Exercises :

3.1.0 : True or False.

a. For each $n \in \mathbb{N}$ the function $(x-a)^n \sin\left(f(x)\,(x-a)^{-n}\right)$ has a limit as $x \to a$?

True, proof.

since $|x^n \sin(x^{-n})| \leq |x|^n$ and $|x|^n \to 0$ as $x \to 0$

By squeeZ Theorem : $x^n \sin(x^{-n}) \to 0$ as $x \to 0$.

---

b. suppose that $\{x_n\}$ is a sequence converging to a with $x_n \neq a$. If $f(x_n) \to L$ as $n \to \infty$ then $f(x) \to L$ as $x \to a$. False.

عطاناه بالشرح

$$f(x) = \begin{cases} \sin \frac{1}{x} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$$

---

c. If f and g are finite valued on the open interval $(a-1, a+1)$ and $f(x) \to 0$ as $x \to a$, then $f(x)\,g(x) \to 0$ as $x \to a$? False.

let $a = 0$, $f(x) = x$ and $g(x) = \frac{1}{x^2}$, $x \neq 0$ and $g(0) = 0$ Then

for $x \neq 0$ we have $f(x)\,g(x) = \frac{1}{x}$ which has No limite as $x \to 0$.

---

d. If $\lim_{x \to a} f(x)$ DNE and $f(x) \leq g(x)$ $\forall x$, then $\lim_{x \to a} g(x)$ DNE ? False

$f(x) = \sin\left(\frac{1}{x}\right)$ and $g(x) = 1$

$\lim_{x \to 0} f(x) = DNE$ But $\lim_{x \to 1} g(x) = 1$.

3.1.1 : use Def. prove that each of the following limits exist.

a. $\lim_{x \to 2} \underbrace{x^2 + 2x - 5}_{f(x)} = 3$ L

$0 < \delta \leq 1 \Rightarrow |x-2| < \delta$

$|x-2| < 1$

$-1 < x-2 < 1$
$+2 \quad +2 \quad +2$

$(1 < x < 3) + 4$

$5 < x+4 < 7$.

$\boxed{x+4 < 7}$

So since $\delta = 1 \rightarrow x+4 < 7$

let $\varepsilon > 0$, set $\delta = \min\{1, \frac{\varepsilon}{7}\}$

If $|x-2| < \delta$ then $|x^2 + 2x - 5 - 3|$

$= |x^2 + 2x - 8|$

$= |(x-2)(x+4)|$

$= |x-2||x+4|$

$< \delta (7)$

$< \frac{\varepsilon}{7} (7)$

$< \varepsilon$

---

b. $\lim_{x \to 1} \frac{x^2 + x - 2}{x-1} = 3$ L

let $\varepsilon > 0$, set $\delta = \varepsilon$

If $|x-1| < \delta$ then $\left|\frac{x^2+x-2}{x-1} - 3\right| = \left|\frac{x^2+x-2-3x+3}{x-1}\right|$

$= \left|\frac{x^2-2x+1}{x-1}\right|$

$= \left|\frac{\cancel{(x-1)}(x-1)}{\cancel{(x-1)}}\right|$

$= |x-1|$

$< \delta$

$< \varepsilon$

c. $\lim_{x\to 1} x^3+2x+1 = 4$

let $\varepsilon>0$ and set $\delta=\min\{1,\frac{\varepsilon}{9}\}$

If $|x-1|<\delta$ then $|x^3+2x+1-4| = |x^3+2x-3|$

$= |(x-1)(x^2+x+3)|$

$\to 0<\delta\le 1 \to |x-1|<\delta$

$= |x-1||x^2+x+3|$

$|x-1|<1$

$< \delta\, 9$

$0<x<2$

$< \frac{\varepsilon}{9}\, 9$

$\Rightarrow x^2+x+3 < 2^2+2+3$

$< \varepsilon$ ☒

$x^2+x+3<9$

---

d. $\lim_{x\to 0} x^3\sin(e^{x^2}) = 0$

let $\varepsilon>0$ and set $\delta=\sqrt[3]{\varepsilon}$

If $|x|<\delta$ then $|x^3\sin(e^{x^2})-0|$

$|x|^3<\delta^3$ $\quad = |x^3|\,|\sin(e^{x^2})| \longrightarrow |\sin(x)|<1$

$< \delta^3 \cdot 1$

$< (\sqrt[3]{\varepsilon})^3$

$< \varepsilon$ ☒

3.2.1: Decide which of the following limits exists and which do not.

a. $\lim_{x \to 0} \tan\left(\frac{1}{x}\right)$ DNE.

$X_n = \frac{4}{(2n+1)\pi}$, $\lim_{n \to \infty} X_n = 0$

But $\tan \frac{1}{X_n} = (-1)^n$ has No limit.

Thus, $\lim_{x \to 0} \tan\left(\frac{1}{x}\right)$ DNE.

---

b. $\lim_{x \to 0} x \cos\left(\frac{x^2+1}{x^3}\right)$ exist. By squeez thm

since $\left| x \cos\left(\frac{x^2+1}{x^3}\right)\right| \leq |x|$ $\forall x \neq 0$.

$\lim_{x \to 0} x = 0$ So $\lim_{x \to 0} x \cos\left(\frac{x^2+1}{x^3}\right) = 1$ ✓

---

c. $\lim_{x \to 1} \frac{1}{\log x}$

If $X_n = 1 + \frac{1}{n}$ then $X_n \to 1$ and $\frac{1}{\log X_n} \to +\infty$ as $n \to \infty$

on the other hand: if $X_n = 1 - \frac{1}{n}$ then $X_n \to 1$ and $\frac{1}{\log X_n} \to -\infty$ as $n \to \infty$

Thus, $\lim_{x \to 1} \frac{1}{\log x}$ DNE.

3.1.3 : Evaluate the following limits using results from this section.

a. $\lim_{x\to1} \frac{x^2+2x-3}{x^3-x} = \lim_{x\to1} \frac{(x-1)(x+3)}{x(x^2-1)} = \lim_{x\to1} \frac{\cancel{(x-1)}(x+3)}{x(x+1)\cancel{(x-1)}} = \frac{4}{2} = 2$

b. $\lim_{x\to1} \frac{x^n-1}{x-1}$, $n \in \mathbb{N}$:

c. $\lim_{x\to1} \frac{\sqrt[3]{x^4-1}}{\cos(1-x)} = \frac{0}{1} = 0$.

تعويض

d. $\lim_{x\to0} \frac{2\sin^2 x + 2x - 2x\cos^2 x}{1-\cos^2(2x)} = \lim_{x\to1} \frac{2(\sin^2 x + x - x\cos^2 x)}{1-\cos^2 2x}$

متطابقة

$= \lim_{x\to1} \frac{2(x(1-\cos^2 x) + \sin^2 x)}{1-\cos^2 2x} = \lim_{x\to1} \frac{2(x\sin^2 x + \sin^2 x)}{1-\cos^2 2x}$

مع بعض متطابقة

$= \lim_{x\to1} \frac{2(x+1)\sin^2 x}{1-\cos^2 2x}$

متطابقة الفرق بين مربعين

$= \lim_{x\to1} \frac{(x+1)\cancel{(1-\cos 2x)}}{\cancel{(1-\cos 2x)}(1+\cos 2x)}$

تعويض

$\lim \frac{x+1}{1+\cos 2x} = \frac{1}{1+1} = \frac{1}{2}$ ✓

e. $\lim_{x\to0} \tan x \sin\left(\frac{1}{x^2}\right) = \lim_{x\to0} \tan x \quad \lim_{x\to0} \sin\left(\frac{1}{x^2}\right) \rightarrow M = 1$

$\underbrace{\sin \frac{1}{x^2} \leq 1}_{\text{upper bound}}$

$= 0 \quad \lim_{x\to0} \sin\left(\frac{1}{x^2}\right)$

$= 0$ ✓

3.1.4 : Prove Theorem 4 : squeez Theorem

ⓐ let $x_n \in I \setminus \{a\}$ converges to a.

By Thm ( ) : $h(x_n) \to L$ as $n \to \infty$.

Hence, By The sequential characterization of Limits,

$h(x) \to L$ as $x \to a$.

Done

ⓑ similarly, By Thm ( ) $f(x_n)g(x_n) \to 0$ $\forall x_n \in I \setminus \{a\}$ which conv. to a.

Hence, By the sequ. charac. of limits,

$f(x) g(x) \to 0$ as $x \to a$.

3.1.5 : prove Theorem 5 : Comparision

let $f(x) \to L$ and $g(x) \to M$ as $x \to a$.

and $x_n \in I \setminus \{a\}$ Converges to a.

By the sequential characterization of limits :

$f(x_n) \to L$ and $g(x_n) \to M$ as $n \to \infty$

Hence, By Thm $L \leq M$

3.1.6 : suppose that f is a real function

a. prove that if $L = \lim_{x \to a} f(x)$ exists, then $|f(x)| \to |L|$ as $x \to a$.

By Archemedian principle :

$$0 \leq \big|\, |f(x)| - |L| \,\big| \leq |f(x) - L|$$

So By squeez thm, $|f(x)| \to |L|$ as $x \to x_0$ through E.

b. show that there is a function such that, as $x \to a$, $|f(x)| \to |L|$ But the limit of $f(x)$ does Not Exist.

If $f(x) = \dfrac{|x|}{x}$ then $|f(x)| = 1 \to 1$ as $x \to 0$

But $f(x)$ has No limit as $x \to 0$.

3.1.7: For each real function f, define the positive part of f by,

$$f^+(x) = \frac{|f(x)| + f(x)}{2} \quad , \; x \in Dom(f) \; .$$

and negative part of f By $f^{-1}(x) = \dfrac{|f(x)| - f(x)}{2}$, $x \in Dom(f)$

a. prove that $f^+(x) \geq 0$, $f^-(x) \geq 0$, $f(x) = f^+(x) - f^-(x)$ and $|f(x)| = f^+(x) + f^-(x)$.
all holds for every $x \in Dom(f)$.

Since $f(x) \leq |f(x)|$ it is clear that $f^+(x) \geq 0$ and $f^-(x) \geq 0$

Also $f^+ - f^- = \dfrac{2f}{2} = f$ and

$$f^+ + f^{-1} = \frac{2|f|}{2} = |f| .$$

b. prove that if $L = \lim_{x \to a} f(x)$ exists, then $f^+(x) \to L^+$ and $f^-(x) \to L^-$ as $x \to a$.

By 3.1.6 :

$|f(x)| \to |L|$ as $x \to x_0$ through E.

Hence By Thm 3.9

$f^+(x) \to L^+$ and $f^-(x) \to L^-$ as $x \to x_0$ through E.