10.6 Alternating Series, Absolute and Conditional Convergence.

Def: A series in which the terms are alternately positive and negative is an Alternating series.

Examples:

1) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots + \frac{(-1)^{n+1}}{n} + \cdots$

2) $\sum_{n=1}^{\infty} \frac{4(-1)^{n}}{2^n} = -2 + 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots + \frac{4(-1)^n}{2^n} + \cdots$

3) $\sum_{n=1}^{\infty} (-1)^{n+1} n = 1 - 2 + 3 - 4 + 5 - \cdots + (-1)^{n+1} n + \cdots$

Remark: (1) The 1st series is called Alternating harmonic series.

(2) The 2nd series is Geometric series.

(3) The 3rd series diverges by the nth term test

Remark: From the previous examples, we see that the Alternating series has the form:

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^{n+1} u_n \quad \text{or} \quad \sum_{n=1}^{\infty} (-1)^n u_n$$

where $u_n = |a_n|$.

Theorem: The Alternating Series Test (Leibniz's Test)

(A.S.T): The series:

$$\sum_{n=1}^{\infty} (-1)^{n+1} u_n = u_1 - u_2 + u_3 - u_4 + \cdots$$

converges if all three of the following conditions are satisfied:

1. The u_n's are all positive, $\forall n$.

2. The positive u_n's are (eventually) nonincreasing:

$$u_n \geq u_{n+1}, \quad \forall n \geq N, \quad N \in \mathbb{Z}$$

3. $u_n \to 0$.

Example: The Alternating Harmonic Series.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

$u_n = \frac{1}{n}$, Clearly:

1. $u_n > 0 \quad , \forall n \geq 1$

2. $u_n' = \frac{-1}{n^2} < 0 \,, \forall n \Rightarrow u_n$ is decreasing

(i.e) $u_{n+1} \leq u_n \quad , \forall n \geq 1$

3. $u_n \longrightarrow 0$ as $n \longrightarrow \infty$.

$\Rightarrow$ By A.S.T the series Converges.

Example: $\sum_{n=1}^{\infty} (-1)^n \left(1 + \frac{1}{n}\right)^n$.

$u_n = \left(1 + \frac{1}{n}\right)^n$

1. $u_n > 0 \quad , \forall n \geq 1$

2. $u_{n+1} \leq u_n \quad , \forall n$

But $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e^1 \neq 0$ (nth term Test)

$\therefore$ By nth term test the series diverges.

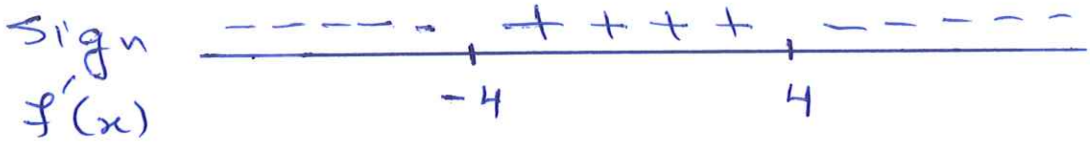
Example: $\sum_{n=1}^{\infty} (-1)^n \frac{10n}{n^2+16}$

$u_n = \frac{10n}{n^2+16}$. Clearly:

1. $u_n > 0 \quad , \forall n \geq 1$.

2. Define $f(x) = \frac{10x}{x^2+16} \quad , x \geq 1$

$$\Rightarrow f'(x) = \frac{(x^2+16)(10) - (10x)(2x)}{(x^2+16)^2} = \frac{10(16-x^2)}{(x^2+16)^2}$$



$\therefore f'(x) \leq 0, \forall x \geq 4$

$\therefore f$ is eventually nonincreasing $(n = 4)$

$\Leftrightarrow u_n$ is nonincreasing for $n \geq 4$

3. $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{10n}{n^2+16} = 0$.

$\therefore$ The series Converges by A.S.T.

Example: Alternating Geometric Series:

$$\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{1}{2}\right)^n = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \cdots$$

$$= \frac{\frac{1}{2}}{1-\left(-\frac{1}{2}\right)} = \frac{1}{3} \Rightarrow \text{(Converges)}$$

Or, Using A.S.T with $u_n = \left(\frac{1}{2}\right)^n$:

1. $u_n = \left(\frac{1}{2}\right)^n > 0, \quad \forall n \geq 1.$

2. Define $f(x) = \left(\frac{1}{2}\right)^x \Rightarrow f'(x) = \ln(0.5)\left(\frac{1}{2}\right)^x < 0$

$\Rightarrow$ $f(x)$ is decreasing $\forall x \geq 1$

$\Rightarrow$ u_n is nonincreasing $\forall n \geq 1$.

3. $\lim_{n\to\infty} \left(\frac{1}{2}\right)^n = 0$.

$\Rightarrow$ The Alternating Geometric Series Converges by A.S.T.

Example: $\sum_{n=1}^{\infty} (-1)^n$

$u_n = 1$, $\lim_{n\to\infty} u_n \neq 0$.

$\therefore$ By the nth term test the series diverges.

The Alternating series Estimation Theorem:

Assume $\sum_{n=1}^{\infty} (-1)^{n+1} u_n = u_1 - u_2 + u_3 - u_4 + \dots$

Converges to L. Then for $n \geq N$

$$S_n = u_1 - u_2 + u_3 - \dots + (-1)^{n+1} u_n$$

approximates L; $S_n \approx L$; with an Error E

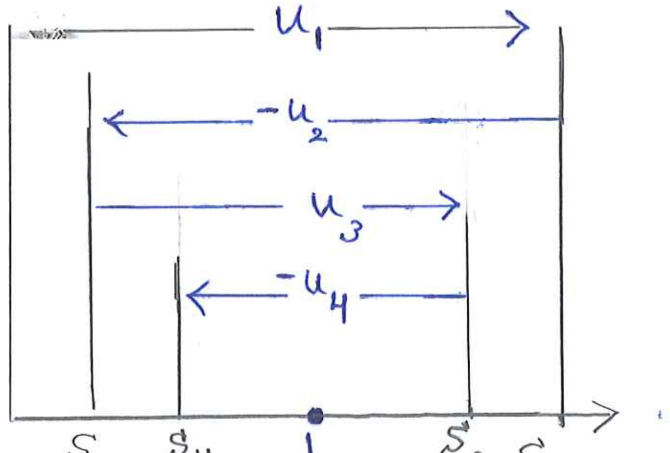
(Remainder) such that $|E| = |L - S_n| < u_{n+1}$,

where $u_{n+1} = |a_{n+1}|$.

Moreover, The sum L lies between any two successive partial sums S_n and S_{n+1}, $\forall n$.

And the Remainder E has the same sign as the first unused term. (The series satisfies !

- $S_1 = u_1 > 0$
- $S_2 = u_1 - u_2 = S_1 - u_2 > 0$
- $S_3 = u_1 - u_2 + u_3 = S_2 + u_3 > 0$
- $S_4 = u_1 - u_2 + u_3 - u_4 = S_3 - u_4$



Example: Use the 4th partial sum S_4 to estimate the sum $\sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1}{2^{n-1}} = 1 - \frac{1}{2} + \frac{1}{4} - \cdots$

Sol: $S_4 = u_1 - u_2 + u_3 - u_4 = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} = \frac{5}{8} = 0.625$

$S_5 = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} = \frac{5}{8} + \frac{1}{16} = \frac{11}{16} \doteq 0.6875$

Remarks:

① Exact Sum: $L = \frac{a}{1-r} = \frac{1}{1+\frac{1}{2}} = \frac{2}{3}$.

Notice that $L = \frac{2}{3} = 0.\overline{666}$ lies between S_4 & S_5

② $|E| = |L - S_4| = \frac{2}{3} - \frac{5}{8} \cong 0.0417$.

By the Estimation theorem:

③ $|E| < u_5 \Rightarrow |E| < \frac{1}{16} \doteq 0.0625$

$\Rightarrow -\frac{1}{16} < E < \frac{1}{16}$

④ Now, $E > 0$, since it has the same sign as the first unused term, (i.e) the same as the sign of a_5 (which is positive).

$\Longrightarrow \quad 0 \leq E < \frac{1}{16} = 0.0625$

Question #49: Estimate the magnitude of the error involved in using the sum of the first four terms to approximate the sum of $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{n}$.

Sol: $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \dots$

$$u_n = \frac{1}{n} \quad , \quad |E| < u_{n+1} = u_5 \overset{|a_5|}{=} \frac{1}{5} = 0.2$$

Question # 53: Determine how many terms should be used to estimate the sum of the series $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{(n^2+3)}$ with error Less than 0.001.

Sol: $|E| < u_{n+1} = \frac{1}{(n+1)^2+3} < 10^{-3}$.

$$(n+1)^2 + 3 > 10^3 \iff (n+1)^2 > 997$$

$$\iff n+1 > \sqrt{997} \iff n > \sqrt{997} - 1 \approx 30.57$$

$\therefore \quad n \geq 31$

Question #57: Approximate the sum $\sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{(2n)!}$ with an error of magnitude less than 5×10^{-6}.

Sol: $\sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{(2n)!} = \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1}{(2(n-1))!}$

$\Rightarrow u_n = \frac{1}{(2(n-1))!}$.

$\Rightarrow |E| < u_{n+1} < 5\times 10^{-6}$

$\Rightarrow \frac{1}{(2n)!} < 5\times 10^{-6} \iff (2n)! > \frac{10^6}{5} = 200,000$

$\iff n \geq 5.$

$\therefore S_5 = 1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \frac{1}{8!} \approx 0.54030.$

Notice that $|E| < u_6 = |a_6| = \frac{1}{9!} = 0.0000027557.$

Absolute and Conditional Convergence.

Def: A series $\sum a_n$ "converges absolutely" (is absolutely convergent) if the corresponding series of absolute values, $\sum |a_n|$, converges.

Example(1). $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ converges absolutely, since the series of absolute values:

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges. (p-series)}$$

Example(2): The Alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is Not absolutely convergent, since

the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Remark: Notice that the Alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges by A.S.T.

($u_n \longrightarrow 0$, $u_n = \frac{1}{n} > 0$, decreasing).

Def: A series that converges but does not converge absolutely "converges conditionally".

Example: Back to example (2), the Alternating harmonic series converges but does not converge absolutely $\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges conditionally.

Theorem(**): The Absolute Convergence Test.

If the series of absolute values $\sum |a_n|$ converges, then the series $\sum a_n$ converges.

(i.e) Absolutely convergent $\Rightarrow$ convergent.

Remark: The converse of the previous theorem(**) is

Not true.

for example: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges by A.S.T

but $\sum_{n=1}^{\infty} \left|\frac{(-1)^{n+1}}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Example : Determine whether the following series converges, converges absolutely, converges conditionally or diverge ?

(1) $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$.

Consider the series of absolute values : $\sum_{n=1}^{\infty} \left|\frac{\sin n}{n^2}\right|$

By Direct Comparision Test :

$$0 \leq |\sin n| \leq 1 \iff 0 \leq \frac{|\sin n|}{n^2} \leq \frac{1}{n^2}$$

$\Rightarrow$ The series is Absolutely convergent, hence it's convergent by theorem(**).

(2) $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$ converges by A.S.T :

(1) $u_n > 0$, $\forall n$, where $u_n = \frac{1}{\sqrt{n}}$.

(2) $\frac{1}{\sqrt{n}} = u_n > \frac{1}{\sqrt{n+1}} = u_{n+1}$, $\forall n$

(3) $\lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0$.

But $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{\sqrt{n}}\right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is divergent (p-series $p<1$)

$\Rightarrow \sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$ is convergent Conditionally.

(3) The Alternating p-series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p} = 1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots$$

The series of absolute values: $\sum_{n=1}^{\infty} \frac{1}{n^p}$

is a p-series which is Convergent if $p > 1$

and divergent if $0 < p \leq 1$.

$\Rightarrow$ The Alternating p-series converges absolutely if $p > 1$.

and if $0 < p \leq 1$ is not converges absolutely

Now, for $0 < p \leq 1$, The Alternating p-series

converges by A.S.T, since: If $u_n = \frac{1}{n^p}$

1) $u_n > 0$, $\forall n$

2) $u_n' = \frac{-1}{n^{p+1}} < 0$, $\forall n$ (i.e) decreasing.

3) $\lim_{n \to \infty} \frac{1}{n^p} = 0$

Conclusion: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p} = \begin{cases} \text{Converges Absolutely, } p > 1 \\ \text{Converges Conditionally, } 0 < p \leq 1 \end{cases}$

(4) $\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{3+n}{5+n}\right)$. The series diverges

since using the nth term test for divergence

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{3+n}{5+n}\right) = 1 \neq 0.$$

Notice that we can't use A.S.T since $\lim_{n\to\infty} u_n \neq 0$.

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(5) $\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{1+n}{n^2}\right)$. The series Converges Conditionally

By A.S.T with $u_n = \frac{1+n}{n^2} = \frac{1}{n^2} + \frac{1}{n}$:

(i) $u_n > 0$, $\forall n$

(ii) $u_n' = -\left(\frac{2}{n^3} + \frac{1}{n^2}\right) < 0$, $\forall n$ (decreasing)

(iii) $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1+n}{n^2} = 0$

So the series converges

But it's not Absolutely convergent since

$$\sum_{n=1}^{\infty} |a_n| = \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^2}}_{\text{convergent p-series}} + \underbrace{\sum_{n=1}^{\infty} \frac{1}{n}}_{\text{divergent harmonic series.}}$$ which is divergent.

(6) $\sum_{n=1}^{\infty} (-1)^n \ln\left(1+\frac{1}{n}\right)$

The series of absolute values : $\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n}\right)$

Take $a_n = \ln\left(1+\frac{1}{n}\right)$ and $b_n = \frac{1}{n}$.

We know $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Now, $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}} \overset{L.H}{=} \lim_{n\to\infty} \frac{\left(\frac{1}{1+\frac{1}{n}}\right)\cancel{\left(\frac{-1}{n^2}\right)}}{\cancel{\left(\frac{-1}{n^2}\right)}} = 1$

Therefore, $\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n}\right)$ diverges by L.C.T

$\Rightarrow \sum_{n=1}^{\infty} (-1)^n \ln\left(1+\frac{1}{n}\right)$ is Not Absolutely Convergent.

Back to the Alternating series : $\sum_{n=1}^{\infty} (-1)^n \ln\left(1+\frac{1}{n}\right)$

Let $u_n = \ln\left(1+\frac{1}{n}\right)$, then:

① $u_n > 0$, $\forall n$

② $u_n' = \left(\frac{1}{1+\frac{1}{n}}\right)\left(\frac{-1}{n^2}\right) < 0$, $\forall n$ (decreasing)

③ $\lim_{n\to\infty} u_n = 0$

$\therefore$ By A.S.T, the series Converges.

(i.e) $\sum_{n=1}^{\infty} (-1)^n \ln\left(1+\frac{1}{n}\right)$ converges Conditionally.

(7) $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{2n+1}$

Notice that $\cos(n\pi) = (-1)^n$, so the series

becomes $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1}$

Now, the series of absolute values : $\sum_{n=1}^{\infty} \frac{1}{2n+1}$

Let $a_n = \frac{1}{2n+1}$ and $b_n = \frac{1}{n}$.

We know that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. By L.C.T:

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/(2n+1)}{1/n} = \frac{1}{2} > 0$. Then

$\sum_{n=1}^{\infty} \frac{1}{2n+1}$ diverges.

Back to $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1}$, Let $u_n = \frac{1}{2n+1}$:

i) $u_n > 0$, $\forall n$

ii) $u_n' = \frac{-2}{(2n+1)^2} < 0$, $\forall n$ (decreasing)

iii) $u_n \longrightarrow 0$ as $n \longrightarrow \infty$.

$\therefore$ By A.S.T, $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1}$ converges Conditionally

(8) $\displaystyle\sum_{n=1}^{\infty} \frac{(-9)^n}{10^n+2n} = \sum_{n=1}^{\infty} \frac{(-1)^n\, 9^n}{10^n+2n}$

The series of absolute values : $\displaystyle\sum_{n=1}^{\infty} \frac{9^n}{10^n+2n}$

Let $a_n = \dfrac{9^n}{10^n+2n}$, $b_n = \left(\dfrac{9}{10}\right)^n$.

Notice that $\displaystyle\sum_{n=1}^{\infty} \left(\frac{9}{10}\right)^n$ is a Convergent Geometric series with $r = \frac{9}{10} < 1$.

Now, $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{9^n}{10^n+2n}\right)\cdot\left(\frac{10}{9}\right)^n$

$$= \lim_{n\to\infty} \frac{10^n}{10^n+2n} = 1 > 0.$$

So by L.C.T : $\displaystyle\sum_{n=1}^{\infty} \frac{9^n}{10^n+2n}$ Converges.

$\Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{(-9)^n}{10^n+2n}$ Converges absolutely.

So by theorem(**), the series converges.

(9) $\sum_{n=1}^{\infty} \frac{(-1)^n 5^{2n+1}}{n^{3n}}$

The series of absolute values : $\sum_{n=1}^{\infty} \frac{5^{2n+1}}{n^{3n}}$

Using root test :

$$\rho = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{5^{\left(\frac{2n+1}{n}\right)}}{n^{\left(\frac{3n}{n}\right)}}$$

$$= \lim_{n\to\infty} \frac{5^{2+\frac{1}{n}}}{n^3} = \lim_{n\to\infty} \frac{5^2 \sqrt[n]{5}}{n^3} = 0 < 1$$

Therefore, the series $\sum_{n=1}^{\infty} \frac{5^{2n+1}}{n^{3n}}$ converges.

Hence, $\sum_{n=1}^{\infty} \frac{(-1)^n 5^{2n+1}}{n^{3n}}$ converges absolutely,

so by theorem(**), the series converges.

Lecture Problems:

Q14) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cdot 3 \cdot \sqrt{n+1}}{\sqrt{n}+1}$

Let $U_n = \dfrac{3 \cdot \sqrt{n+1}}{\sqrt{n}+1}$

$$\Rightarrow \lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{3 \cdot \sqrt{n+1}}{\sqrt{n}+1} \overset{L.H}{=} \lim_{n\to\infty} \frac{\frac{3}{\cancel{2}\sqrt{n+1}}}{\frac{1}{\cancel{2}\sqrt{n}}}$$

$$= \lim_{n\to\infty} \frac{3 \cdot \sqrt{n}}{\sqrt{n+1}} = 3 \cdot \sqrt{\lim_{n\to\infty} \frac{n}{n+1}}$$

$$= 3 \cdot 1 = 3 \neq 0 .$$

So by the nth term test for divergence, the series diverges.

Q19) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cdot n}{n^3+1}$

The series of absolute values : $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^3+1}$

$$\frac{n}{n^3+1} < \frac{n}{n^3} = \frac{1}{n^2}$$

We know that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a Convergent p-series $(P=2)$

So by D.C.T : $\sum_{n=1}^{\infty} \frac{n}{n^3+1}$ Converges.

$\therefore \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{n}{n^3+1}$ Converges absolutely $\Rightarrow$ Converges.

(Q28) $\sum_{n=2}^{\infty} (-1)^{n+1} \cdot \frac{1}{n \ln n}$.

Let $u_n = \frac{1}{n \ln n}$. Then:

(i) $u_n > 0$, $\forall n$.

(ii) $u_n' = \frac{-(n \cdot \frac{1}{n} + \ln n)}{(n \ln n)^2} = \frac{-(1+\ln n)}{(n \ln n)^2} < 0, \forall n \geq 2$

(iii) $\lim_{n \to \infty} \frac{1}{n \ln n} = 0$.

$\therefore$ By A.S.T : The series Converges.

Now, we need to check wether it's absolutely converges or Conditionally Converges.

The series of absolute values: $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

By Integral test, the series of absolute values diverges, since:

$$\int_2^{\infty} \frac{1}{x \ln x}\,dx = \lim_{A\to\infty} \int_2^{A} \frac{1}{x \ln x}\,dx = \lim_{A\to\infty} \ln(\ln x)\Big|_2^{A}$$

$$= \lim_{A\to\infty} \ln(\ln A) - \ln(\ln 2) = \infty .$$

$\Rightarrow \sum_{n=2}^{\infty} (-1)^{n+1} \frac{1}{n \ln n}$ Converges Conditionally.

Q29) $\sum_{n=1}^{\infty} (-1)^n \frac{\tan^{-1} n}{n^2+1}$

The series of absolute values: $\sum_{n=1}^{\infty} \frac{\tan^{-1} n}{n^2+1}$

By Integral Test, the series of absolute values converges, since:

Let $u = \tan^{-1} x$

$du = \frac{1}{x^2+1}\,dx$

$$\int_1^{\infty} \frac{\tan^{-1} x}{x^2+1}\,dx = \lim_{A\to\infty} \int_1^{A} \frac{\tan^{-1} x}{x^2+1}\,dx$$

$$= \lim_{A\to\infty} \frac{(\tan^{-1} x)^2}{2}\Big|_1^{A} = \lim_{A\to\infty} \frac{(\tan^{-1} A)^2}{2} - \frac{(\tan^{-1} 1)^2}{2} = \frac{\left(\frac{\pi}{2}\right)^2}{2} - \frac{\left(\frac{\pi}{4}\right)^2}{2}$$

$\Rightarrow$ The Alternating series Converges.

Summary of Tests:

1. The nth-Term Test: Unless $a_n \to 0$, the series diverges.

2. Geometric Series: $\sum ar^n$ converges if $|r|<1$. Otherwise it diverges.

3. P-series: $\sum \frac{1}{n^p}$ converges if $p>1$. Otherwise it diverges.

4. Series with nonnegative terms: Try the Integral Test, Ratio Test, or Root Test. Try comparing to a known series with the Comparison Test or the Limit Comparison Test.

5. Series with some negative terms: Does $\sum |a_n|$ converge? If Yes, so does $\sum a_n$.

6. Alternating series: $\sum a_n$ converges if the series satisfies the Conditions of A.S.T.

The following diagram shows how to deal with series that contains negative Terms.

$\sum a_n$

$\downarrow$

$\sum |a_n|$

Converges $\rightarrow$ $\sum a_n$ Converges Absolutely

diverges $\rightarrow$ Is $\sum a_n$ Alternating

- Yes $\rightarrow$ A.S.T
  - satisfied $\rightarrow$ $\sum a_n$ converges Conditionally.
  - Not satisfied $\rightarrow$ $\lim\limits_{n\to\infty} u_n \neq 0$ $\rightarrow$ $\sum a_n$ diverges
- NO $\rightarrow$ $\lim\limits_{n\to\infty} a_n \neq 0$ $\rightarrow$ $\sum a_n$ diverges