operation is componentwise.

(اشرحه بالكتاب).

exp1 :

بطلع 16 عناصر

$U(8) \oplus U(10) =$ $\quad U(8) = \{1, 3, 5, 7\}$, $\quad U(10) = \{1, 3, 7, 9\}$. بضربهم مع بعض

$= \{(1,1), (1,3), (1,7), (1,9), (3,1), (3,3), (3,7), (3,9),$
$(5,1), (5,3), (5,7), (5,9), (7,1), (7,3), (7,7), (7,9)\}$

exp2 :

$\mathbb{Z}_2 \oplus \mathbb{Z}_3 = \{0,1\} \oplus \{0,1,2\}$

$= \{(0,0), (0,1), (0,2), (1,0), \underline{(1,1)}, (1,2)\}$

$(1,1)$ عشان أشوف اذا cyclic

$(1,1)^1 = (1,1)$.

$(1,1)^2 = ((1+1)_2, (1+1)_3) = (0,2)$

$(1,1)^3 = (1,0)$

$(1,1)^4 = (0,1)$

$(1,1)^5 = (1,2)$

$(1,1)^6 = \underline{(0,0)}$ identity

So its cyclic.

→ $\mathbb{Z}_2 \oplus \mathbb{Z}_3$ the order equal 6, $\boxed{\text{is cyclic ??}}$ yes

→ $(\mathbb{Z}_6, \oplus)$ the order equal 6 and cyclic.

$\Rightarrow$ So $(\mathbb{Z}_2 \oplus \mathbb{Z}_3)$ isomorphic to $(\mathbb{Z}_6, \oplus)$.

But since order of $(S_3, \otimes) = 6$

$(\mathbb{Z}_2 \oplus \mathbb{Z}_3)$ Not isomorphic to $(S_3, \otimes)$.

→ The groups of order 6 up isomorphisim = 2

2 : ① $(\mathbb{Z}_6, +) = (\mathbb{Z}_2 \oplus \mathbb{Z}_3)$

② $(S_3, \circ)$.

exp3:

$\mathbb{Z}_2 \oplus \mathbb{Z}_2 = \{(0,0), (0,1), (1,0), (1,1)\}$.

↳ Not cyclic: $(0,1)^1 = (0,1)$

$(0,1)^2 = (0,0)$

$|(0,1)| = 2$.

← أي جروب order الـ 4 ح يكون isomorphic لـ $(\mathbb{Z}_4, \oplus)$ و $(\mathbb{Z}_2 \oplus \mathbb{Z}_2)$.

Thm 8.1: order of an element in a Direct product.

The order of an element in a direct product of a finite number of finite groups is the least common multiple of the orders of the components of the element. In symbols: $|(g_1, g_2, \ldots g_n)| = LCM(|g_1|, |g_2|, \ldots |g_n|)$.

exp: $G = \mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_5 = \{$ .... فيه 40 واحد $\}$

$|G| = 4 \times 2 \times 5 = 40$

→ let for exp $|(2,1,3)| = LCM(|2|_{\text{on } \mathbb{Z}_4}, |1|_{\text{on } \mathbb{Z}_2}, |3|_{\text{on } \mathbb{Z}_5})$

$= LCM(2, 2, 5)$

$|(2,1,3)| = 10$

exp 4: number of element of order 5 in $\mathbb{Z}_{25} \oplus \mathbb{Z}_5$

$|(a,b)| = 5$

case 1 : $|a| = 1$ and $|b| = 1$ or 5.

$|a|$ on $\mathbb{Z}_{25}$:   كم element في $\mathbb{Z}_{25}$ الها order 5

a : 5, 10, 15, 20

b : 0, 1, 2, 3, 4   كم element في $\mathbb{Z}_5$ الها order 1 أو 5

5 element on a and 4 element on b $= 4 \times 5 = 20$.

case 2 : $|a| = 1$ and $|b| = 5$

a : 0
b : 1, 2, 3, 4 $\rightarrow 1 \times 4 = 4$

Thus, $\mathbb{Z}_{25} \oplus \mathbb{Z}_5$ has 24 element of order 5.

24 element of order 5 من أصل $25 \times 5 = 125$ element

↲ كم cyclic subgroup في $(\mathbb{Z}_{25} \oplus \mathbb{Z}_5)$ الها order يساوي 5 ؟؟

6 groups

every cyclic group of order 5 contains 4 elements of order 5. (24\4)

exp:

H: e, a, b, c, d

prime number

the group H have $\underline{5}$ elements

$|H| = \underline{5}$ so H is cyclic since 5 is prime

on H any $x \neq e \Rightarrow |x| \setminus |H|$

$|x| \setminus 5$ diviser of 5 → 1, 5

$|x| = 5$ $x \neq e$ → order $\neq 1$

so $|a| = |b| = |c| = |d| = 5$.

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exp 5+6.

on 5: since $|H| = 10$, $H = \langle a \rangle$

so the generators: a^1, a^3, a^7, a^9. ارجع للاطار :

Thm 8.2: Criterion for $G \oplus H$ to be cyclic:

let G and H be finite cyclic groups. Then $G \oplus H$ is cyclic iff $|G|$ and $|H|$ are relatively prime.

exp: $\mathbb{Z}_5 \oplus \mathbb{Z}_7$

$|\mathbb{Z}_5| = 5$, $|\mathbb{Z}_7| = 7$ and Both are cyclic.

→ $|\mathbb{Z}_5 \oplus \mathbb{Z}_7| = 35 \Rightarrow (\mathbb{Z}_5 \oplus \mathbb{Z}_7)$ is cyclic of order 35.

→ $\mathbb{Z}_5 \oplus \mathbb{Z}_7 = \langle (3,3) \rangle, \langle (1,1) \rangle, \langle (2,3) \rangle, \ldots$

→ Relatively prime when there are No common factors other than 1.

$\rightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_3 \simeq \mathbb{Z}_6$ and cyclic.

$\rightarrow \mathbb{Z}_3 \oplus \mathbb{Z}_4 \simeq \mathbb{Z}_{12}$ and cyclic.

Corollary 1: criterion for $G_1 \oplus G_2 \oplus \dots \oplus G_n$ to be cyclic.

An external direct product $G_1 \oplus G_2 \oplus G_3 \oplus \dots \oplus G_n$ of a finite number of finite cyclic groups is cyclic iff $|G_i|$ and $|G_j|$ are relatively prime when $i \neq j$.

$\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5 \overset{\text{isomorphism}}{\simeq} \mathbb{Z}_{30}$ and cyclic.

$(1,1,1)$ its the generator.

$\phi: (1,1,1) \rightarrow 1$ the isomorphisin.

$+(1,1,1)$ $(0,2,2) \rightarrow 2$

$+(1,1,1)$ $(1,0,3) \rightarrow 3$

$\vdots$ $\vdots$

Corollary 2: criterion for $\mathbb{Z}_{n_1 n_2 \dots n_k} \simeq \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \dots \oplus \mathbb{Z}_{n_k}$.

let $m = n_1 n_2 \dots n_k$. Then $\mathbb{Z}_m$ is isomorphic to $\mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \dots \oplus \mathbb{Z}_{n_k}$ iff n_i and n_j are relatively prime when $i \neq j$.

$\mathbb{Z}_3 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_7 \oplus \mathbb{Z}_{11} = \mathbb{Z}_{3.5.7.11}$.

Dalle.