1.4) Singularity Functions

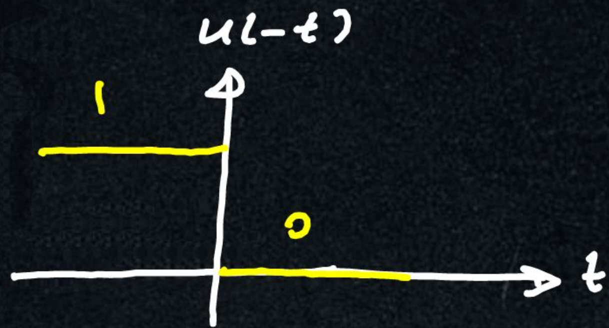
① Unit step function

$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

unit step

$$u(-t) = \begin{cases} 1 & t < 0 \\ 0 & t > 0 \end{cases}$$

reflection of u(t)



$$sgn(t) = u(t) - u(-t) = \begin{cases} 1 & t > 0 \\ -1 & t < 0 \end{cases}$$

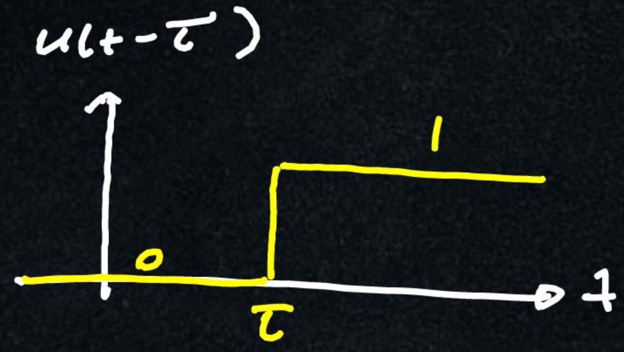
signum function

$sgn(t) = u(t) - u(-t)$

$sgn(t) = 2u(t) - 1$

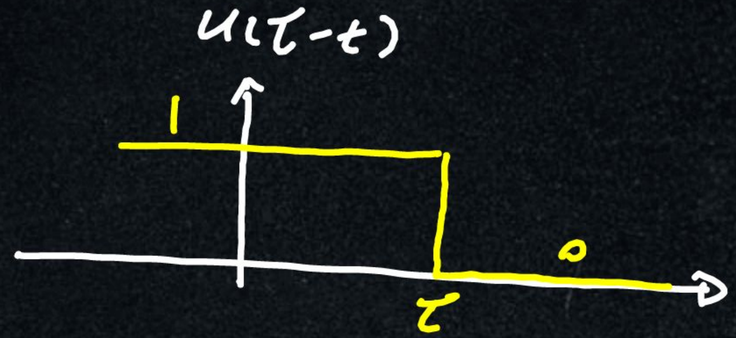
shifting operation:-

$$u(t-\tau) = \begin{cases} 1 & t > \tau \\ 0 & t < \tau \end{cases}$$



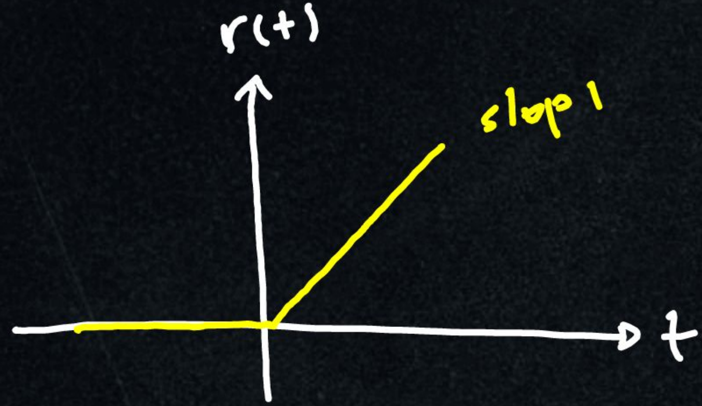
$$u(\tau-t) = \begin{cases} 1 & t < \tau \\ 0 & t > \tau \end{cases}$$

reflection of $u(t-\tau)$



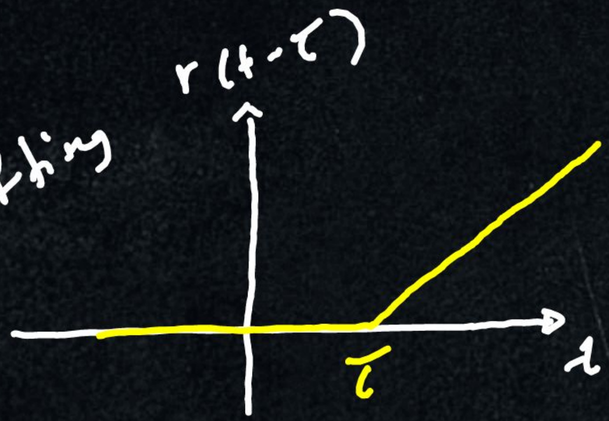
② Ramp Function

$$r(t) = \begin{cases} t & \geq 0 \\ 0 & t < 0 \end{cases}$$



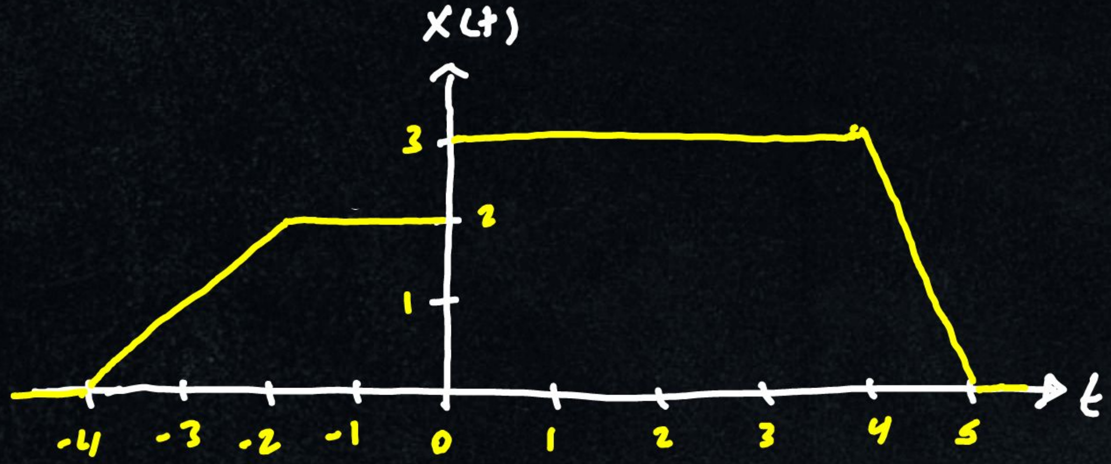
$$r(t) = \int_{-\infty}^{t} u(t')\,dt' \quad \text{or} \quad u(t) = \frac{dr(t)}{dt}$$

shifting



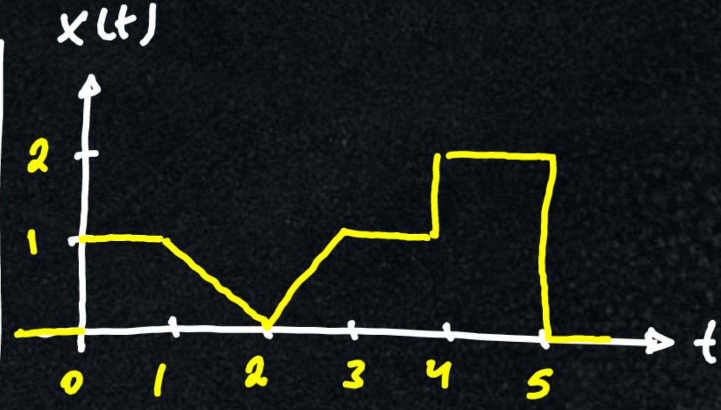
shifting with reflection

EX:- write an expression for the signals using singularity functions



$$x(t) = (1-0)\,r(t+4) + (0-1)\,r(t+2) + u(t) + (-3-0)\,r(t-4) + (0--3)\,r(t-5)$$

$$\boxed{x(t) = r(t+4) - r(t+2) + u(t) - 3r(t-4) + 3r(t-5)}$$

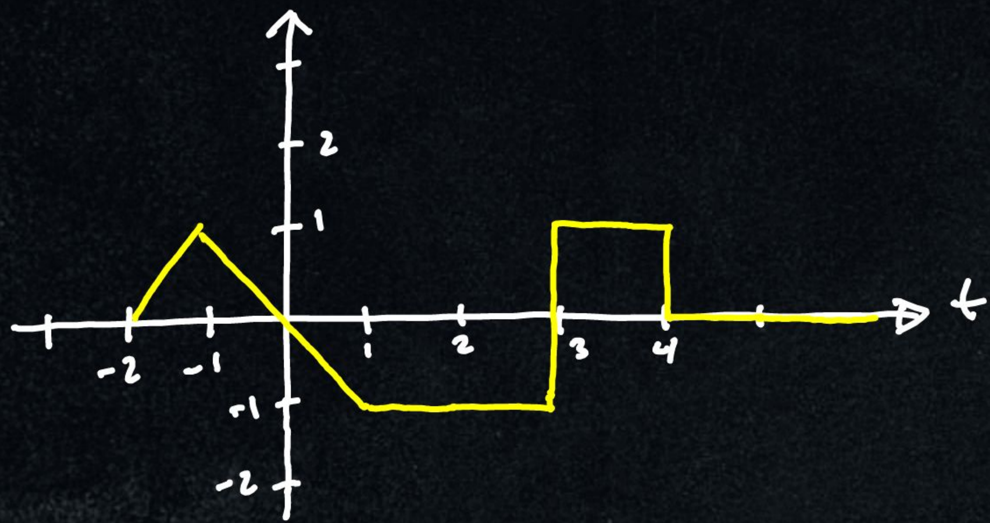


$$x(t) = u(t) + (-1-0)\,r(t-1) + (1--1)\,r(t-2) + (0-1)\,r(t-3) + u(t-4) + (0-2)\,u(t-5)$$

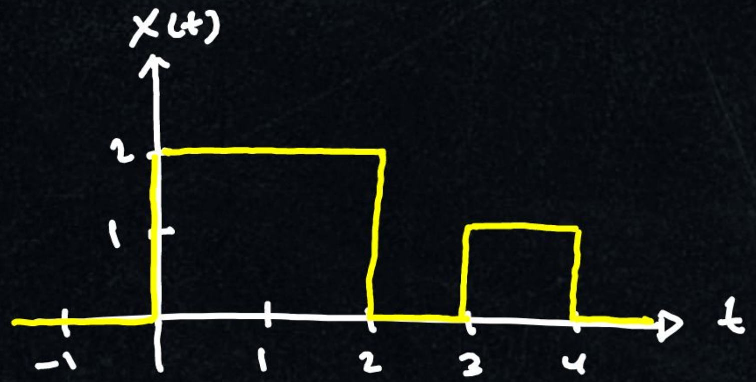
$$\boxed{x(t) = u(t) - r(t-1) + 2r(t-2) - r(t-3) + u(t-4) - 2u(t-5)}$$

EX :- Sketch the following signals :-

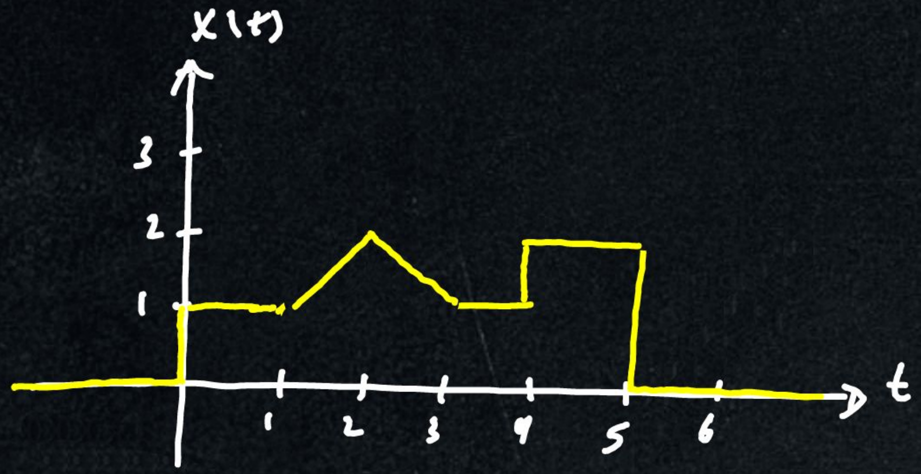
① $x(t) = r(t+2) - 2r(t+1) + r(t-1) + 2u(t-3) - u(t-4)$



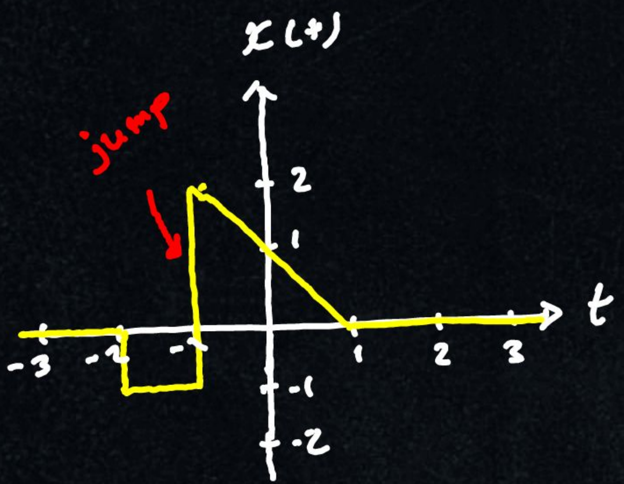
② $x(t) = 2u(t) - 2u(t-2) + u(t-3) - u(t-4)$

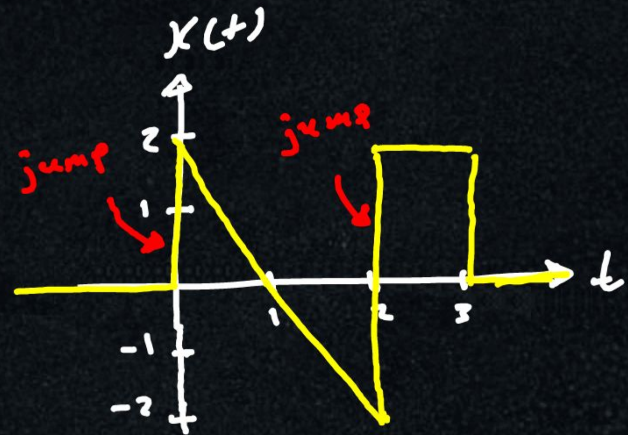


③ $x(t) = u(t) + r(t-1) - 2r(t-2) + r(t-3) + u(t-4) - 2u(t-5)$



EX:- Write an expression for the signals using singularity functions

① 

② 

① $x(t) = \left[-u(t+2)\right] + \left[3u(t+1) - r(t+1)\right] + \left[r(t-1)\right]$

due to the jump

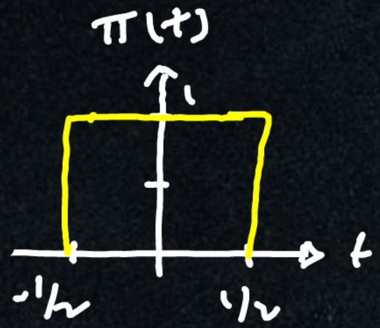
② $x(t) = \left[2u(t) - 2r(t)\right] + \left[4u(t-2) + 2r(t-2)\right] + \left[-2u(t-3)\right]$

due to the jump

due to the jump

③ Unit pulse function

$$\Pi(t) = \begin{cases} 1 & -\frac{1}{2} < t < \frac{1}{2} \\ 0 & \text{o.w} \end{cases}$$

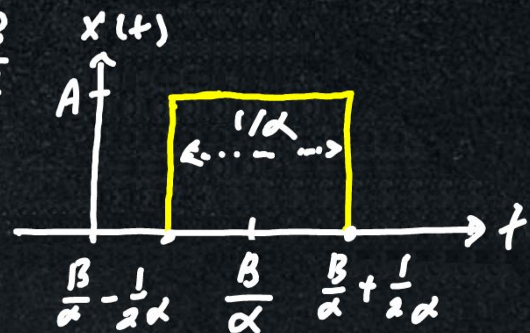


$$\Pi(t) = u(t+1/2) - u(t-1/2)$$

EX : sketch the following signal

$$x(t) = A\,\Pi(\alpha t - \beta) = A\,\Pi(\alpha(t - \beta/\alpha)) \quad \text{where } \beta > 0 \ \& \ \alpha > 0$$

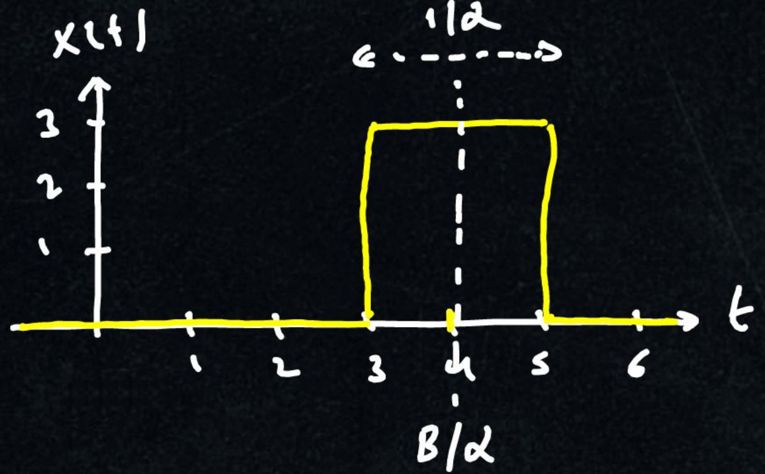
$$x(t) = \begin{cases} A & -\frac{1}{2} < \alpha t - \beta < \frac{1}{2} \\ 0 & \text{o.w} \end{cases} = \begin{cases} A & -\frac{1}{2\alpha} + \frac{\beta}{\alpha} < t < \frac{1}{2\alpha} + \frac{\beta}{\alpha} \\ 0 & \text{o.w} \end{cases}$$



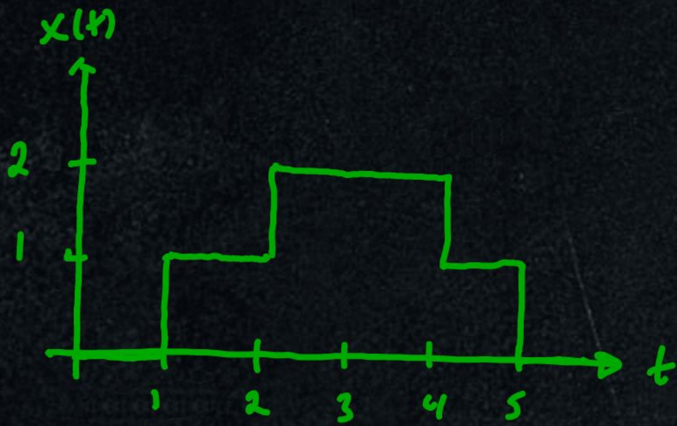
EX:- Sketch $x(t) = 3\Pi\left(\frac{1}{2}t - 2\right)$

$\alpha = \frac{1}{2}$ $\quad A = 3$

$\beta = 2$



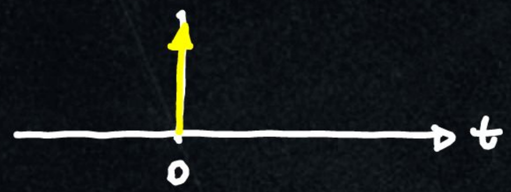
EX:- Express $x(t)$ in terms of pulse function



$x(t) = \Pi\left(\frac{1}{4}(t-3)\right) + \Pi\left(\frac{1}{2}(t-3)\right)$

④ Unit impulse function (Dirac delta function)

$$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases}$$



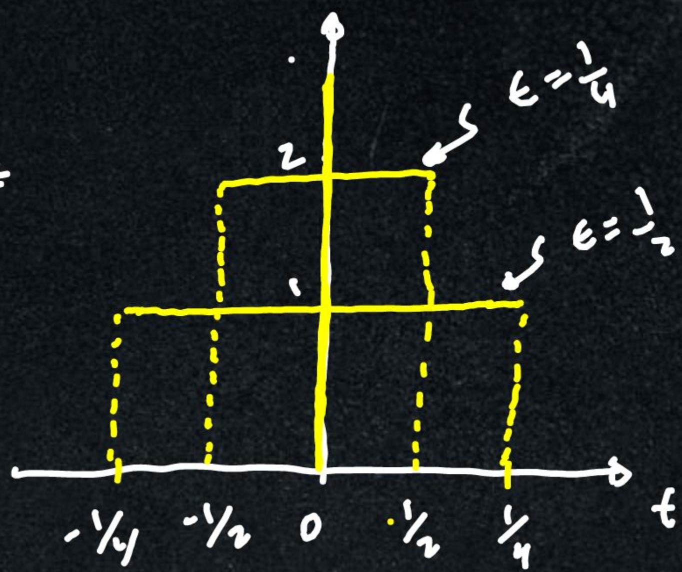
→ The area of unit impulse funⁿ is always 1

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

Test functions for $\delta(t)$:

1) $$\delta_\epsilon(t) = \frac{1}{2\epsilon}\,\Pi\left(\frac{t}{2\epsilon}\right) = \begin{cases} \frac{1}{2\epsilon} & -\epsilon < t < \epsilon \\ 0 & \text{o.w} \end{cases}$$

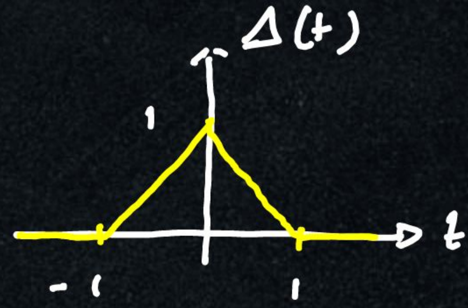
$$\int_{-\infty}^{\infty} \delta_\epsilon(t)\, dt = 1 \qquad \lim_{\epsilon \to 0} \delta_\epsilon(t) = \infty$$

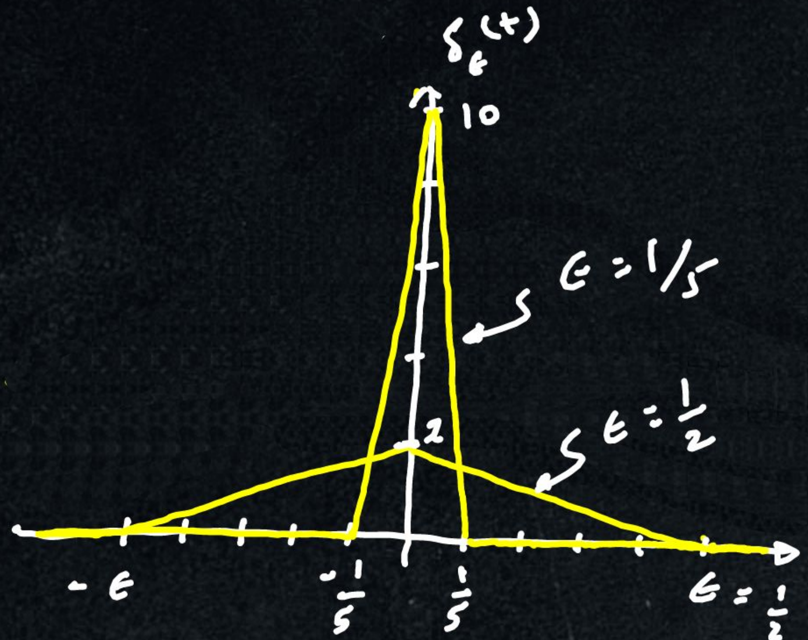


2) $\delta_\epsilon(t) = \frac{1}{\epsilon}\Delta(t/\epsilon)$

where $\Delta(t/\epsilon)$ is a unit triangular funuction

$$\Delta(t) = \begin{cases} 1-|t| & -1 \le t \le 1 \\ 0 & \text{o.w} \end{cases}$$

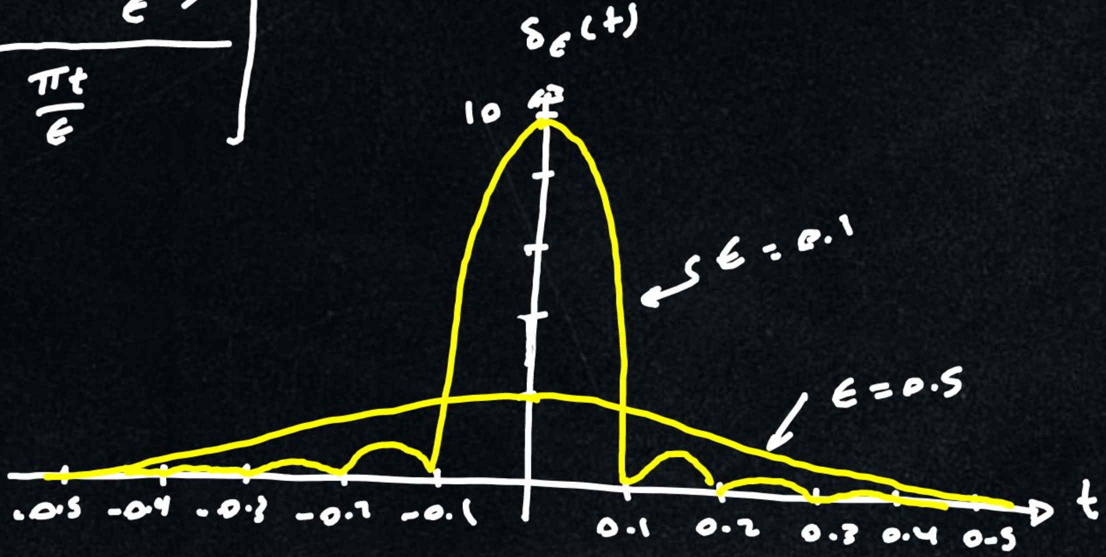




$$\int_{-\infty}^{\infty} \delta_\epsilon(t)\,dt = \frac{1}{2}(2\epsilon)\frac{1}{\epsilon} = 1$$

$$\lim_{\epsilon \to 0} \delta_\epsilon(t) = \infty$$

2) $\delta_\epsilon(t) = \frac{1}{\epsilon}\left[\frac{\sin\left(\frac{\pi t}{\epsilon}\right)}{\frac{\pi t}{\epsilon}}\right]^2$



$$\lim_{\epsilon\to 0}\delta_\epsilon(t)\Big|_{t=0} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\frac{\sin\frac{\pi t}{\epsilon}}{\frac{\pi t}{\epsilon}}\right]^2\Bigg|_{t=0} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\,\frac{\frac{\pi}{\epsilon}\cos\frac{\pi t}{\epsilon}}{\frac{\pi}{\epsilon}}\Bigg|_{t=0} = \lim_{\epsilon\to 0}\frac{1}{\epsilon} = \infty$$

L'Hopital rule

$$\int_{-\infty}^{\infty}\delta_\epsilon(t)\,dt = \int_{-\infty}^{\infty}\frac{1}{\epsilon}\,\frac{\sin^2\frac{\pi t}{\epsilon}}{\left(\frac{\pi t}{\epsilon}\right)^2}\,dt \qquad \tau = \frac{\pi t}{\epsilon} \Rightarrow dt = \frac{\epsilon}{\pi}d\tau \Rightarrow \frac{1}{\pi}\cancelto{\pi}{\int_{-\infty}^{\infty}\frac{\sin^2\tau}{\tau^2}\,d\tau} = 1$$

properties of $\delta(t)$

① **scaling property**

$$\delta(at) = \frac{1}{|a|}\delta(t)$$

② Even function

$$\delta(-t) = \delta(t$$

③ **point property**

$$x(t)\,\delta(t-t_0) = x(t_0)\,\delta(t-t_0)$$

EX : Evaluate

$$x(t) = (3t^2+2)\,\delta(-2t+6)$$

$$x(t) = (3t^2+2)\,\delta(-2(t-3))$$

$$x(t) = (3t^2+2)\,\delta(2(t-3))$$

$$x(t) = \frac{(3t^2+2)}{2}\,\delta(t-3)$$

$$\boxed{x(t) = \frac{29}{2}\,\delta(t-3}$$

4) Sampling property

$$\int_{-\infty}^{\infty} x(t)\,\delta(t-t_0)\,dt = x(t_0)$$

$$\boxed{\delta(t-t_0) = \frac{du(t-t_0)}{dt}}$$

$$\frac{d}{dt}\left[x(t)\,u(t-t_0)\right] = \frac{dx(t)}{dt}\,u(t-t_0) + x(t)\,\frac{du(t-t_0)}{dt}$$

$$\int_{-\infty}^{\infty} \frac{d}{dt}\left[x(t)\,u(t-t_0)\right] = \int_{-\infty}^{\infty} \frac{dx(t)}{dt}\,u(t-t_0) + \int_{-\infty}^{\infty} x(t)\,\delta(t-t_0)$$

$$x(t)\,u(t-t_0)\Big|_{-\infty}^{\infty} = \int_{x(t_0)}^{x(\infty)} dx(t) + \int_{-\infty}^{\infty} x(t)\,\delta(t-t_0)\,dt$$

$$x(\infty) = x(\infty) - x(t_0) + \int_{-\infty}^{\infty} x(t)\,\delta(t-t_0)\,dt \Rightarrow \int_{-\infty}^{\infty} x(t)\,\delta(t-t_0)\,dt = x(t_0)$$

EX: Eavaluate $\int_{-2}^{4} (3t^2+2)\,\delta(-2t+6)\,dt$

$$\int_{-2}^{4} (3t^2+2)\,\delta(-2(t-3))\,dt = \int_{-2}^{4} (3t^2+2)\,\delta(2(t-3))\,dt$$

$$= \int_{-2}^{4} (3t^2+2)\,\frac{1}{2}\,\delta(t-3)\,dt \quad \rightsquigarrow \quad \underline{-2 < 3 < 4}$$

$$= \frac{3(3)^2+2}{2} = \boxed{\frac{29}{2}}$$

5) Derivative of $\delta(t)$

$$\int_{t_1}^{t_2} x(t)\,\delta^{(n)}(t-t_0)\,dt = \begin{cases} (-1)^n\, x^{(n)}(t_0) & t_1 < t_0 < t_2 \\ 0 & \text{o.W} \end{cases}$$

$$\frac{d}{dt}\left[x(t)\,\delta(t-t_0)\right] = \dot{x}(t)\,\delta(t-t_0) + x(t)\,\dot{\delta}(t-t_0) = 0$$

$$\int_{t_1}^{t_2} \dot{x}(t)\,\delta(t-t_0)\,dt + \int_{t_1}^{t_2} x(t)\,\dot{\delta}(t-t_0)\,dt = 0$$

$$\int_{t_1}^{t_2} x(t)\,\delta^{(n)}(t-t_0)\,dt = (-1)^n\, x^{(n)}(t_0)$$

$$\int_{t_1}^{t_2} x(t)\,\dot{\delta}(t-t_0)\,dt = -\int_{t_1}^{t_2} \dot{x}(t)\,\delta(t-t_0)\,dt$$

$$\int_{t_1}^{t_2} x(t)\,\dot{\delta}(t-t_0)\,dt = -\dot{x}(t_0)$$

In general for higher derivatives (n^{th} derivatives)

EX :- Eavaluate $\int_{-2}^{3} (3t^2+3t)\,\dot{\delta}(-2t+2)\,dt$

$$\int_{-2}^{3} (3t^2+3t)\,\dot{\delta}(-2(t-1))\,dt = \int_{-2}^{3} \frac{3t^2+3t}{2}\,\dot{\delta}(t-1)$$

$$= (-1)^1 \left(\frac{6t+3}{2}\right)\Big|_{t=1} = \boxed{\frac{-9}{2}}$$

EX :- Evaluate $\int_{-2}^{2} \cos(20\pi t)\,\dot{\delta}(t+1)\,dt$

$$= (-1)^1 \left(-20\pi \sin(20\pi t)\right)\Big|_{t=-1} = \boxed{0}$$