10.2 Part 2

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* The Binomial series of $f(x) = (1+x)^m$ is "using Taylor series exp."

$$(1+x)^m = 1 + \sum_{k=1}^{\infty} \binom{m}{k} x^k \quad , \quad |x| < 1 \quad , \quad m \text{ is constant}$$

x^4 $\quad$ x^4

where the series converges absolutely.

where $\binom{m}{1} = m$, $\binom{m}{2} = \frac{m(m-1)}{2}$

$$\binom{m}{k} = \frac{m(m-1)(m-2)\cdots(m-k+1)}{k!} \quad \text{for } k \geq 3$$

Approximating Nonelementary Integrals

Exp Use series to estimate the following integrals with an error of magnitude less than 10^{-3}

$$\boxed{1} \int_0^{0.1} \frac{dx}{\sqrt{1+x^4}} = \int_0^{0.1} (1+x^4)^{-\frac{1}{2}} dx \qquad m = -\frac{1}{2}$$

$$= \int_0^{0.1} \left[1 + \sum_{k=1}^{\infty} \binom{-\frac{1}{2}}{k} (x^4)^k \right] dx$$

$$= \int_0^{0.1} \left[1 + \binom{-\frac{1}{2}}{1} x^4 + \binom{-\frac{1}{2}}{2} x^8 + \binom{-\frac{1}{2}}{3} x^{12} + \cdots \right] dx$$

$\binom{-\frac{1}{2}}{1} = -\frac{1}{2}$

$\binom{-\frac{1}{2}}{2} = \frac{-\frac{1}{2}(-\frac{1}{2}-1)}{2} = \frac{-\frac{1}{2}(-\frac{3}{2})}{2} = \frac{3}{8}$

$\binom{-\frac{1}{2}}{3} = \frac{-\frac{1}{2}(-\frac{1}{2}-1)(-\frac{1}{2}-2)}{3!} = \frac{(-\frac{1}{2})(-\cancel{\frac{3}{2}})(-\frac{5}{2})}{\cancel{3} \times 2} = -\frac{5}{16}$

$$= \int_0^{0.1} \left[1 - \frac{x^4}{2} + \frac{3}{8} x^8 + \cdots \right] dx$$

$$= x - \frac{x^5}{10} + \frac{3x^9}{(8)(9)} + \cdots \Big|_{x=0}^{x=0.1} = \left. \boxed{x \; - \frac{x^5}{10}} + \frac{3x^9}{(8)(9)} + \cdots \right|_{x=0.1}$$

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$$\boxed{\int_0^{0.1} \frac{dx}{\sqrt{1+x^4}} \approx x \Big|_{x=0.1} = 0.1}$$

with error $|E| \leq \left| \frac{x^5}{10} \right| \approx \frac{(0.1)^5}{10} \approx 0.000001 < 10^{-3}$

less

$$\int_0^{0.1} \frac{dx}{\sqrt{1+x^4}} \approx x - \frac{x^5}{10} \Big|_{x=0.1}$$

with $|E| < \left| \frac{3x^9}{(8)(9)} \right|_{x=0.1} \approx \frac{3(0.1)^9}{(8)(9)}$

$$\boxed{2} \quad \int_0^{0.2} \frac{e^{-x} - 1}{x} \, dx$$

Maclurin Series $\boxed{x=0}$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^{-x} = 1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \cdots$$

$$= \int_0^{0.2} \frac{1}{x} \left[\cancel{1} - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots \cancel{-1} \right] dx$$

$$= \int_0^{0.2} \frac{-x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots}{x} \, dx$$

$$= \int_0^{0.2} \left(-1 + \frac{x}{2} - \frac{x^2}{6} + \cdots \right) dx$$

$$= \int_0 \left(-1 + \frac{x}{2} - \frac{x}{6} + \cdots\right) dx$$

$$= -x + \frac{x^2}{4} - \frac{x^3}{18} + \frac{x^4}{4(4!)} + \cdots \Bigg|_{x=0}^{x=0.2}$$

$$= -x + \frac{x^2}{4} - \frac{x^3}{18} + \frac{x^4}{4(4!)} + \cdots \Bigg|$$

$$\int_0^{0.2} \frac{e^{-x}-1}{x}\,dx \approx -x\Big|_{x=0}^{x=0.2} \text{ with } E \le \left|\frac{x^2}{4}\right|_{x=0.2} = \frac{(0.2)^2}{4} = \frac{0.04}{4} = 0.01 > 10^{-3}$$

$$\int_0^{0.2} \frac{e^{-x}-1}{x}\,dx \approx -x + \frac{x^2}{4}\Bigg|_{x=0.2} \approx -0.2 + 0.01 = -0.19 \text{ with error}$$

$$E \le \left|\frac{x^3}{18}\right|_{x=0.2} \approx \frac{(0.2)^3}{18} = \frac{0.008}{18} \approx 0.0004 < 0.001$$

$$\int_0^{0.2} \frac{e^{-x}-1}{x}\,dx \approx -x + \frac{x^2}{4} - \frac{x^3}{18}\Bigg|_{x=0.2} \approx -0.19044 \text{ with}$$

$$\text{Error} \le \left|\frac{x^4}{4(4!)}\right|_{x=0.2} \approx 0.00002 \ll 10^{-3}$$

Indeterminate forms

Exp Use Series to find this limit

1) $\lim_{x \to 1} \frac{\ln x}{x-1} = \lim_{x \to 1} \frac{(x-1) - \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} - \cdots}{x-1}$

$\frac{\ln 1}{1-1} = \frac{0}{0}$

$\lim_{x \to 1} \frac{\frac{1}{x}}{1}$

$\lim_{x \to 1} \frac{1}{x} = 1$

$= \lim_{x \to 1} \left(1 - \cancel{\frac{x-1}{2}} + \cancel{\frac{(x-1)^2}{6}} - \cancel{\cdots} \right)$

$= 1 - 0 + 0 \ \cdots$

$= 1$

2) $\lim_{x \to 0} \frac{e^x - (1+x)}{x^2}$

$\frac{e^0 - (1+0)}{0^2} = \frac{1-1}{0} = \frac{0}{0}$ / $\lim_{x \to 0} \frac{e^x - 1}{2x}$ $\frac{0}{0}$

$\lim_{x \to 0} \frac{e^x}{2} = \frac{e^0}{2} = \frac{1}{2}$

$\lim_{x \to 0} \frac{\cancel{1} + \cancel{x} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \cancel{-1} - \cancel{x}}{x^2} = \lim_{x \to 0} \frac{\frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{4!} + \cdots}{x^2}$

$= \lim_{x \to 0} \left(\frac{1}{2} + \cancel{\frac{x}{6}}^{\,0} + \cancel{\frac{x^2}{4!}}^{\,0} + \cancel{\cdots} \right) = \frac{1}{2}$

3) $\lim_{y \to 0} \frac{y - \tan^{-1} y}{y^3}$

Maclrin series $f(x) = \tan^{-1} x$

$f' = \frac{1}{1+x^2}$

$f'' = \cdots$

$y \to 0$ $\quad y$

$f''= \cdots$

$f' = \cdots$

$x=0$

$$f(x) = \tan^{-1} x = \int f'(x)\,dx = \int \frac{1}{1+x^2}\,dx$$

$$= \int \left[1 - x^2 + x^4 - x^6 + \cdots \right] dx$$

a

$a = 1$
$r = -x^2$

$$\frac{a}{1-r} = \frac{1}{1--x^2} = \frac{1}{1+x^2}$$

$$= \boxed{x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots}$$

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$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7!} + \cdots$$

$$\lim_{y \to 0} \frac{y - \tan^{-1} y}{y^3}$$

$$\frac{0-0}{0} = \frac{0}{0} \quad \cdots$$

$$\lim_{y \to 0} \frac{\cancel{y} - \left(\cancel{y} - \frac{y^3}{3} + \frac{y^5}{5} - \frac{y^7}{7} + \cdots\right)}{y^3}$$

$$\lim_{y \to 0} \frac{\frac{y^3}{3} - \frac{y^5}{5} + \frac{y^7}{7} - \cdots}{y^3}$$

$$\lim_{y\to 0} \frac{\quad}{y^3}$$

$$\lim_{y\to 0}\left(\frac{1}{3}-\cancel{\frac{y^2}{5}}+\cancel{\frac{y^4}{7}}+\cancel{\cdots}\right)=\frac{1}{3}$$

Euler formula

$i=\sqrt{-1}$

$i^2=-1$

$i^3=i\,i^2=-i$

$i^4=i^2\,i^2=(-1)(-1)=1$

$i^5=i\,i^4=i$

$\vdots$

MS

$$\boxed{e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots}$$

$$e^{i\theta}=1+i\theta+\frac{(i\theta)^2}{2!}+\frac{(i\theta)^3}{3!}+\frac{(i\theta)^4}{4!}+\frac{(i\theta)^5}{5!}+\frac{(i\theta)^6}{6!}+\cdots$$

$$=1+i\theta-\frac{\theta^2}{2!}-\frac{\theta^3 i}{3!}+\frac{\theta^4}{4!}+\frac{\theta^5 i}{5!}+\cdots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)$$

$$\boxed{e^{i\theta} = \cos\theta + i \sin\theta}$$ Euler's formula

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$a + bi$

$$e^{i\pi} = -1 \Rightarrow \boxed{e^{i\pi} + 1 = 0}$$

$$e^{i\pi} = \cos\pi + i \cancel{\sin\pi}^{\,0}$$

$$= -1$$

$$e^{i\frac{\pi}{4}} = \cos\frac{\pi}{4} + i \sin\frac{\pi}{4}$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i$$