$\boxed{10.7}$ Power Series

- Power Series are sum of infinite polynomials.

Def • A power series about $x=a$ has the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots$$

where $c_0, c_1, c_2, \ldots, c_n, \ldots$ constant coefficients and a is the center.

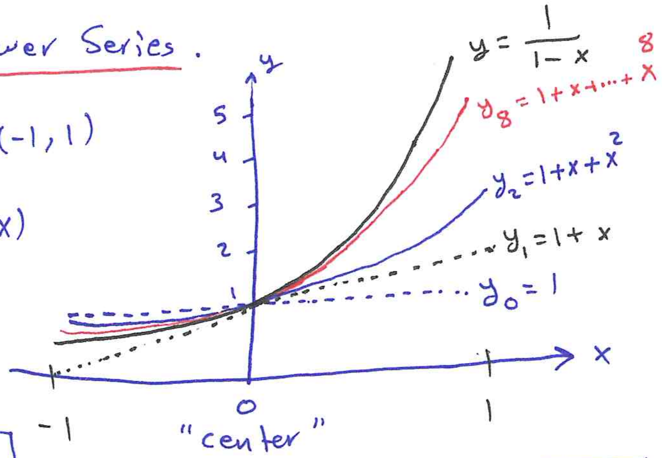
- A power series about $x=0$ is then given by

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots \quad \cdots (1)$$

Exp If $c_0 = c_1 = \cdots = c_n = \cdots = 1$ in (1), then we get the geometric power series $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots = \frac{1}{1-x}$ if $-1<x<1$

which we call the Reciprocal Power Series.

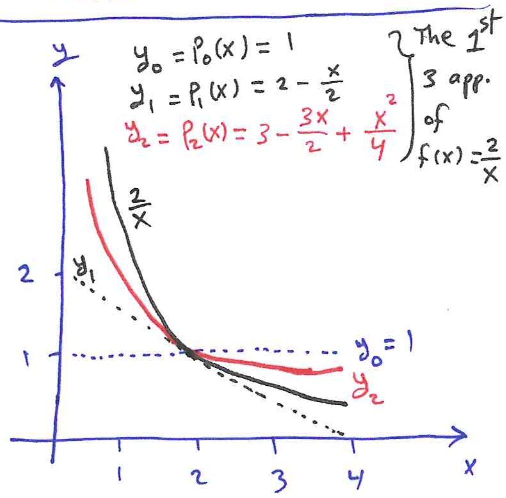
- To approximat $f(x) = \frac{1}{1-x}$ on $(-1,1)$ we use the partial sums $y_n = P_n(x)$
- The approximations works only on $(-1,1)$ in which $f(x)$ is continuous.



Exp Consider the power series

$$1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 + \cdots + \left(-\frac{1}{2}\right)^n (x-2)^n + \cdots$$

- The center: $a=2$
- The coefficients: $c_0 = 1$, $c_1 = \frac{-1}{2}$, $c_2 = \frac{1}{4}, \ldots, c_n = \left(\frac{-1}{2}\right)^n$
- The geometric series converges to $\frac{1}{1-r}$ where $|r| = \left|\frac{-1}{2}(x-2)\right| < 1 \iff 0<x<4.$
- The sum is $\frac{1}{1-r} = \frac{1}{1+\frac{x-2}{2}} = \frac{2}{x}$
- So, $\frac{2}{x} = 1 - \frac{(x-2)}{2} + \frac{(x-2)^2}{4} + \cdots + \left(-\frac{1}{2}\right)^n (x-2)^n + \cdots, \quad 0<x<4$



Exp Find the radius of convergence and the interval of convergence for the following power series:

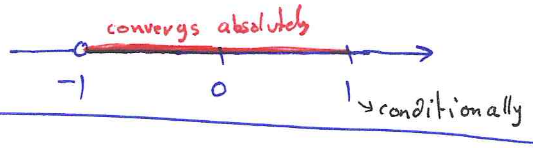
$\boxed{1}$ $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$ Apply Ratio Test to $\sum |u_n|$

- $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{n+1} \cdot \frac{n}{x^n}\right| = |x| \lim_{n\to\infty} \frac{n}{n+1} = |x| < 1$
- Thus, the radius of convergence is $R = 1$.
- To find the interval of convergence: $-1 < x < 1$
  - The series converges absolutely for $-1 < x < 1$
  - The series diverges for $|x| > 1$
  - To check the endpoints:

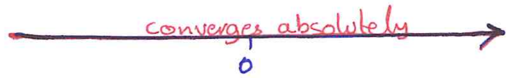
→ converges conditionally • when $x = 1 \Rightarrow \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$ " alternating harmonic series which converges"

- when $x = -1 \Rightarrow \sum_{n=1}^{\infty} (-1)^{2n-1} \frac{1}{n} = -\sum_{n=1}^{\infty} \frac{1}{n}$ " negative of harmonic series which diverges."

- Thus, the series converges for $-1 < x \le 1$

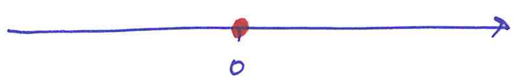


$\boxed{2}$ $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ Apply Ratio Test

- $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| = |x| \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1$ for all x
- Thus, $R = \infty$ → • The series converges absolutely for all x.
- 

$\boxed{3}$ $\sum_{n=0}^{\infty} n!\, x^n$ Apply Ratio Test: $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!\, x^{n+1}}{n!\, x^n}\right|$

$= |x| \lim_{n\to\infty} (n+1) = \infty$ except $x = 0$

- Thus, $R = 0$
- The series diverges for all values of x except $x = 0$
- 

$\boxed{4}$ $\sum\limits_{n=0}^{\infty} \frac{(x-2)^n}{10^n}$ Apply Ratio Test $\lim\limits_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim\limits_{n\to\infty}\left|\frac{(x-2)^{n+1}}{10^{n+1}}\cdot\frac{10^n}{(x-2)^n}\right|$

$= \frac{|x-2|}{10} < 1$

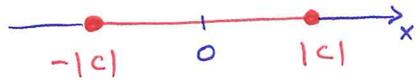
- Thus, $R = 10$
- The series converges absolutely for $|x-2| < 10 \iff -8 < x < 12$
- when $x = -8 \Rightarrow \sum\limits_{n=0}^{\infty}(-1)^n$ which diverges since $a_n \not\to 0$ as $n\to\infty$
- when $x = 12 \Longrightarrow \sum\limits_{n=0}^{\infty} 1$ which diverges

$\boxed{\text{center} = 2}$

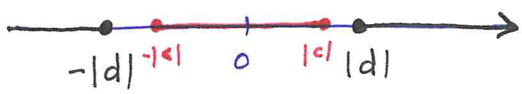
- Thus,

Th Consider the power series $\sum\limits_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$

- If the series converges at $x = c \neq 0$, then it converges absolutely for all x with $|x| < |c|$



- If the series diverges at $x = d$, then it diverges for all x with $|x| > |d|$



Exp $\boxed{2}$ the series converges at $x = 3 \Rightarrow$ it converges absolutely for $|x| < 3 \iff -3 < x < 3$ ✓

Exp $\boxed{3}$ the series diverges at $x = 3 \Rightarrow$ it diverges for $|x| > 3$

Exp $\boxed{4}$ the series converges at $x = 3 \Rightarrow$ it converges absolutely for $|x-2| < 3 \iff -1 < x < 5$

Th If $\sum\limits_{n=0}^{\infty} a_n x^n = A(x)$ and $\sum\limits_{n=0}^{\infty} b_n x^n = B(x)$ converge absolutely for $|x| < R$,

then $\left(\sum\limits_{n=0}^{\infty} a_n x^n\right)\left(\sum\limits_{n=0}^{\infty} b_n x^n\right)$ converges absolutely to $A(x)\,B(x)$ for $|x| < R$:

That is: $A(x)\,B(x) = \left(\sum\limits_{n=0}^{\infty} a_n x^n\right)\left(\sum\limits_{n=0}^{\infty} b_n x^n\right) = \sum\limits_{n=0}^{\infty} c_n x^n$, where

$c_n = \sum\limits_{k=0}^{n} a_k b_{n-k} = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_{n-1} b_1 + a_n b_0$

Th 20 If $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely on $|x| < R$, then **72**

$\sum_{n=0}^{\infty} a_n [f(x)]^n$ converges absolutely on $|f(x)| < R$ for any continuous function f.

Exp $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ converges absolutely for $|x| < 1$

$\Rightarrow \frac{1}{1-4x^2} = \sum_{n=0}^{\infty} (4x^2)^n$ converges absolutely for $|4x^2| < 1 \Leftrightarrow |x| < \frac{1}{2}$

Exp Use Th 20 to find the interval of convergence and the sum of the series as function of x: $\sum_{n=0}^{\infty} \left(\frac{x^2+1}{3}\right)^n$ Apply Ratio Test

- $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left| \frac{(x^2+1)^{n+1}}{3^{n+1}} \cdot \frac{3^n}{(x^2+1)^n} \right| = \frac{|x^2+1|}{3} < 1 \Leftrightarrow x^2 < 2 \Leftrightarrow -\sqrt{2} < x < \sqrt{2}$
- At $x = \pm\sqrt{2} \Rightarrow \sum_{n=0}^{\infty} (1)^n$ which diverges.
- Thus, the interval of convergence is $-\sqrt{2} < x < \sqrt{2}$
- The series $\sum_{n=0}^{\infty} \left(\frac{x^2+1}{3}\right)^n$ is convergent geometric series on $-\sqrt{2}<x<\sqrt{2}$
- The sum is $\frac{1}{1-\frac{x^2+1}{3}} = \frac{3}{2-x^2}$

Th (Term by Term Differentiation)
Assume that $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ converges absolutely on $|x-a| < R$.

Then f has derivatives of all orders on $|x-a| < R$. That is,

$f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$, $f''(x) = \sum_{n=2}^{\infty} n(n-1) c_n (x-a)^{n-2}$, ...

converge at every point on $|x-a| < R$.

Exp $f(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$ on $|x| < 1$ converges absolutely if $|\text{ratio}| < 1$

$f'(x) = \sum_{n=1}^{\infty} n x^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \cdots$ on $|x| < 1$

$f''(x) = \sum_{n=2}^{\infty} n(n-1) x^{n-2} = 2 + 6x + 12x^2 + \cdots$ on $|x| < 1$

Th (Term by Term Integration Theorem)

suppose that $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ converges for $|x-a| < R$.

Then, $\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$ converges for $|x-a| < R$ and $\int f(x)dx = c + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$.

Exp Identify the function $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$ $\quad |x| \le 1$. $*$

Note that $f'(x) = 1 - x^2 + x^4 - x^6 + \cdots \quad |x| < 1$

$f'(x) = \frac{1}{1+x^2}$ geometric series

$\boxed{\frac{\pi}{4} = \tan^{-1} 1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}}$ Leibniz's Formula

Hence, $f(x) = \int f'(x)dx = \tan^{-1} x + c$.

To find $c \Rightarrow$ From $*$ $f(0) = 0 \Leftrightarrow 0 = \tan^{-1} 0 + c \Leftrightarrow \boxed{c = 0}$

Hence, $f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \tan^{-1} x \quad , \quad |x| \le 1$

Exp The series $\frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots$ converges on $|t| < 1$.

Therefore $\ln(1+x) = \int_0^x \frac{dt}{1+t} = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots \Big|_0^x$

$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

or $\quad \ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} \quad , \quad |x| < 1$

$$\ln(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$