

10.9 Convergence of Taylor Series

77

Th (Taylor's Theorem)

Assume $f, f', f'', \dots, f^{(n)}$ are continuous on $[a, b]$ and f is differentiable on (a, b) , then there exists a number $c \in (a, b)$ such that:

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}$$

* Note that Taylor's Theorem is a generalization of the MVT.

* If we change b by x , we get Taylor's formula:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)$$

where

$P_n(x)$

• $R_n(x)$ is the Remainder of order n "or the error term" results from approximating f by $P_n(x)$ on an open interval I contains a given by

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x-a)^{n+1} \quad \text{where } c \in (a, x)$$

* Convergence of Taylor Series:

If $\lim_{n \rightarrow \infty} R_n(x) = 0$ for all $x \in I$, then the Taylor Series

generated by f at $x=a$ converges to f on I :

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k$$

Exp show that the Taylor series generated by $f(x) = e^x$ at $x=0$ converges to $f(x)$ for every real value x . 78

$f(x) = e^x$ is smooth on $\mathbb{R} = (-\infty, \infty)$. Thus, has all orders of all derivatives. Taylor's formula is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + R_n(x), \text{ where}$$

$c \in (0, x)$

$$R_n(x) = \frac{e^c}{(n+1)!} x^{n+1}. \text{ Note that } \lim_{n \rightarrow \infty} R_n(x) = e^c \lim_{n \rightarrow \infty} \frac{x^{n+1}}{(n+1)!} = 0$$

Thus, the series converges to e^x for every $x \Rightarrow$ Th 5 Sec. 10.1

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^k}{k!} + \dots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

*Note that $e = \sum_{k=0}^{\infty} \frac{1}{k!}$ as a series: $c \in (0, 1)$
 $0 < c < 1$
 $e^0 < e^c < e^1$
 < 3

$$e = 1 + 1 + \frac{1}{2!} + \dots + \frac{1}{n!} + R_n(1), \text{ where } R_n(1) = \frac{e^c}{(n+1)!} < \frac{3}{(n+1)!}$$

Th (The Remainder Estimation Theorem)

Given Taylor's formula: $f(x) = P_n(x) + R_n(x)$.

If $|f^{(n+1)}(t)| \leq M$ for all $t \in (a, x)$, then the Remainder

$$|R_n(x)| \leq \frac{M}{(n+1)!} |x - a|^{n+1} \text{ *}$$

Further, if * holds for every n ,

then the series converges to $f(x)$.

Exp show that the Taylor series for $\sin x$ at $x=0$ converges for all x .

Recall that Taylor's formula for $\sin x$ at $x=0$ is $c \in (a, x) = (0, x)$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + R_{2n+1}(x)$$

Now apply The Remainder Estimation Theorem:

$$|R_{2n+1}(x)| = \left| \frac{f^{(2n+2)}(c)}{(2n+2)!} x^{2n+2} \right| \leq \frac{|x|^{2n+2}}{(2n+2)!}$$

But $\lim_{n \rightarrow \infty} |R_{2n+1}(x)| \leq \lim_{n \rightarrow \infty} \frac{|x|^{2n+2}}{(2n+2)!} = 0 \forall x$
 \Rightarrow The Maclaurin series for $\sin x$ converges to $\sin x \forall x$.
 $\Rightarrow \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

Exp Show that Taylor series for $\cos x$ at $x=0$ converges to $\cos x$ for every value of x .

(79)

The Taylor's formula for $\cos x$ at $x=0$ is

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + (-1)^n \frac{x^{2n}}{(2n)!} + R_{2n}(x).$$

Since $|\cos x|$ and $|$ its all derivative $| < 1$ we apply the Remainder Estimation Theorem with $M=1$

$$|R_{2n}(x)| \leq \frac{|x|^{2n+1}}{(2n+1)!}. \text{ Now for every } x \text{ we have } R_{2n}(x) \rightarrow 0 \text{ as } n \rightarrow \infty \text{ by } \underline{\text{Th 5 sect 10.1}}$$

Therefore, the series converges to $\cos x$ for every x . Thus,

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

Exp Find the first four nonzero terms in the Maclaurin series for the functions:

$$\begin{aligned} \text{[1]} \quad \frac{1}{3}(2x + x \cos x) &= \frac{2x}{3} + \frac{x}{3} \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \right) \\ &= \frac{2x}{3} + \frac{x}{3} - \frac{x^3}{2! \cdot 3} + \frac{x^5}{3(4!)} - \frac{x^7}{3(6!)} + \dots \\ &= x - \frac{x^3}{6} + \frac{x^5}{72} - \frac{x^7}{2160} + \dots \end{aligned}$$

$$\begin{aligned} \text{[2]} \quad e^x \cos x &= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \right) \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \right) \\ &= \left(\underline{1+x} + \cancel{\frac{x^2}{2!}} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \right) - \left(\cancel{\frac{x^2}{2!}} + \frac{x^3}{2!} + \frac{x^4}{2! \cdot 2!} + \frac{x^5}{2! \cdot 3!} + \dots \right) \\ &\quad + \left(\frac{x^4}{4!} + \frac{x^5}{4!} + \frac{x^6}{2! \cdot 4!} + \dots \right) \\ &= \underline{1+x} - \frac{x^3}{3} - \frac{x^4}{6} + \dots \end{aligned}$$

3] $\cos 2x$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$\cos 2x = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} x^{2n}}{(2n)!} = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \dots$$

$$= 1 - \frac{2^2 x^2}{2!} + \frac{2^4 x^4}{4!} - \frac{2^6 x^6}{6!} + \dots$$

Exp For what values of x can we replace $\sin x$ by $x - \frac{x^3}{3!}$ with an error of magnitude no more than 3×10^{-4} ?

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

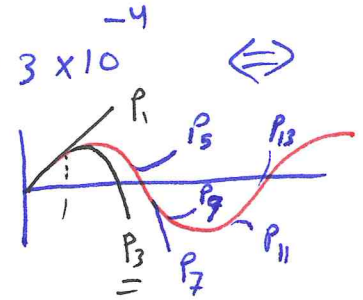
Using the Alternating Series Estimation Theorem:

The error after $\frac{x^3}{3!} < \left| \frac{x^5}{5!} \right|$. Therefore the error

will be less than 3×10^{-4} $U_{n+1} = |a_{n+1}|$

$$|x| < \sqrt[5]{360 \times 10^{-4}} \approx 0.514$$

$$\frac{|x|^5}{120} < 3 \times 10^{-4} \Leftrightarrow$$



Exp Estimate the error if $P_3(x) = x - \frac{x^3}{6}$ is used to estimate the value of $\sin x$ at $x=0.1$

$$f(x) = \sin x = P_3(x) + R_3(x), \text{ where } R_3(x) = \frac{f^{(4)}(c)}{4!} (x-0)^4$$

Apply The Remainder Estimation Th. with $M=1$

$$\text{Error} = |R_3(x)| \leq \frac{|x^4|}{4!} = \frac{(0.1)^4}{24} < 4.2 \times 10^{-6}$$

$$= \frac{f^{(4)}(c)}{4!} x^4$$

$$M \leq |f^{(4)}(c)| \leq 1$$