Chapter 4: Motion in Two and Three Dimensions

Position Vector The location of a particle relative to the origin of a coordinate system is given by a *position vector* $\vec{r}$, which in unit-vector notation is

$$\vec{r} = x\hat{\mathrm{i}} + y\hat{\mathrm{j}} + z\hat{\mathrm{k}}.$$

Here $x\hat{\mathrm{i}}$, $y\hat{\mathrm{j}}$, and $z\hat{\mathrm{k}}$ are the vector components of position vector $\vec{r}$, and x, y, and z are its scalar components (as well as the coordinates of the particle). A position vector is described either by a magnitude and one or two angles for orientation, or by its vector or scalar components.

Displacement If a particle moves so that its position vector changes from $\vec{r}_1$ to $\vec{r}_2$, the particle's *displacement* $\Delta\vec{r}$ is

$$\Delta\vec{r} = \vec{r}_2 - \vec{r}_1.$$

The displacement can also be written as

$$\Delta\vec{r} = (x_2 - x_1)\hat{\mathrm{i}} + (y_2 - y_1)\hat{\mathrm{j}} + (z_2 - z_1)\hat{\mathrm{k}}$$
$$= \Delta x\hat{\mathrm{i}} + \Delta y\hat{\mathrm{j}} + \Delta z\hat{\mathrm{k}}.$$

Average Velocity and Instantaneous Velocity If a particle undergoes a displacement $\Delta\vec{r}$ in time interval Δt, its *average velocity* $\vec{v}_{avg}$ for that time interval is

$$\vec{v}_{avg} = \frac{\Delta\vec{r}}{\Delta t}.$$

velocity or the *instantaneous velocity* $\vec{v}$:

$$\vec{v} = \frac{d\vec{r}}{dt},$$

which can be rewritten in unit-vector notation as

$$\vec{v} = v_x\hat{\mathrm{i}} + v_y\hat{\mathrm{j}} + v_z\hat{\mathrm{k}},$$

where $v_x = dx/dt$, $v_y = dy/dt$, and $v_z = dz/dt$. The instantaneous velocity $\vec{v}$ of a particle is always directed along the tangent to the particle's path at the particle's position.

Average Acceleration and Instantaneous Acceleration If a particle's velocity changes from $\vec{v}_1$ to $\vec{v}_2$ in time interval Δt, its *average acceleration* during Δt is

$$\vec{a}_{avg} = \frac{\vec{v}_2 - \vec{v}_1}{\Delta t} = \frac{\Delta\vec{v}}{\Delta t}.$$

As Δt in Eq. 4-15 is shrunk to 0, $\vec{a}_{avg}$ reaches a limiting value called either the *acceleration* or the *instantaneous acceleration* $\vec{a}$:

$$\vec{a} = \frac{d\vec{v}}{dt}.$$

In unit-vector notation,

$$\vec{a} = a_x\hat{\mathrm{i}} + a_y\hat{\mathrm{j}} + a_z\hat{\mathrm{k}},$$

where $a_x = dv_x/dt$, $a_y = dv_y/dt$, and $a_z = dv_z/dt$.

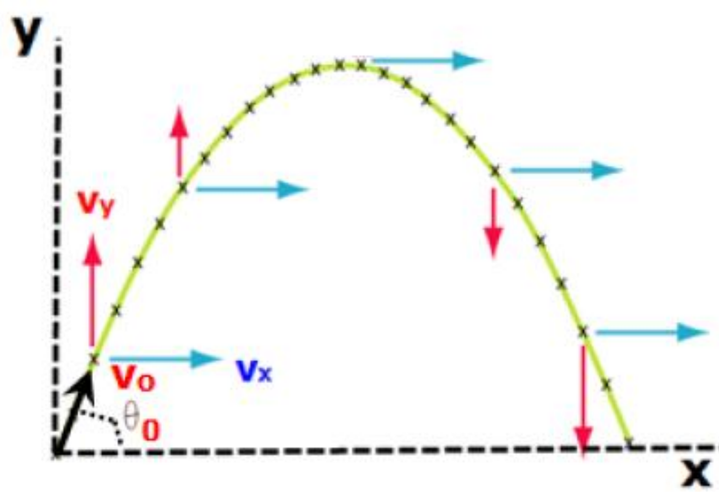
Projectile Motion (2D motion)

The initial velocity

$\vec{v}_{o}=v_{ox}\hat{\imath}+v_{oy}\hat{\jmath}$

$v_{0x}=v_0\cos(\theta_o)$

$v_{0y}=v_0\sin(\theta_o)$



Acceleration

$a_x=0,$
$a_y=-g.$

Velocity

$v_x=v_{0x}$

$v_y=v_{0y}-gt$

$v=\sqrt{v_x^2+v_y^2}$

$v^2{}_y=v^2{}_{0y}-2g(y-y_o)$

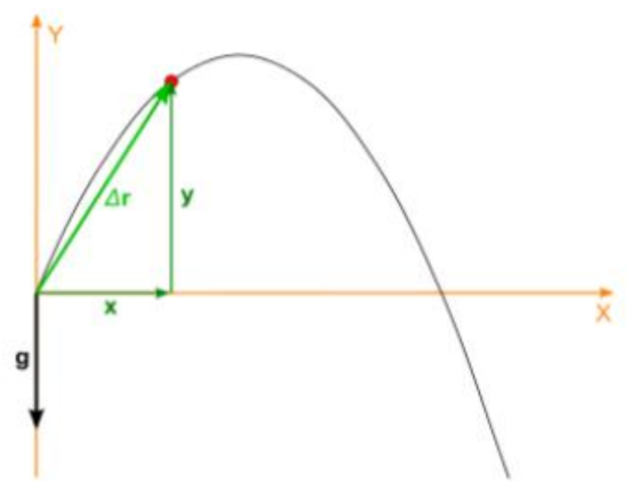
Displacement

$x=v_0t\cos(\theta),$

$y=v_0t\sin(\theta)-\frac{1}{2}gt^2.$

The magnitude of the displacement:

$\Delta r=\sqrt{x^2+y^2}.$



Displacement and coordinates of parabolic throwing

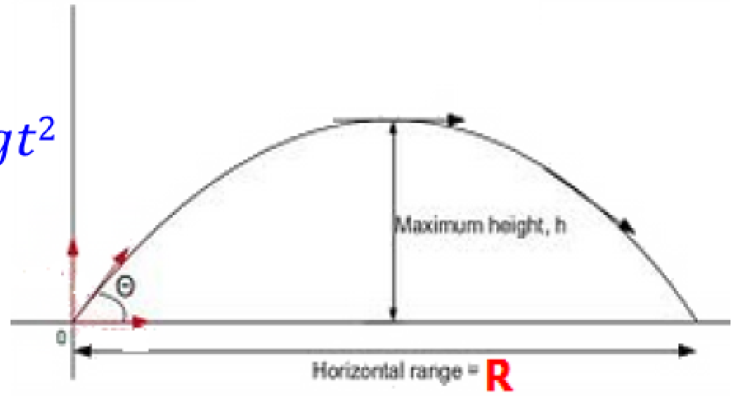
The Horizontal Range

$R = v_{0x} t_f$

$0 - 0 = v_{0y} t - ½ g t^2$

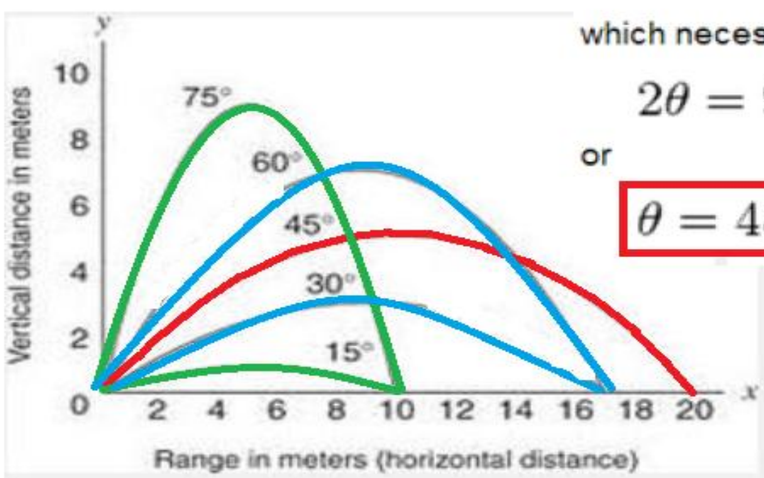
$t_f = 2v_{0y}/g$

$R = 2v_{0x} v_{0y}/g$



$$R = \frac{2v^2_0 \cos(\theta_o) \sin(\theta_o)}{g} = \frac{v^2_0 \sin(2\theta_o)}{g}$$

Maximum Range



R has its maximum value when

$$\sin 2\theta = 1,$$

which necessarily corresponds to

$$2\theta = 90°,$$

or

$$\theta = 45°.$$

Same Range with two different angles

$$\boldsymbol{R_1 = R_2}$$

$$\sin 2\theta_1 = \sin 2\theta_2$$

$$2\theta_1 + 2\theta_2 = \pi$$

$$\theta_1 + \theta_2 = \pi/2$$

The maximum height of projectile

The highest height which the object will reach is known as the peak of the object's motion. The increase of the height will last, until $v_y = 0$, that is,

$$0 = v_0 \sin(\theta) - g t_h.$$

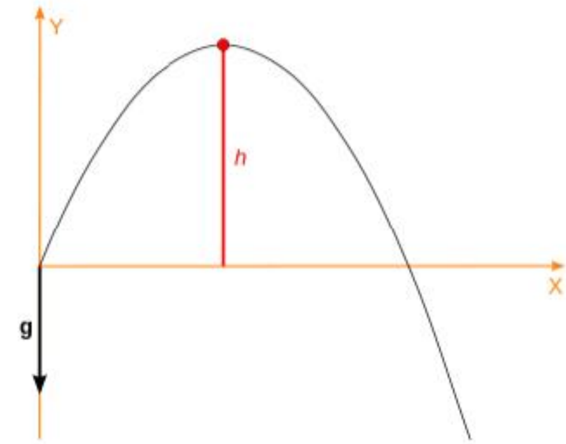
Time to reach the maximum height:

$$t_h = \frac{v_0 \sin(\theta)}{g}.$$

From the vertical displacement

$$h = v_0 t_h \sin(\theta) - \frac{1}{2} g t_h^2$$

$$h = \frac{v_0^2 \sin^2(\theta)}{2g}.$$



Maximum height of projectile

The relation between the range (R) on the horizontal plane and the maximum height (h) is:

$$h = \frac{v_0^2 \sin^2(\theta)}{2g}. \qquad \mathbf{R} = \frac{v_0^2 \sin(2\theta)}{g}.$$

$$h = \frac{R \tan\theta}{4}$$

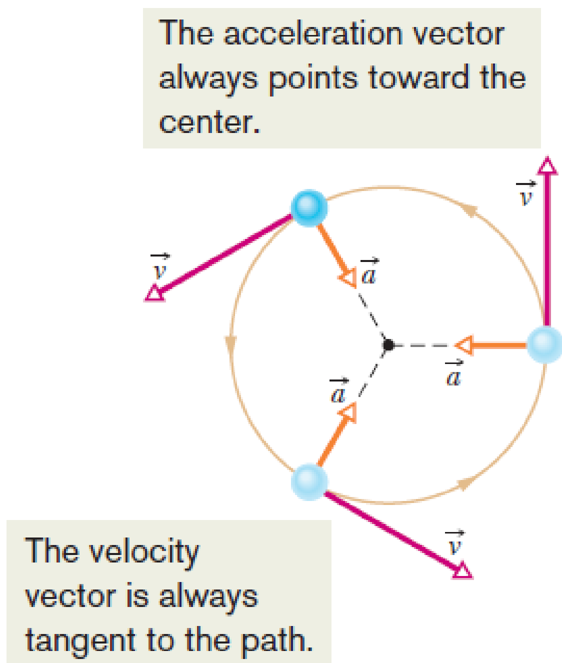
Uniform Circular Motion If a particle travels along a circle or circular arc of radius r at constant speed v, it is said to be in *uniform circular motion* and has an acceleration $\vec{a}$ of constant magnitude

$$a = \frac{v^2}{r} \quad \text{(centripetal acceleration)},$$

The direction of $\vec{a}$ is toward the center of the circle or circular arc, and $\vec{a}$ is said to be *centripetal*. The time for the particle to complete a circle is

$$T = \frac{2\pi r}{v} \quad \text{(period)}.$$

T is called the *period of revolution*, or simply the *period*, of the motion.



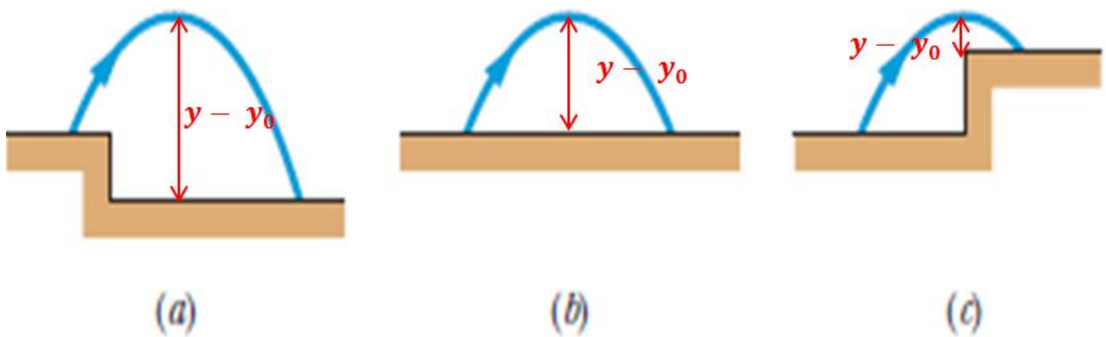
Relative Motion When two frames of reference A and B are moving relative to each other at constant velocity, the velocity of a particle P as measured by an observer in frame A usually differs from that measured from frame B. The two measured velocities are related by

$$\vec{v}_{PA} = \vec{v}_{PB} + \vec{v}_{BA},$$

where $\vec{v}_{BA}$ is the velocity of B with respect to A. Both observers measure the same acceleration for the particle:

$$\vec{a}_{PA} = \vec{a}_{PB}.$$

Q-5)The below figure shows three situations in which identical projectiles are launched (at the same level) at identical initial speeds and angles. The projectiles do not land on the same terrain, however. Rank the situations according to the final speeds of the projectiles just before they land, greatest first.



Maximum height → Land

$$y - y_0 = v_{0y}\, t - \frac{1}{2} g\, t^2$$

At the maximum height $v_{0y} = 0$

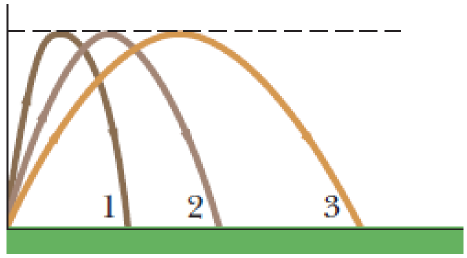
$$y - y_0 = -\frac{1}{2} g\, t^2$$

$$t_a > t_b > t_c$$

$v_y = v_{0y} - gt; v_{0y} = 0$

$$\boldsymbol{v_{y,a} > v_{y,b} > v_{y,c}}$$

Q-9) The below figure shows three paths for a football kicked from ground level. Ignoring the effects of air, rank the paths according to (a) time of flight, (b) initial vertical velocity component, (c) initial horizontal velocity component, and (d) initial speed, greatest first.



In the three paths, the football reached the same maximum height (h) so the initial vertical velocity component is the same in the three situations

$$h = \frac{v_{0y}^2}{2g}$$

$$\text{Time of flight} = \frac{2\,v_{0y}}{g}$$

(a) All tie (the same tome of flight)

(b) All tie (the same initial vertical velocity component)

(c) Initial horizontal velocity component:

The Horizontal Range (R):

$$R = v_{0x}t$$

$$R_3 > R_2 > R_1$$

$$\boldsymbol{v_{0x,3} > v_{0x,2} > v_{0x,1}}$$

(d) The initial speed:

$$v = \sqrt{(\boldsymbol{v_{0x}})^2 + (\boldsymbol{v_{0y}})^2}$$

$$v_3 > v_2 > v_1$$

Q -13) (a) Is it possible to be accelerating while traveling at constant speed? Is it possible to round a curve with (b) zero acceleration and (c) a constant magnitude of acceleration?

(a) Yes
(b) No
(c) Yes

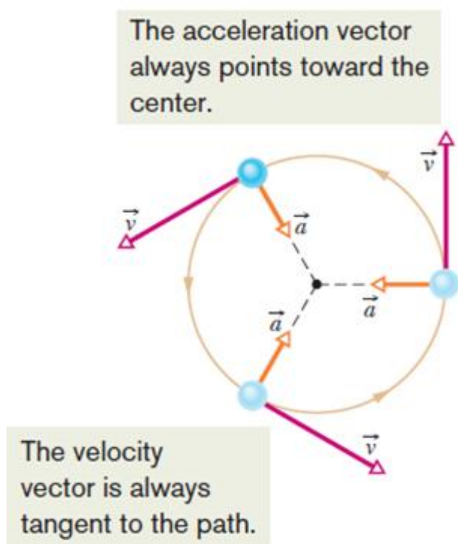
Uniform Circular Motion If a particle travels along a circle or circular arc of radius r at constant speed v, it is said to be in *uniform circular motion* and has an acceleration $\vec{a}$ of constant magnitude

$$a = \frac{v^2}{r} \quad \text{(centripetal acceleration)},$$

The direction of $\vec{a}$ is toward the center of the circle or circular arc, and $\vec{a}$ is said to be *centripetal*. The time for the particle to complete a circle is

$$T = \frac{2\pi r}{v} \quad \text{(period)}.$$

T is called the *period of revolution*, or simply the *period*, of the motion.



P-4) The minute hand of a wall clock measures 10 cm from its tip to the axis about which it rotates. The magnitude and angle of the displacement vector of the tip are to be determined for three time intervals. What are the (a) magnitude and (b) angle from a quarter after the hour to half past, the (c) magnitude and (d) angle for the next half hour, and the (e) magnitude and (f) angle for the hour after that?

Case1: from a quarter after the hour to half past

a) The magnitude

$\vec{r}_i = 10\hat{i}$

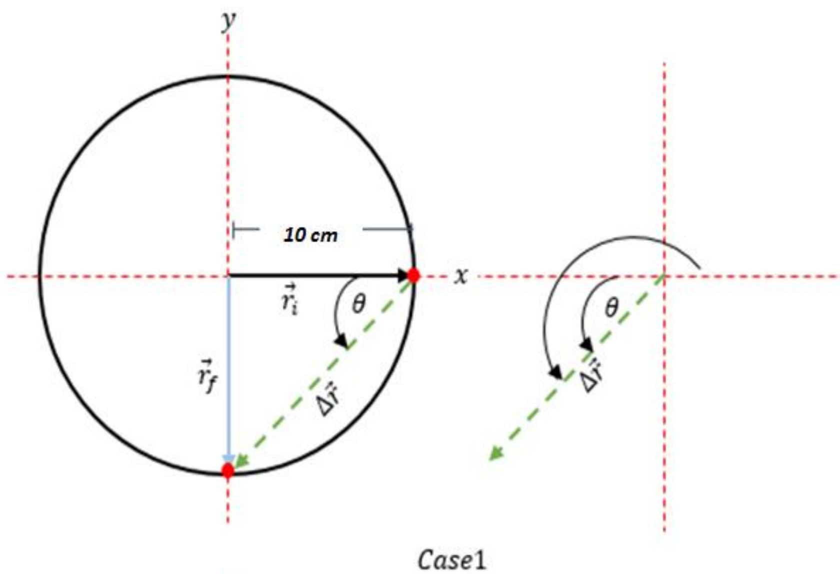
$\vec{r}_f = -10\hat{j}$

$\Delta\vec{r} = \vec{r}_f - \vec{r}_i = -10\hat{j} - 10\hat{i}$

$\Delta\vec{r} = -10\hat{i} - 10\hat{j}$ cm

$|\Delta\vec{r}| = \sqrt{(-10)^2 + (-10)^2}$ cm

$|\Delta\vec{r}| = 14.1$ cm



b) The angle

$$\theta = \tan^{-1}\left(\frac{-10}{-10}\right)$$

$\theta = 45^\circ$ Counter clockwise relative to $-x$ or $\theta = 225^\circ$ Counter clockwise relative to $+x$

Case2: the next half hour

c) The magnitude

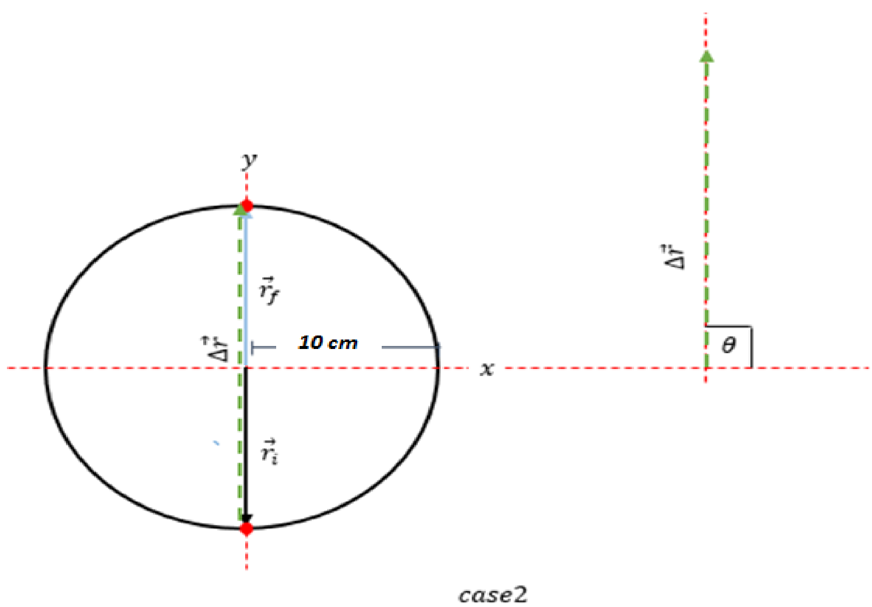
$\vec{r}_i = -10\hat{j}$

$\vec{r}_f = +10\hat{j}$

$\Delta\vec{r} = \vec{r}_f - \vec{r}_i = 10\hat{j} - (-10\hat{j})$

$\Delta\vec{r} = 20\hat{j}$ cm

$|\Delta\vec{r}| = 20$ cm



d) The angle

$$\theta = \tan^{-1}\left(\frac{20}{0}\right)$$

$\theta = 90^\circ$ Counter clockwise relative to $+x$

Case3: the hour after that

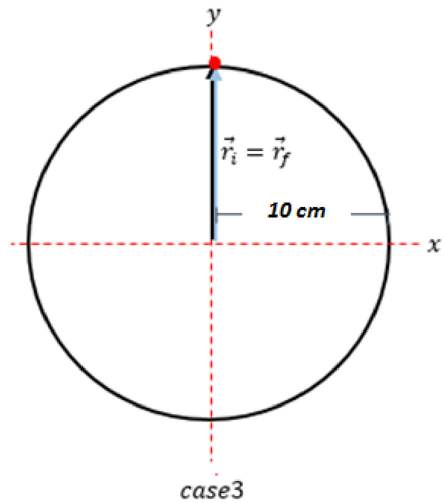
e) The magnitude

$\vec{r}_i = +10\hat{j}$

$\vec{r}_f = +10\hat{j}$

$\Delta\vec{r} = \vec{0}$

$|\Delta\vec{r}| = \text{zero}$



f) The angle

$\theta = \text{zero}$

P-6) An electron's position is given by $\vec{r} = 3.00t\hat{\imath} - 4.00t^2\hat{\jmath} + 2.00\hat{k}$, with t in seconds and $\vec{r}$ in meters, (a) In unit-vector notation, what is the electron's velocity $\vec{v}(t)$? At t = 2.00s, what is $\vec{v}$ (b) in unit-vector notation and as (c) a magnitude and (d) an angle relative to the positive direction of the x axis ?

(a) $\vec{v}(t) = \frac{d\vec{r}(t)}{dt} = (3.00\,\mathrm{m/s})\hat{\imath} - (8.00t\,\mathrm{m/s^2})\hat{\jmath}$

(b) $\vec{v}(t = 2.00s) = (3.00\hat{\imath} - 16.00\hat{\jmath})\ \mathrm{m/s}$

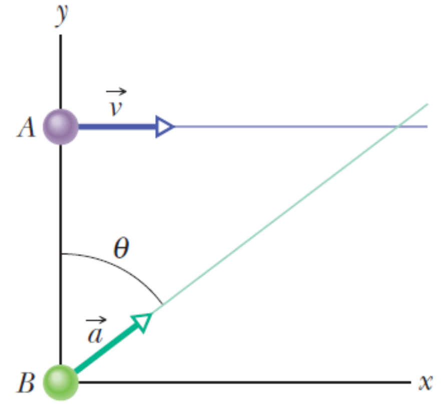
(c) The magnitude of the velocity at t = 2.00s:

The speed at t = 2.00s: $|\vec{v}(t = 2.00s)| = \sqrt{(3)^2 + (-16)^2} = 16.3\,\mathrm{m/s}$

(d) The angle of the velocity at that moment:

$$\theta = \tan^{-1}\left(\frac{-16}{3}\right) = -79.38^{\circ}$$

$\theta = 280.62^{\circ}$ Counter clockwise relative to $+x$ (79.38°clockwise from the +x; the velocity vector at t = 2.00s is in the fourth quadrant) p-20) In the below figure, particle A moves along the line y = 30 m with a constant velocity $\vec{v}$ of magnitude 3.0 m/s and parallel to the x axis. At the instant particle A passes the y axis, particle B leaves the origin with a zero initial speed and a constant acceleration $\vec{a}$ of magnitude 0.40 m/s^2.What angle θ between $\vec{a}$ and the positive direction of the y axis would result in a collision?



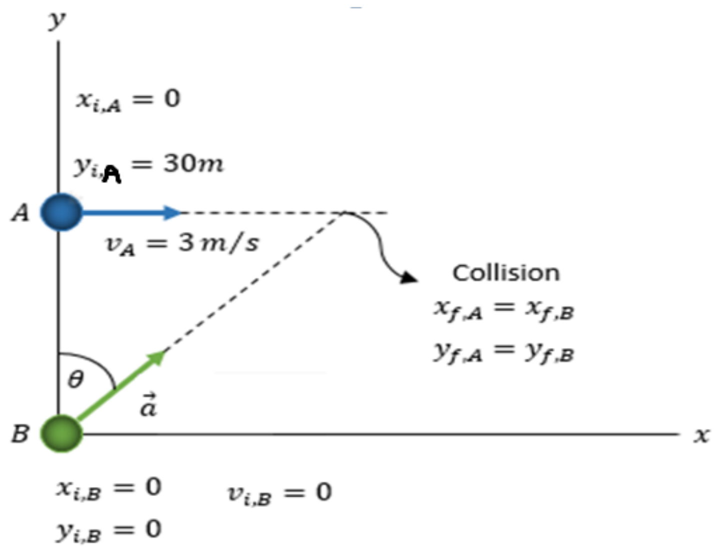
Particle A:

Constant velocity $\vec{v}$ →→ ZERO acceleration

$x_{f,A} - x_{i,A} = v_A t$

$x_{f,A} = (3t)\ m$

$y_{f,A} = y_{i,A} = 30\ m$



Particle B:

$x_{f,B} - x_{i,B} = v_{ix,B} t + \frac{1}{2} a_x t^2$

$x_{f,B} = \frac{1}{2} a(\sin\theta) t^2$

$x_{f,B} = \frac{1}{2} * 0.4 * (\sin\theta) t^2$

$x_{f,B} = 0.2(\sin\theta) t^2$

$y_{f,B} - y_{i,B} = v_{iy,B} t + \frac{1}{2} a_y t^2$

$y_{f,B} = \frac{1}{2} a(\cos\theta) t^2$

$y_{f,B} = \frac{1}{2} * 0.4 * (\cos\theta) t^2$

$y_{f,B} = 0.2(\cos\theta) t^2$

At collision the two particles will be in the same position, thus

$$\mathbf{x_{f,A}} = \mathbf{x_{f,B}}$$

$$3t = 0.2(\sin\theta)t^2$$

$$t = \frac{15}{\sin\theta} \dots\dots\dots\dots\dots\dots (1)$$

And

$$\mathbf{y_{f,A}} = \mathbf{y_{f,B}}$$

$$30 = 0.2(\cos\theta)t^2$$

Substituting t from (1) you get

$$30 = 0.2(\cos\theta)\left(\frac{15}{\sin\theta}\right)^2$$

$$\frac{2}{3} = \frac{\cos\theta}{\sin^2\theta}$$

$$\frac{2}{3} = \frac{\cos\theta}{1-\cos^2\theta}$$

$$2\cos^2\theta + 3\cos\theta - 2 = 0$$

$$\cos\theta = \frac{-3 \pm \sqrt{3^2 - 4*2*(-2)}}{4}$$

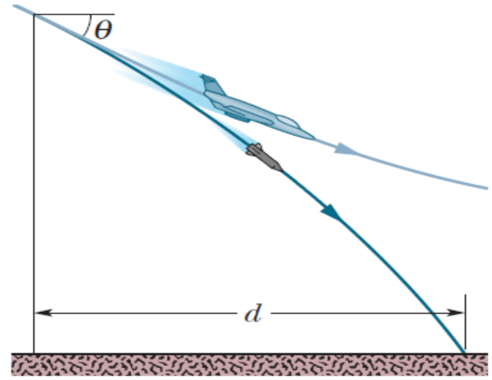
Thus,

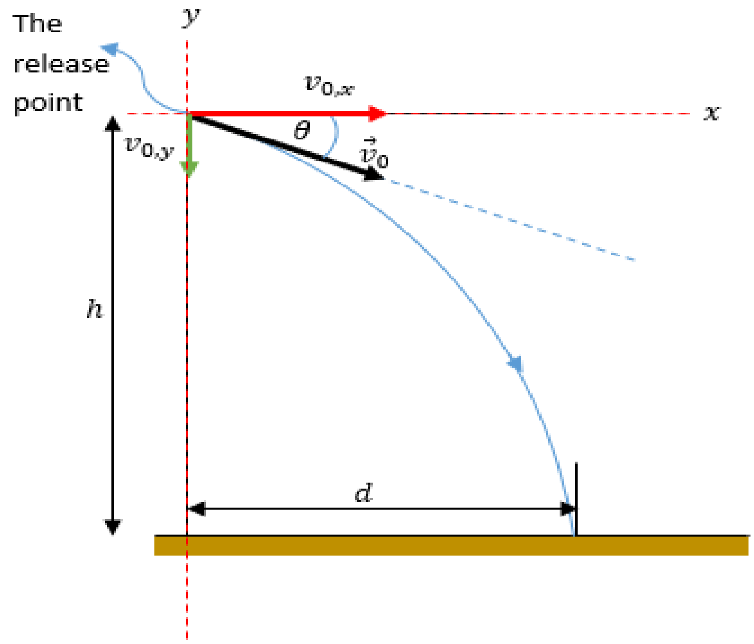
$$\cos\theta = \frac{1}{2} \Rightarrow \theta = 60^{\circ}$$

Or

$\cos\theta = -2$, this case is impossible

4-27) A certain airplane has a speed of 290.0 km/h and is diving at an angle of $\theta = 30.0°$ below the horizontal when the pilot releases a radar decoy. The horizontal distance between the release point and the point where the decoy strikes the ground is d =700 m. (a) How long is the decoy in the air? (b) How high was the release point?





$\theta = 30^{\circ}$

$v_0 = 290\ \mathrm{Km/h} = 80.6\ \mathrm{m/s}$

a) How long is the decoy in the air ?

$$d = v_{0,x} t = v_0 \cos\theta_0\, t$$

$$t = \frac{d}{v_0 \cos\theta_0} = \frac{700}{80.6 \cos(30^{\circ})} = 10\ \mathrm{sec}$$

b) How high was the release point ?

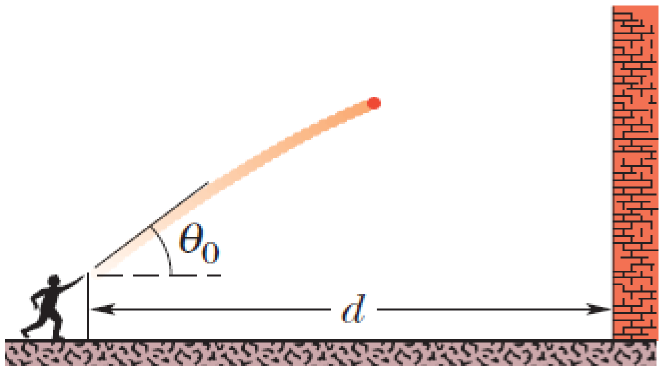
$$y_f - y_0 = v_{0,y} t + \frac{1}{2} g t^2$$

$$-h - 0 = -v_0 \sin\theta_0\, t + \frac{1}{2} g t^2$$

$$-h = -80.6 * (\sin 30^{\circ}) * (10) + \frac{1}{2}(-10) * (10)^2$$

$$h = 903\ \mathrm{m}$$

p-32) You throw a ball toward a wall at speed 25.0 m/s and at angle $\theta_\circ = 40.0°$ above the horizontal. The wall is distance d = 22.0 m from the release point of the ball. (a) How far above the release point does the ball hit the wall? What are the (b) horizontal and (c) vertical components of its velocity as it hits the wall? (d) When it hits, has it passed the highest point on its trajectory?



$v_{0x} = 25.0\cos(40.0°)\frac{m}{s}$, $v_{0y} = 25.0\sin(40.0°)\frac{m}{s}$

d = 22.0 m is the horizontal range of the ball

$d = v_{0x}t = v_0\cos(\theta_0)\,t$

$$t = \frac{d}{v_0\cos(\theta_0)} = \frac{22.0\,m}{25.0\cos(40.0°)\,m/s} = 1.15\,s$$

$t = 1.15\,s\ is\ the\ ball\ flight\ time$

(a) $y_f - y_0 = v_{0,y}t + \frac{1}{2}gt^2$

$$\Delta y = 25.0\sin(40.0°)\,(1.15) + \frac{1}{2}(-9.8)(1.15)^2 = 12.0\,m$$

(b)The horizontal component of ball's velocity when it hits the wall equals the initial horizontal component of the velocity $(a_x = zero)$

$$v_x = 25.0\cos(40.0°)\,m/s = 19.15\,m/s$$

(c) The vertical component of the ball's velocity when it hits the wall:

$$v_y = v_{0y} + gt = 25.0\sin(40.0°) - 9.8\,(1.15) = 4.8\,m/s$$

The ball's velocity when it hits the wall is $(19.15\,m/s\,\hat{\imath} + 4.8\ m/s\,\hat{\jmath})$
The ball's speed when it hits the wall is $19.74\ m/s$

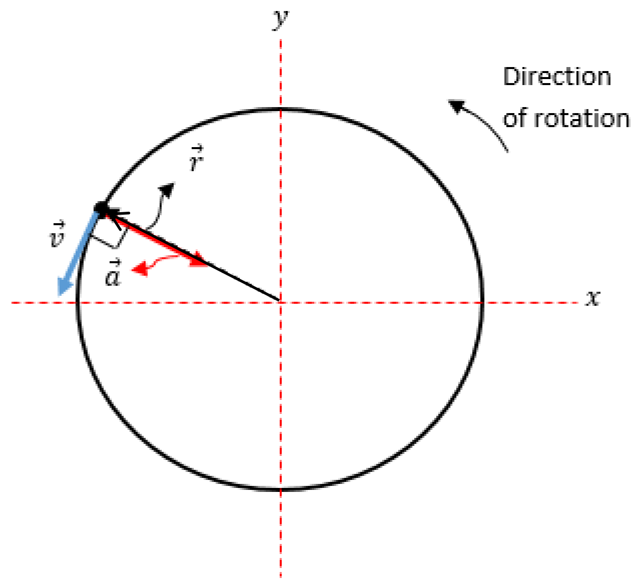
(d) The vertical component of the ball's velocity when it hits the wall is positive ($v_y > 0$), so it has not reached the highest point yet.

P-60) A centripetal-acceleration addict rides in uniform circular motion with radius r = 3.00 m. At one instant his acceleration is $\vec{a} = (6.00\ m/s^2)\hat{\imath} + (-4.00\ m/s^2)\hat{\jmath}$. At that instant, what are the values of (a) $\vec{v}.\vec{a}$ and (b) $\vec{r} \times \vec{a}$?

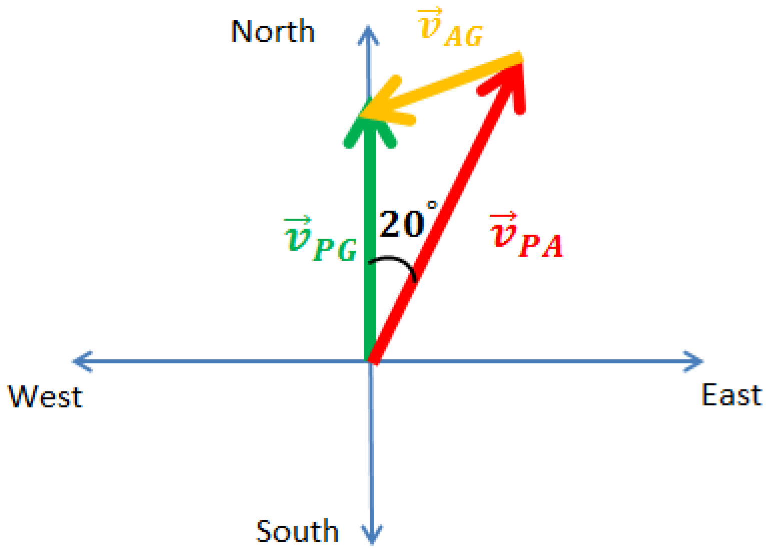
As you can see in the figure the angle between $\vec{v}$ and $\vec{a}$ is 90°, and between $\vec{r}$ and $\vec{a}$ is 180°

a) $\mathbf{\vec{v}.\vec{a} = v\, a \cos\,(90^{\circ}) = 0}$

b) $\mathbf{\vec{r} \times \vec{a} = r\, a \sin\,(180^{\circ}) = 0}$



p-76) A light plane attains an airspeed of 500 km/h. The pilot sets out for a destination 800 km due north but discovers that the plane must be headed 20.0° east of due north to fly there directly. The plane arrives in 2.00 h. What were the (a) magnitude and (b) direction of the wind velocity?



$\vec{v}_{PA} = 500\ Km/h\ (\cos 70^\circ\,\hat{\imath} + \sin 70^\circ\,\hat{\jmath})$

$\vec{v}_{PA} = (\,171.0\,\hat{\imath} + 469.8\,\hat{\jmath}\,)\ Km/h$

$\vec{v}_{PG} = \dfrac{800\ Km}{2.00\ h} = 400\ Km/h\ \hat{\jmath}$

$$\vec{v}_{PG} = \vec{v}_{PA} + \vec{v}_{AG}$$

$$400\ Km/h\ \hat{\jmath} = (\,171.0\,\hat{\imath} + 469.8\,\hat{\jmath}\,)\ Km/h + \vec{v}_{AG}$$

$$\vec{v}_{AG} = (-171.0\,\hat{\imath} - 69.8\,\hat{\jmath}\,)\ Km/h$$

(a) $v_{AG} = \sqrt{(-171.0)^2 + (-69.8)^2} = 184.7\ Km/h$

(b) $\theta = tan^{-1}\left(\frac{-69.8}{-171.0}\right) = 22.2^\circ$ **South of West**