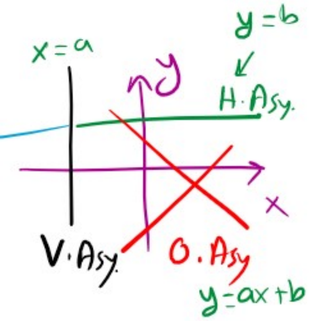
Q. How can we draw the rational function

$$f(x) = \frac{g(x)}{h(x)} ? \qquad g, h \text{ poly}$$

A. If we understand Asymptotes, then it helps to draw $f(x)$

lines



$f(x) = \frac{g(x)}{h(x)}$

Asymptotes

- Horizontal Asy. (H. Asy) [1]
- Vertical Asy. (V. Asy) [2]
- Oblique Asy. (O. Asy) [3]

[1] H. Asy: $y = b$ is H. Asy for $f(x)$ if

$$\lim_{x \to \infty} f(x) = \underset{(\text{عدد ثابت})}{b} \quad \text{or} \quad \lim_{x \to -\infty} f(x) = \underset{(\text{عدد ثابت})}{b}$$

Exp $f(x) = \frac{1}{x}$ Find H. Asy.

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{1}{x} = 0 \Rightarrow \boxed{y = 0} \text{ H. Asy.}$$

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{1}{x} = 0$$

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{1}{x} = 0$$

$\boxed{2}$ V. Asy : $x = a$ is V. Asy for $f(x)$ if

check zeros of $h(x)$

$$\lim_{x \to a^+} f(x) = \infty \text{ or } -\infty$$

Or

$$\lim_{x \to a^-} f(x) = \infty \text{ or } -\infty$$

Exp $f(x) = \frac{1}{x}$ Find V. Asy.

$x = 0$

check $x = 0$

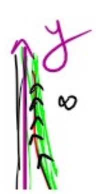
$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{1}{x} = \frac{1}{\text{small } +} = +\infty$$

so $x = 0$ is V. Asy.

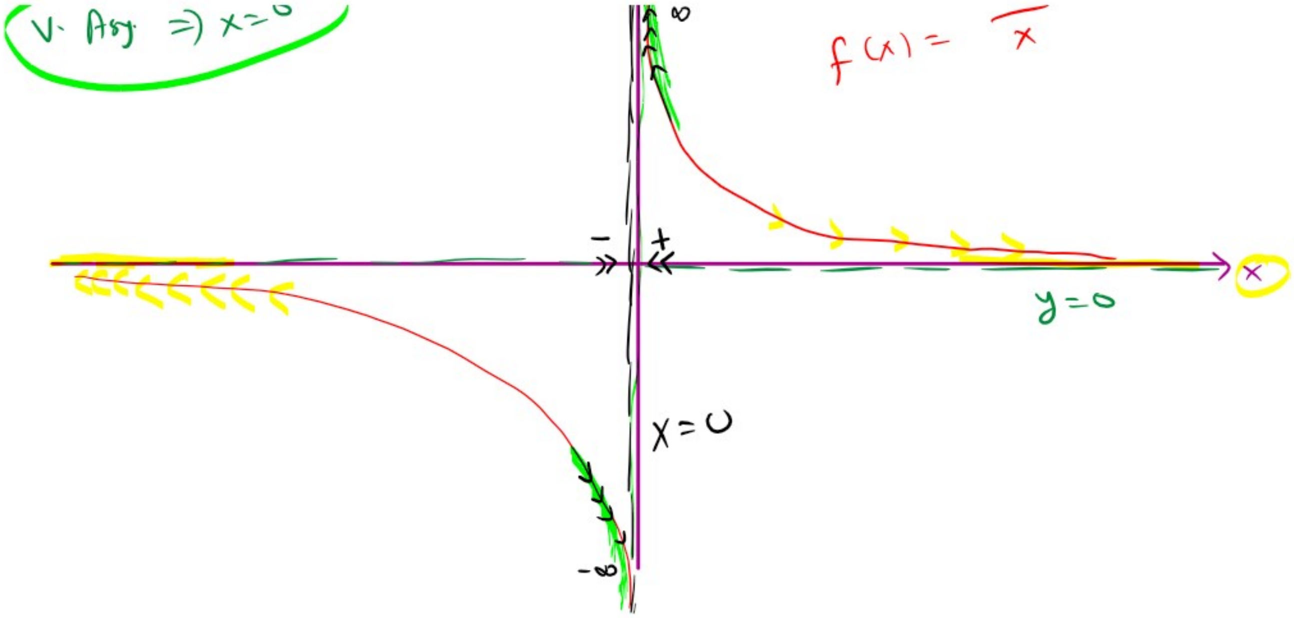
$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{1}{x} = \frac{1}{\text{small } -} = -\infty$$

$\boxed{\text{H. Asy} \Rightarrow y = 0}$

V. Asy $\Rightarrow x = 0$



$f(x) = \frac{1}{x}$



3) O. Asy: $y = ax+b$ is O. Asy. if

مائل

degree $g(x)$ is one greater than degree $h(x)$

$f(x) = \frac{g(x)}{h(x)} = \frac{x^2}{x+1}$ ✓ $g(x)$: numerator, $h(x)$: denomenator

✓ $\exists$ oblique Asy. $\Rightarrow \nexists$ H. Asy. $\rightarrow \lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^2}{x+1} = \infty$

$f(x) = \frac{1}{x}$

$$\boxed{\nexists \text{ O. Asy} \Rightarrow \exists \text{ H. Asy.}}$$

$f(x) = \frac{x^4}{x+1}$ has no O. Asy. has no. H. Asy

To find O. Asy. $\Rightarrow$ we use long division
القسمة الطويلة

Exp ① $f(x) = \dfrac{x^3+1}{x^2-1}$ Find All Asy. Sketch.

② $f(x) = \dfrac{x+1}{x-1}$

② H. Asy. $\Rightarrow$ $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \dfrac{x+1}{x-1} = \lim_{x\to\infty} \dfrac{1+\frac{1}{x}}{1-\frac{1}{x}}$

$= \dfrac{1+0}{1-0} = \dfrac{1}{1} = 1$

$\boxed{y=1}$ H. Asy.

$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \dfrac{x+1}{x-1} = 1$

V. Asy $\Rightarrow$ check $x=1$

0.999 1 1.001

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \dfrac{x+1}{x-1} = \dfrac{2}{\text{small}+} = \infty$

$\boxed{x=1}$ is V. Asy

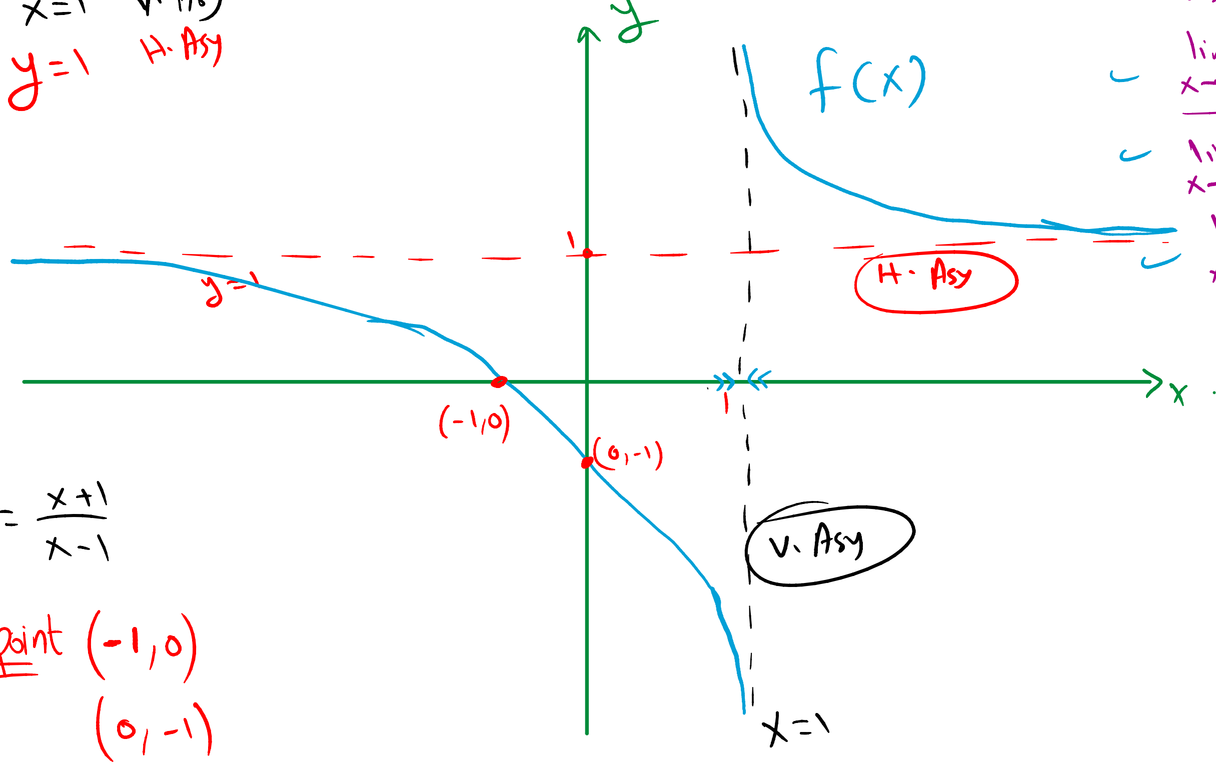
$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \frac{x+1}{x-1} = \frac{2}{\text{small} -} = -\infty$$

3) f has no O. Asy. since f has H. Asy.

$$f(x) = \frac{x^{①}+1}{x^{①}-1} \quad \times \quad \nexists \text{ O. Asy.}$$

x=1 V. Asy

y=1 H. Asy



$\lim_{x \to \infty} f(x) = 1$

$\lim_{x \to -\infty} f(x) = 1$

$\lim_{x \to 1^+} f(x) = \infty$

$\lim_{x \to 1^-} f(x) = -\infty$

$f(x) = \frac{x+1}{x-1}$

Keypoint (-1,0)

(0,-1)

H. Asy $\lim_{x \to \infty} f(x) = \text{عدد}$

X

O. Asy. (deg num = deg den + 1) ⇒ lim ( ) = ∞ or −∞

O. Asy. $\dfrac{\text{درجة البسط أعلى من درجة المقام}}{\text{المقام}}$ $\Rightarrow$ $\lim\limits_{x \to \infty}$ ( ) $= \infty$ or $-\infty$)

If f has H. Asy. $\Rightarrow$ $\nexists$ O. A.

درجة البسط

$f(x) = \dfrac{x}{x+1}$

درجة المقام

$\nexists$ O. A.

رقم درجة البسط أقل

$= \dfrac{\quad}{x^2+5}$

درجة المقام

H. + V.

H. + O.

V. + O.

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2) $f(x) = \dfrac{x^3+1}{x^2-1}$

$\exists$ O. Asy $\Rightarrow$ $\nexists$ H. Asy.

1) $\nexists$ H. Asy

O. Asy $\Rightarrow$ long division

$\boxed{x}$

2) $\exists$ O. Asy. $\Rightarrow$ long division

$$\begin{array}{r} x \\ x^2-1 \overline{\smash{)}\, x^3+1} \\ -x^3+x \\ \hline x+1 \end{array}$$

$$f(x) = \frac{x^3+1}{x^2-1} = x + \frac{x+1}{x^2-1}$$

O. As

$y = x$ is O. Asy

3) V. Asy Check $x^2-1=0$

$x^2 = 1$

$\boxed{x = \pm 1}$

1.001

1

Check $\boxed{x=1}$ $\Rightarrow \lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} \frac{x^3+1}{x^2-1} = \frac{2}{\text{small}+} = \infty$

$\Rightarrow$ $x=1$ is V. As

0.999

1

$\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} \frac{x^3+1}{x^2-1} = \frac{2}{\text{small}-} = -\infty$

check $\boxed{x=-1}$ $\Rightarrow \lim\limits_{x\to -1^+} f(x) = \lim\limits_{x\to -1^+} \frac{x^3+1}{x^2-1} = \lim\limits_{x\to -1^+} \frac{3x^2}{2x}$ $\frac{0}{0}$

$= \frac{3}{-2} = -\frac{3}{2}$

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} \frac{x^3+1}{x^2-1} = \lim_{x \to -1^-} \frac{3x^2}{2x} = -\frac{3}{2}$$

Hence, $x = -1$ is not V. Asy

So $x = -1$ is removable discont.

$y = x$ O. Asy.

$x = 1$ V. Asy

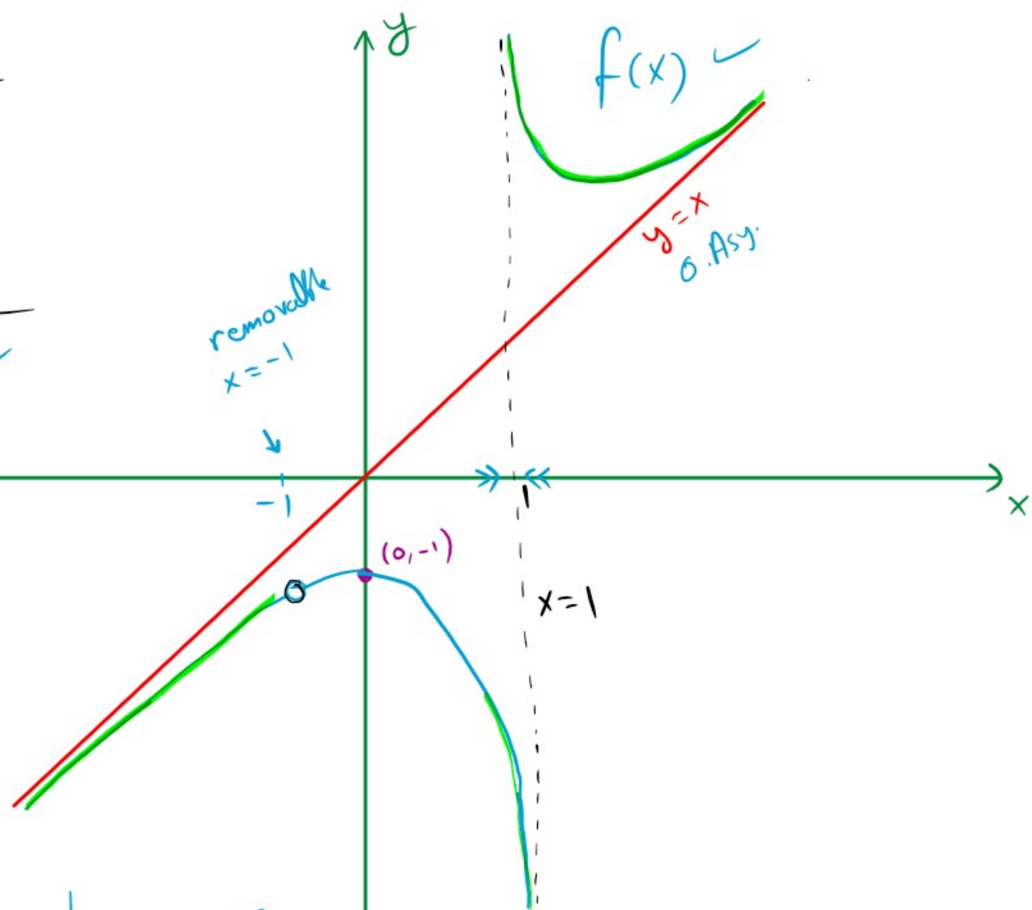
$$f(x) = \frac{x^3+1}{x^2-1}$$

$D(f) = \mathbb{R} \setminus \{\pm 1\}$

$\lim_{x \to 1^+} f(x) = \infty$

$\lim_{x \to 1^-} f(x) = -\infty$

Keypoint $(0, -1)$



Find cont. extension for $f(x)$ at $x = -1$

$$F(x) = \begin{cases} f(x) & \text{if } x \neq -1 \\ \lim_{x \to -1} f(x) & \text{if } x = -1 \end{cases} = \begin{cases} \frac{x^3+1}{x^2-1} & \text{if } x \neq -1 \\ -\frac{3}{2} & \text{if } x = -1 \end{cases}$$

$x \to -1$

$\hookrightarrow$ cont. at $x = -1$

$f(x) = \frac{2x^3 + 1}{x^2 - 5}$ Find O. Asy

$\frac{2x^3}{x^2} = 2x$

$= 2x + \frac{10x + 1}{x^2 - 5}$

$y = 2x$ ✓

$$\begin{array}{r} 2x \\ x^2 - 5 \overline{\smash{)}\, 2x^3 + 1} \\ -2x^3 + 10x \\ \hline 10x + 1 \end{array}$$

✓ $\lim\limits_{x \to \text{عدد}} \frac{\square}{\square} = \frac{\text{رقم غير الصفر}}{\text{صفر}}$ ∞ , $-\infty$

$\frac{1}{\text{small}+}$

$\frac{1}{\text{small}-}$

Exp $\lim\limits_{x \to 2} \frac{3}{x-2} = \text{DNE}$

$\frac{3}{0}$ ∞ , $-\infty$

$\lim\limits_{x \to 2^+} \frac{3}{x-2}$

2.001

2

$\lim\limits_{x \to 2^-} \frac{3}{x-2}$

$\frac{3}{\text{small}-}$

1.99

2

$x \to 2^+$

$$\frac{3}{\text{small}+} = \infty$$

$$\frac{-}{\text{small}-} = -\infty$$

$$\lim_{x \to 2} \frac{\square}{\square}$$

$\frac{0}{0}$ or $\frac{\infty}{\infty}$

removable disc. $\Rightarrow$ not V. Asy.

Exp

$$\lim_{x \to -1^+} \frac{x}{x+1} = \cancel{\frac{-1}{0}} = \infty$$

-0.999

-1

$$\frac{1}{\text{small}+} = \infty$$

$$\frac{\text{رقم غير الصفر}}{\text{صفر}} \to \text{small}+ ,\quad \text{small}-$$

$\frac{0}{0}, \frac{\infty}{\infty} \longrightarrow$ نطبق L'Hopital Rule

Remark : The graph of $f(x)$ might cross the Asy.

Exp $f(x) = \frac{\sin x}{x}$ Find Asy. sketch f

① H. Asy.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\sin x}{x}$$

$$\frac{-1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}$$

$$-\lim_{x\to\pm\infty} \frac{1}{x} \le \lim_{x\to\pm\infty} \frac{\sin x}{x} \le \lim_{x\to\pm\infty} \frac{1}{x}$$

$$0 \qquad\qquad 0 \text{ by S.T} \qquad\qquad 0$$

$$\lim_{x\to\pm\infty} \frac{\sin x}{x} = 0 \implies \boxed{y=0 \text{ is H. Asy}}$$

$\nexists$ O. Asy.

② V. Asy. $f(x) = \frac{\sin x}{x} \implies$ check $\boxed{x=0}$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{\sin x}{x} = \lim_{x \to 0^+} \frac{\cos x}{1} = \cos 0 = 1 \qquad \frac{0}{0}$$

مش ∞ , $-\infty$ X

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{\sin x}{x} = \lim_{x \to 0^-} \frac{\cos x}{1} \qquad \frac{0}{0}$$

$$= \cos 0 = 1 \qquad X$$

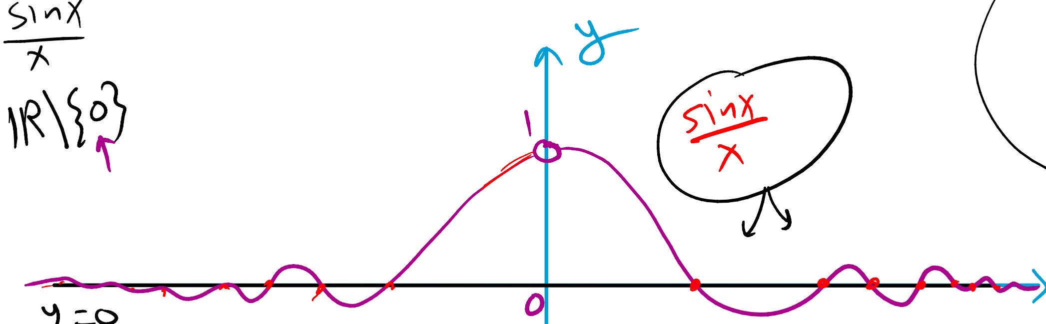
$\boxed{x = 0 \text{ is not V. Asy}}$

$\Downarrow$

$x = 0$ removable disc.

$f(x) = \frac{\sin x}{x}$

$D(f) = \mathbb{R} \setminus \{0\}$



$\lim_{x \to 0^+} f(x) = 1$

$\lim_{x \to 0^-} f(x) = 1$

$\Downarrow$

$\lim_{x \to 0} f(x) = 1$

$y = 0$ H. Asy.

but $x = 0$ is not V. Asy

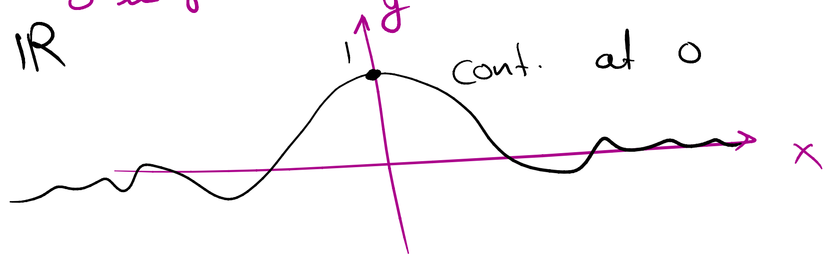
$x = 0$ removal

Find cont. extension $\Rightarrow$ $F(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ \boxed{1} & \text{if } x = 0 \end{cases}$

if $x = 0$

0 is fein

$D(F) = \mathbb{R}$



cont. at 0

$$f(x) = \frac{x-2}{x^2+1} \Rightarrow$$ Keypoint $\underline{\underline{(2,0)}}$, $(0,-2)$

0, 1, 2, −2.

$g(x) =$.