16.2 Vector Fields and Line Integrals

Def A vector fied is a function that assigns a vector to each point in its domain. A vector field in space has the form: $\vec{F}(x,y,z) = M(x,y,z)\,\vec{i} + N(x,y,z)\,\vec{j} + P(x,y,z)\,\vec{k}$

- The vector fied $\vec{F}$ is continuous if the component functions M, N, P are continuous.
- The vector fied $\vec{F}$ is differentiable if the component functions M, N, P are differentiable.
- A vector field of two-dimensional has the form:

$$\vec{F}(x,y) = M(x,y)\,\vec{i} + N(x,y)\,\vec{j}$$

Examples of vector fields:

- The tangent vectors $\vec{T}$ and normal vectors $\vec{N}$ for a curve in space.
- *1 Velocity vector field $\vec{v}(t) = f(t)\,\vec{i} + g(t)\,\vec{j} + h(t)\,\vec{k}$ along the streamlines (خطوط انسيابية) of water/air moving through a contracting channel.
- *2 If we attach the gradient vector $\vec{\nabla} f$ of a scalar function $f(x,y,z)$ to each point of a level surface of the function, we obtain a 3-dimensional vector field on the surface.
- If we attach the velocity vector $\vec{v}$ to each point of a flowing fluid, we obtain a 3-dimensional vector field defined on a region in space.

Def (Gradient Fields)

The gradient vector $\vec{\nabla} f$ of a differentiable scalar-valued function $f(x,y,z)$ at point gives the direction of greatest increase of the function and defined by:

$$\vec{\nabla} f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$$

Exp Suppose the temperature T at each point (x,y,z) in a region of space is given by $T = 100 - x^2 - y^2 - z^2$. Find the gradient fields of T.

$$\vec{F} = \vec{\nabla} T = T_x \vec{i} + T_y \vec{j} + T_z \vec{k}$$

by $*^2$

$$= -2x\vec{i} - 2y\vec{j} - 2z\vec{k}$$

At each point in space, $\vec{\nabla} T$ gives the direction for which the increase in T is the greatest.

line Integrals of Vector Fields

- In section 16.1, we defined the line integral of a scalar function $f(x,y,z)$ over a path C.
- Now, we will define the line integral of a vector field $\vec{F}$ along the curve C.

Def Let $\vec{F} = M(x,y,z)\vec{i} + N(x,y,z)\vec{j} + P(x,y,z)\vec{k}$ be a vector field with continuous components M, N, P defined along a smooth curve C parametrized by

$$\vec{r}(t) = g(t)\vec{i} + h(t)\vec{j} + k(t)\vec{k}, \quad a \le t \le b.$$

Then, the line integral of $\vec{F}$ along C is

$$\int_C \vec{F} \cdot \vec{T}\, ds = \int_C \left(\vec{F} \cdot \frac{d\vec{r}}{ds}\right) ds = \int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F} \cdot \frac{d\vec{r}}{dt}\, dt$$

$\vec{T} = \frac{d\vec{r}}{ds} = \frac{\vec{v}}{|\vec{v}|}$ is a unit vector tangent to the path C and pointing a forward direction.

$\vec{v} = \frac{d\vec{r}}{dt}$ is the velocity vector

Note that $x = g(t)$, $y = h(t)$, $z = k(t)$

Exp Find the line integrals of
$\vec{F} = 3y\,\vec{i} + 3x\,\vec{j} + 4z\,\vec{k}$ along the curve
$\vec{r}(t) = t\,\vec{i} + t^2\,\vec{j} + t^3\,\vec{k}$, $0 \le t \le 1$.

- Note that $x = t$, $y = t^2$, $z = t^3$

$$\boxed{\vec{F} = 3t^2\,\vec{i} + 3t\,\vec{j} + 4t^3\,\vec{k}}$$

- $$\boxed{\frac{d\vec{r}}{dt} = \vec{i} + 2t\,\vec{j} + 3t^2\,\vec{k}}$$

- $$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 \left(\vec{F}\cdot\frac{d\vec{r}}{dt}\right)dt = \int_0^1 (3t^2 + 6t^2 + 12t^5)\,dt$$
$$= \int_0^1 (9t^2 + 12t^5)\,dt = \left(3t^3 + 2t^6\right)\Big|_0^1 = 5$$

* line Integrals w.r.t the xyz coordinates:

- Given the curve C parametrized by
$\vec{r}(t) = g(t)\,\vec{i} + h(t)\,\vec{j} + k(t)\,\vec{k}$, $a \le t \le b$

$x = g(t)$ $y = h(t)$ $z = k(t)$
$dx = g'(t)\,dt$ $dy = h'(t)\,dt$ $dz = k'(t)\,dt$
$\vec{F} = M\vec{i} + N\vec{j} + P\vec{k}$

- We define the line integral of M over C w.r.t x-axis as
$$\int_C M(x,y,z)\,dx = \int_a^b M(g(t),h(t),k(t))\,g'(t)\,dt$$
and the line integral of N over C w.r.t y-axis as
$$\int_C N(x,y,z)\,dy = \int_a^b N(g(t),h(t),k(t))\,h'(t)\,dt$$
and the line integral of P over C w.r.t z-axis as
$$\int_C P(x,y,z)\,dz = \int_a^b P(g(t),h(t),k(t))\,k'(t)\,dt$$

Exp Evaluate the integral ① $\int_C (x+y-z)\,dx$ along the curve C parametrized by $\vec{r}(t) = t\,\vec{i} - \vec{j} + t^2\,\vec{k}$, $0 \le t \le 1$

$x = t$
$y = -1$
$z = t^2$
$dx = dt$

$$\int_C (x+y-z)\,dx = \int_0^1 (t - 1 - t^2)\,dt$$

$$= \frac{t^2}{2} - t - \frac{t^3}{3} \Big|_0^1 = -\frac{5}{6}$$

$dy = 0\,dt$
$dz = 2t\,dt$

② $\int_C -y\,dx + z\,dy + 2x\,dz$

$$= \int_0^1 dt + t^2(0) + 2t\,(2t)\,dt$$

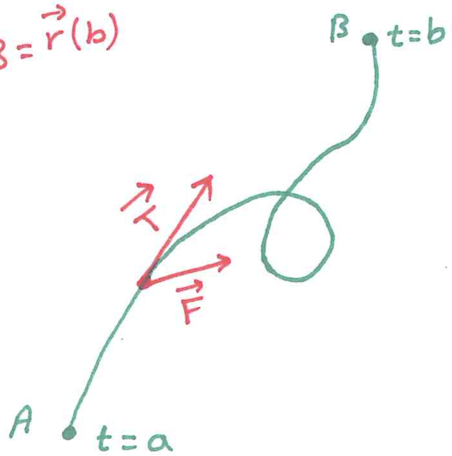
$$= \int_0^1 (1 + 4t^2)\,dt = t + 4\frac{t^3}{3} \Big|_0^1 = 1 + \frac{4}{3} = \frac{7}{3}$$

Def (Work done by a force over a curve in Space)

- Let C be a smooth curve parametrized by $\vec{r}(t) = g(t)\,\vec{i} + h(t)\,\vec{j} + k(t)\,\vec{k}$, $a \le t \le b$.
- Let $\vec{F} = M(x,y,z)\,\vec{i} + N(x,y,z)\,\vec{j} + P(x,y,z)\,\vec{k}$ be a continuous force field over a region containing C.
- Then, the work done in moving an object from the point $A = \vec{r}(a)$ to the point $B = \vec{r}(b)$ along the curve C is

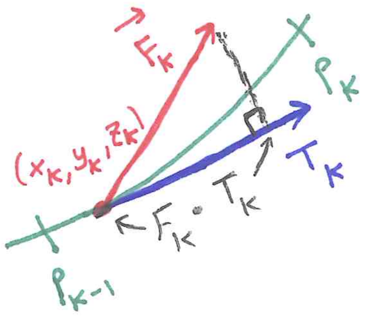
$$W = \int_C \vec{F}\cdot\vec{T}\,ds \quad \cdots \$$$

which is just the line integral of $\vec{F}$ along C.



- The work done along the subarc is approximated by

$$w_K = \vec{F}_K \cdot \vec{T}_K \, \Delta s_K$$



- The total work done in moving the object from point A to point B is then approximated by

$$W = \sum_{K=1}^{n} w_K = \sum_{K=1}^{n} \vec{F}(x_K, y_K, z_K) \cdot \vec{T}(x_K, y_K, z_K) \, \Delta s_K$$

- As $n \to \infty \Rightarrow \Delta s_K \to 0$ and the sum approachs the line integral:

$$\lim_{n\to\infty} W = \lim_{n\to\infty} \sum_{K=1}^{n} w_K = \int_C \vec{F} \cdot \vec{T} \, ds$$ given in \$.

* Different forms for the work integral:

$$\vec{W} = \int_C \vec{F} \cdot \vec{T} \, ds$$ The definition

$$= \int_C \vec{F} \cdot d\vec{r}$$ Vector differential form

$$= \int_a^b \vec{F} \cdot \frac{d\vec{r}}{dt} \, dt$$ Parametric vector evaluation

$$= \int_a^b \left(M \frac{dx}{dt} + N \frac{dy}{dt} + P \frac{dz}{dt} \right) dt$$ Parametric scalar evaluation

$$= \int_a^b M \, dx + N \, dy + P \, dz$$ scalar differential form

Exp Find the work done by the force field $\vec{F} = \vec{i} + x\vec{j} + y\vec{k}$ along the curve
$\vec{r}(t) = (\sin t)\vec{i} + (\cos t)\vec{j} + t\vec{k}$, $0 \le t \le 2\pi$

- $x = \sin t$, $y = \cos t$, $z = t$
- $\vec{F} = \vec{i} + (\sin t)\vec{j} + (\cos t)\vec{k}$
- $\dfrac{d\vec{r}}{dt} = (\cos t)\vec{i} - (\sin t)\vec{j} + \vec{k}$
- work is $W = \int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \left(\vec{F}\cdot\dfrac{d\vec{r}}{dt}\right) dt$

$$= \int_0^{2\pi} (\cos t - \sin^2 t + \cos t)\, dt$$

$$= -\int_0^{2\pi} \left(\frac{1-\cos 2t}{2}\right) dt = -\pi$$

Backward direction

Def (Flow Integrals and Circulation for Velocity Fields)

- Suppose $\vec{F}$ represents the velocity field of a fluid flowing through a region (channel) in space.
- Let C be a smooth curve in the domain of a continuous velocity field $\vec{F} = M(x,y,z)\vec{i} + N(x,y,z)\vec{j} + P(x,y,z)\vec{k}$ that is parametrized by
  $\vec{r}(t) = g(t)\vec{i} + h(t)\vec{j} + k(t)\vec{k}$, $a \le t \le b$
- Then, the flow integral along the curve from $A = \vec{r}(a)$ to $B = \vec{r}(b)$ is Flow $= \int_C \vec{F}\cdot\vec{T}\, ds$.
- If the curve starts and ends at same point $(A = B)$, then the flow is called circulation around the curve.

Exp Find the flow of the velocity field

$\vec{F} = xy\,\vec{i} + (y-x)\,\vec{j}$ over the straight line from $(1,1)$ to $(2,3)$

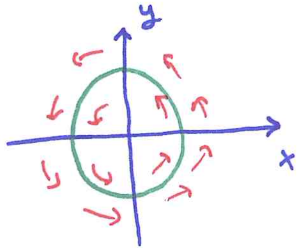
- $\vec{r}(t) = (t+1)\,\vec{i} + (2t+1)\,\vec{j}$ $\quad 0 \le t \le 1$ is a possible parametrization for the line.
- $x = t+1$ and $y = 2t+1$
- $\vec{F} = (t+1)(2t+1)\,\vec{i} + (2t+1-t-1)\,\vec{j}$

  $= (t+1)(2t+1)\,\vec{i} + t\,\vec{j}$
- $\text{Flow} = \int_C \vec{F}\cdot\vec{T}\,ds = \int_0^1 \left((t+1)(2t+1)\,\vec{i} + t\,\vec{j}\right)\cdot\left(\vec{i} + 2\vec{j}\right) dt$

  $= \int_0^1 \left(2t^2 + 3t + 1 + 2t\right) dt = \int_0^1 \left(2t^2 + 5t + 1\right) dt = \dfrac{25}{6}$

Exp Find the circulation of the field $\vec{F} = -y\,\vec{i} + x\,\vec{j}$ around the circle $\vec{r}(t) = (\cos t)\,\vec{i} + (\sin t)\,\vec{j}$, $0 \le t \le 2\pi$

- $x = \cos t$ and $y = \sin t$
- $\vec{F} = (-\sin t)\,\vec{i} + (\cos t)\,\vec{j}$
- $\dfrac{d\vec{r}}{dt} = (-\sin t)\,\vec{i} + (\cos t)\,\vec{j}$
- $\text{Circulation} = \int_0^{2\pi} \vec{F}\cdot\dfrac{d\vec{r}}{dt}\,dt = \int_0^{2\pi} \left(\sin^2 t + \cos^2 t\right) dt = 2\pi$



---

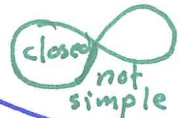
a fluid with this velocity field is circulating counterclockwise around the circle

تدفق Flux Across a Simple Plane Curve

Def

- A curve in the xy-plane is simple if it does not cross itself.

- If the curve starts and ends at the same point, it is called a closed curve or loop.



- How to find the rate at which a fluid is entering or leaving a region enclosed by a smooth simple closed curve C in the xy-plane

---

Def Let C be a smooth simple closed curve in the domain of a continuous vector field $\vec{F} = M(x,y)\,\vec{i} + N(x,y)\,\vec{j}$ in the xy-plane. If $\vec{n}$ is the outward-pointing unit normal vector on C, then the flux of $\vec{F}$ across C is

♡ $$\text{Flux of } \vec{F} \text{ across } C = \oint_C M\,dy - N\,dx = \int_C \vec{F}\cdot\vec{n}\,ds$$

Notes

- we put a directed circle ⟲ on the last integral to mean integration around closed curve C in the counterclockwise direction.
- The flux of F across C is the line integral w.r.t arc length of $\vec{F}\cdot\vec{n}$ (scalar component of $\vec{F}$ in direction of $\vec{n}$).
- The circulation of $\vec{F}$ across C is the line integral w.r.t arc length of $\vec{F}\cdot\vec{T}$ (scalar component of $\vec{F}$ in direction of unit $\vec{T}$) tangent
- We calculate the flux in ♡ from any parametrization $x = g(t)$ and $y = h(t)$, $a \le t \le b$ that traces C counterclockwise exactly once.

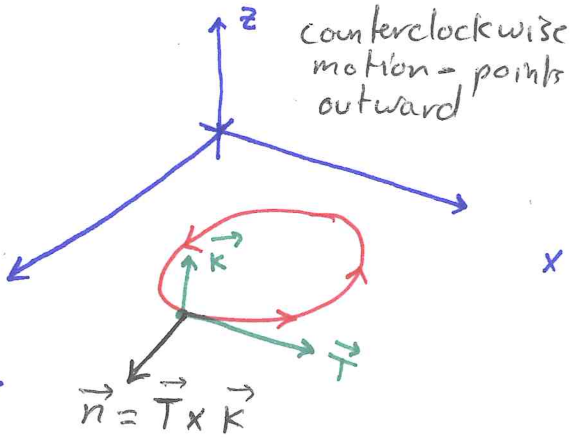
Exp Find the flux of the field $\vec{F} = x\vec{i} + y\vec{j}$ cross the curve

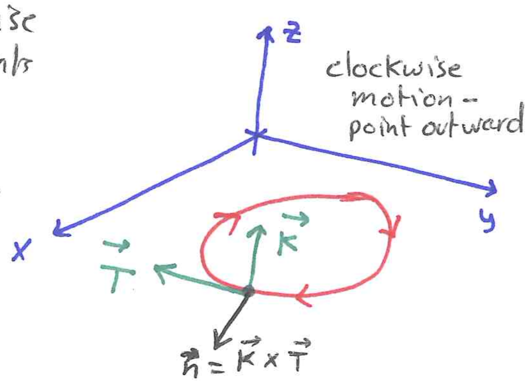
$$\vec{r}(t) = (\cos t)\vec{i} + (4\sin t)\vec{j}, \quad 0 \le t \le 2\pi$$

- $x = \cos t$ and $y = 4\sin t$ $\qquad dx = -\sin t \, dt$
- $\vec{F} = (\cos t)\vec{i} + (4\sin t)\vec{j}$ $\qquad dy = 4\cos t \, dt$
- $M = \cos t$ and $N = 4\sin t$
- Flux of $\vec{F}$ across $C = \oint M\,dy - N\,dx$

$$= \int_0^{2\pi} (4\cos^2 t + 4\sin^2 t)\,dt = 8\pi$$

Proof ♡ :





- $\vec{n} = \vec{T} \times \vec{K}$

$$= \left(\frac{dx}{ds}\vec{i} + \frac{dy}{ds}\vec{j}\right) \times \vec{K}$$

$$= \frac{dy}{ds}\vec{i} - \frac{dx}{ds}\vec{j}$$

- we have $\vec{F} = M(x,y)\vec{i} + N(x,y)\vec{j}$
- So $\vec{F} \cdot \vec{n} = M\frac{dy}{ds} - N\frac{dx}{ds}$
- Hence,

$$\int_C \vec{F} \cdot \vec{n}\,ds = \oint M\,dy - N\,dx$$

Note that

$$\frac{d\vec{r}}{ds} = \frac{d\vec{r}}{dt}\frac{dt}{ds} = \vec{v}\,\frac{1}{|\vec{v}|} = \vec{T}$$

$$\vec{r}(t) = g(t)\vec{i} + h(t)\vec{j} = x\vec{i} + y\vec{j}$$

$$\frac{d\vec{r}}{ds} = \frac{dx}{ds}\vec{i} + \frac{dy}{ds}\vec{j}$$