CH3: Fourier Series

→ It is used to determine the spectral representation of periodic signal

3.1) Forms of Fourier series

① sinusoidal form → It is used for single-sided representation

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + \sum_{n=1}^{\infty} b_n \sin(n\omega t)$$

a_0, a_n, and b_n are real parameters (cofficients)

or $$x(t) = a_0 + \sum_{n=1}^{\infty} c_n \cos(n\omega t - \theta_n) \;;$$

where $c_n = \sqrt{a_n^2 + b_n^2}$ and $\theta_n = \tan^{-1}(b_n / a_n)$

Fundamental frequency $\omega = 2\pi f \Rightarrow f = 1/T$ → period

② complex exponential form ⇝ double-sided representation

$$x(t) = \sum_{n=-\infty}^{n=\infty} X_n e^{jn\omega t}$$

X_n is a complex parameter

$$X_n = \begin{cases} a_0 & n=0 \\ (a_n - jb_n)/2 & n>1 \\ (a_{-n} + jb_{-n})/2 & n<1 \end{cases}$$

proof :-

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + \sum_{n=1}^{\infty} b_n \sin(n\omega t)$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \left(\frac{e^{jn\omega t} + e^{-jn\omega t}}{2} \right) + \sum_{n=1}^{\infty} b_n \left(\frac{e^{jn\omega t} - e^{-jn\omega t}}{2j} \right)$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left(\frac{a_n - jb_n}{2} \right) e^{jn\omega t} + \sum_{n=1}^{\infty} \left(\frac{a_n + jb_n}{2} \right) e^{-jn\omega t}$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left(\frac{a_n - jb_n}{2} \right) e^{jn\omega t} + \sum_{-\infty}^{n=-1} \left(\frac{a_{-n} + jb_{-n}}{2} \right) e^{jn\omega t} = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega t}$$

3.2] Fourier series cofficients

① $$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + \sum_{n=1}^{\infty} b_n \sin(n\omega t)$$

$$a_0 = \frac{1}{T}\int_T x(t)\,dt$$ → DC component or average

$$a_n = \frac{2}{T}\int_T x(t)\cos(n\omega t)\,dt$$

$$b_n = \frac{2}{T}\int_T x(t)\sin(n\omega t)\,dt$$

$n=1$ ⇒ Main or Fundamental Component of $x(t)$

$n=2,3,\ldots$ ⇒ Harmonics

② $$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega t}$$

$$X_n = \begin{cases} a_0 & n = 0 \\ (a_n - jb_n)/2 & n > 1 \\ (a_{-n} + jb_{-n})/2 & n < 1 \end{cases}$$

$$X_n = \frac{1}{T}\int_T x(t)\, e^{-jn\omega t}\, dt$$

properties of FS coefficients

$$X_n = \frac{1}{T}\int_T x(t)\, e^{-jn\omega t}\, dt = |X_n| \angle \theta_n$$

$$X_n = \frac{1}{T}\int_T x(t)\cos(n\omega t)\,dt - j\,\frac{1}{T}\int_T x(t)\sin(n\omega t)\,dt$$

$$X_{-n} = \frac{1}{T}\int_T x(t)\cos(n\omega t)\,dt + j\,\frac{1}{T}\int_T x(t)\sin(n\omega t)\,dt$$

1) $X_{-n} = X_n^* \Rightarrow \begin{cases} |X_{-n}| = |X_n| & \text{even symmetry} \\ \theta_{-n} = -\theta_n & \text{odd symmetry} \end{cases}$

2) $x(t)$ is even $\Rightarrow$ X_n is real and even

3) $x(t)$ is odd $\Rightarrow$ X_n is imaginary and odd

$$a_n = \frac{2}{T}\int_T x(t)\cos(n\omega t)\,dt$$

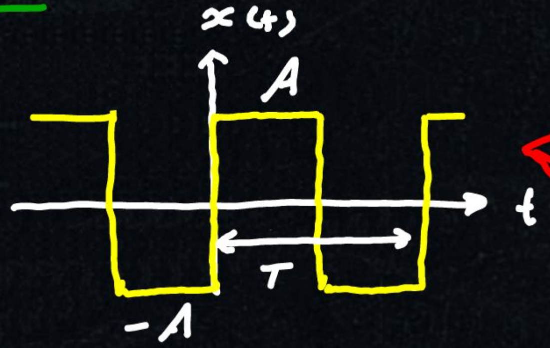
$$b_n = \frac{2}{T}\int_T x(t)\sin(n\omega t)\,dt$$

4) $x(t)$ is even $\Rightarrow a_n \neq 0$ & $b_n = 0$

5) $x(t)$ is odd $\Rightarrow a_n = 0$ & $b_n \neq 0$

6) $x(t)$ is half-wave symmerty $\Rightarrow a_n = 0$ & $b_n = 0$ for n even

EX:- Calculate the FS coefficients for $x(t)$



odd $\Rightarrow a_n = 0$

Half-wave symmetry $\Rightarrow b_n = 0$ for n even

zero average $\Rightarrow a_0 = 0$

$$b_n = \frac{2}{T}\int_0^T x(t)\sin(n\omega t)\,dt = \frac{1}{\pi}\int_0^{2\pi} x(t)\sin(n\omega t)\,d(\omega t)$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} A\sin(n\omega t)\,d(\omega t) - \frac{1}{\pi}\int_{\pi}^{2\pi} A\sin(n\omega t)\,d(\omega t)$$

$$b_n = \frac{A}{\pi}\left[\left.\frac{-\cos(n\omega t)}{n}\right|_0^{\pi} + \left.\frac{\cos(n\omega t)}{n}\right|_{\pi}^{2\pi}\right]$$

$$b_n = \frac{A}{n\pi}\left[\left(1-\cos(n\pi)\right) + \left(\overbrace{\cos(2n\pi)}^{1 \text{ for all } n} - \cos(n\pi)\right)\right]$$

$$b_n = \frac{A}{n\pi}\left[2 - 2\cos(n\pi)\right] = \frac{2A}{n\pi}\left[1-\cos(n\pi)\right]$$

$$\frac{1}{2}\left(1-\cos x\right) = \sin^2 2x$$

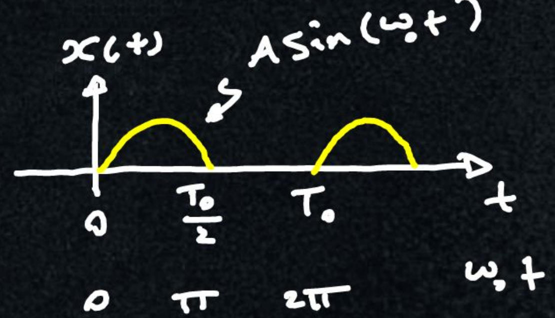
$$\therefore \quad b_n = \frac{4A}{n\pi}\sin^2(2n\pi) = \begin{cases} 0 & n \text{ is even} \\ \dfrac{4A}{n\pi} & n \text{ is odd} \end{cases}$$

$$X_n = \frac{-jb_n}{2} = -j\frac{2A}{n\pi} \quad \text{for } n \text{ is odd}$$

do it

$$\text{or} \quad X_n = \frac{1}{T}\int_T x(t)\, e^{-jn\omega t}\, dt = -j\frac{2A}{n\pi} \quad \text{for } n \text{ is odd}$$

Ex :- Determine the FS cofficients of the following periodic signal :-



$$X_n = \frac{1}{2\pi}\int_0^{\pi} A\sin(\omega_0 t)\, e^{-jn\omega_0 t}\, d(\omega_0 t)$$

$$X_n = \frac{1}{2\pi}\int_0^{\pi} \frac{A}{2j}\left[e^{j\omega_0 t} - e^{-j\omega_0 t}\right] e^{-jn\omega_0 t}\, d(\omega_0 t)$$

$$X_n = \frac{A}{4j\pi}\left[\int_0^{\pi} e^{j\omega_0 t(1-n)}\, d(\omega_0 t) - \int_0^{\pi} e^{-j\omega_0 t(1+n)}\, d(\omega_0 t)\right]$$

$$X_n = -\frac{A}{4\pi}\left[\frac{1}{(1-n)}\left(e^{j\pi(1-n)} - 1\right) + \frac{1}{(1+n)}\left(e^{-j\pi(1+n)} - 1\right)\right] \qquad n \neq \mp 1$$

$$e^{j\pi(1-n)} : \cos\pi(1-n) + j\sin\pi(1-n)$$

$$e^{-j\pi(1+n)} : \cos\pi(1+n) - j\sin\pi(1+n)$$

$$= \begin{cases} 1 & n \text{ odd} \\ -1 & n \text{ even} \end{cases}$$

$$\boxed{X_n = \frac{A}{\pi}\,\frac{1}{1-n^2} \quad \text{for } n \text{ even}}$$

$$\boxed{X_n = 0 \quad \text{for } n \text{ odd } \& \ n \neq \mp 1}$$

when $n = \mp 1$

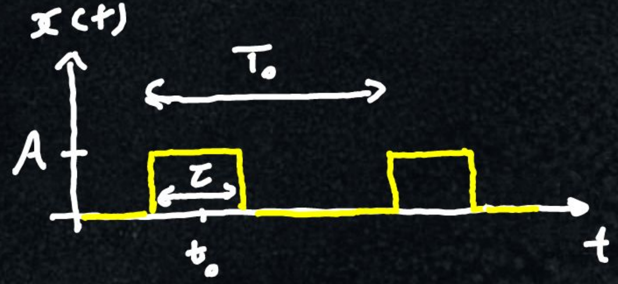
$$X_1 = \frac{A}{2\pi}\int_0^{\pi}\left(\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}\right) e^{-j\omega_0 t}\, d(\omega_0 t)$$

$$X_1 = \frac{A}{4j\pi}\int_0^{\pi}\left(1 - e^{-2j\omega_0 t}\right) d(\omega_0 t) = \frac{A}{4j}$$

$$\Rightarrow \boxed{X_1 = \frac{A}{4j}} \qquad \boxed{X_{-1} = X_1^* = \frac{-A}{4j}}$$

EX:- Determine the FS coefficients of the following signal :-

$$x(t) = \sum_{m=-\infty}^{\infty} A\,\Pi\left(\frac{t-t_0-mT_0}{\tau}\right)$$



$$X_n = \frac{1}{2\pi}\int_{\omega_0(t_0-\tau/2)}^{\omega_0(t_0+\tau/2)} A\, e^{-jn\omega_0 t}\, d(\omega_0 t)$$

$$\operatorname{sinc} x = \frac{\sin \pi x}{\pi x}$$

$$X_n = \frac{A}{2\pi}\left[\frac{1}{-jn}\, e^{-jn\omega_0(t_0+\tau/2)} - \frac{1}{-jn}\, e^{-jn\omega_0(t_0-\tau/2)}\right]$$

$$X_n = \frac{A}{n\pi}\, e^{-jn\omega_0 t_0}\left[\frac{e^{jn\omega_0\tau/2} - e^{-jn\omega_0\tau/2}}{2j}\right] = \frac{A}{n\pi}\, e^{-jn\omega_0 t_0}\, \sin(n\omega_0\tau/2)$$

$$X_n = \frac{A}{n\pi}\, e^{-jn\omega_0 t_0}\, \sin(n\pi f_0\tau) = \boxed{\frac{A\tau}{T_0}\, e^{-jn\omega_0 t_0}\, \operatorname{sinc}(nf_0\tau)} \leftarrow X_n$$