COMP 233 Discrete Mathematics

Chapter 7
Functions

1

Functions

7.1 Introduction to Functions

In this lecture:

- ❑ Part 1: **What is a function**
- ❑ Part 2: Equality of Functions
- ❑ Part 3: Examples of Functions
- ❑ Part 3: Checking Well Defined Functions

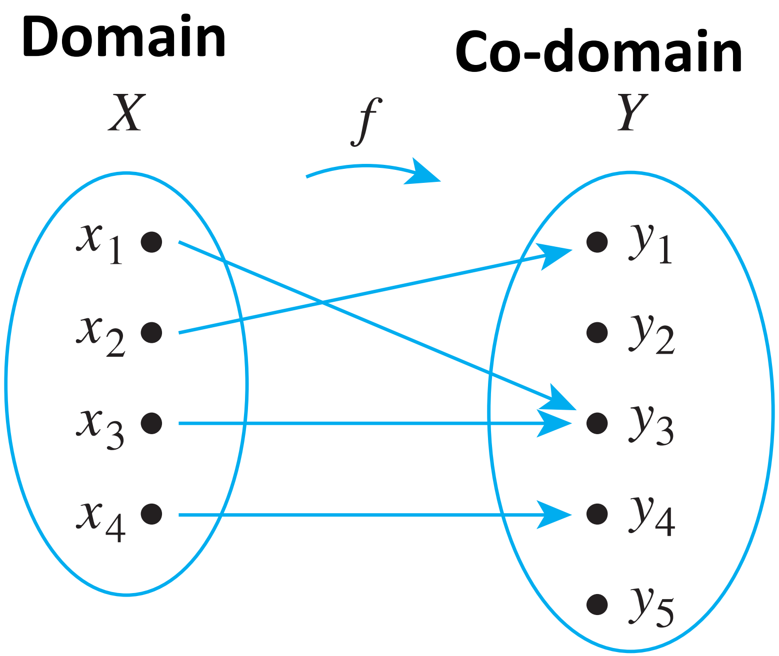
Motivation

Many issues in life can be mathematized and used as functions:

- Div(x), mod(x), ….
- FatherOf(x), TruthTable (x)

- In this chapter we focus on **discrete functions**

What is a Function



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كل عنصر في المجال يجب ان
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A function is a relation from X, the domain, to Y, the co-domain, that satisfies 2 properties:

1) Every element x is related to some element in Y;
2) No element in X is related to more than one element in Y

Function Definition

• Definition

A **function f from a set X to a set Y**, denoted $f: X \to Y$, is a relation from X, the **domain**, to Y, the **co-domain**, that satisfies two properties: (1) every element in X is related to some element in Y, and (2) no element in X is related to more than one element in Y. Thus, given any element x in X, there is a unique element in Y that is related to x by f. If we call this element y, then we say that "f sends x to y" or "f maps x to y" and write $x \xrightarrow{f} y$ or $f: x \to y$. The unique element to which f sends x is denoted

$\boldsymbol{f(x)}$ and is called **f of x**, or
the output of f for the input x, or
the value of f at x, or
the image of x under f.

The set of all values of f taken together is called the *range of f* or the *image of X under f*. Symbolically,

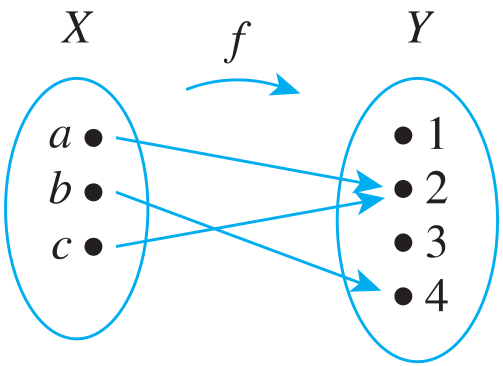
$$\textbf{range of } f = \textbf{image of } X \textbf{ under } f = \{y \in Y \mid y = f(x), \text{ for some } x \text{ in } X\}.$$

Given an element y in Y, there may exist elements in X with y as their image. If $f(x) = y$, then x is called **a preimage of y** or **an inverse image of y**. The set of all inverse images of y is called *the inverse image of y*. Symbolically,

$$\textbf{the inverse image of } y = \{x \in X \mid f(x) = y\}.$$

Example

Let $X = \{a, b, c\}$ and $Y = \{1,2,3,4\}$. Define a function $\boldsymbol{f}$ from X to Y



a. Write the domain and co-domain of f.

b. Find $f(a)$, $f(b)$, and $f(c)$.

c. What is the range of f?

d. Is c an inverse image of 2? Is b an inverse image of 3?

e. Find the inverse images of 2, 4, and 1.

f. Represent f as a set of ordered pairs.

Solution

a. domain of f = {a, b, c}, co-domain of f = {1, 2, 3, 4}

b. f (a) = 2, f (b) = 4, f (c) = 2

c. range of f = {2, 4}

d. Yes, No

e. inverse image of 2 = {a, c}

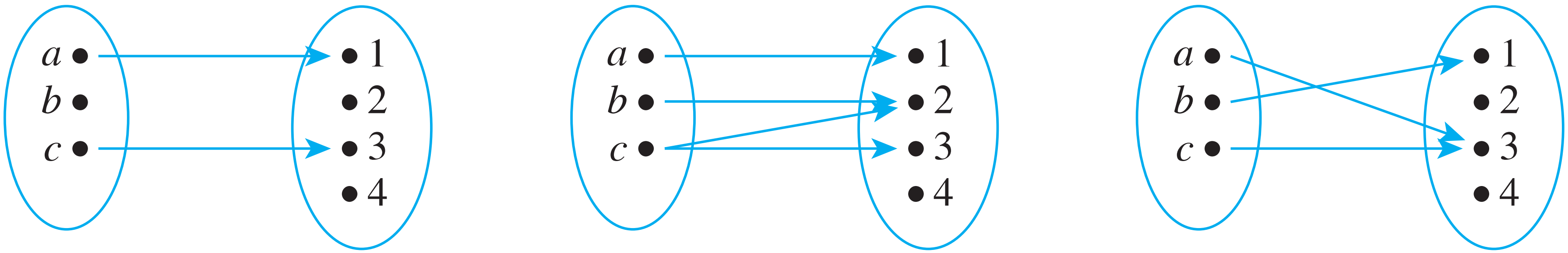
inverse image of 4 = {b}

inverse image of 1 = ∅ (since no arrows point to 1)

f. {(a, 2), (b, 4), (c, 2) }

Example

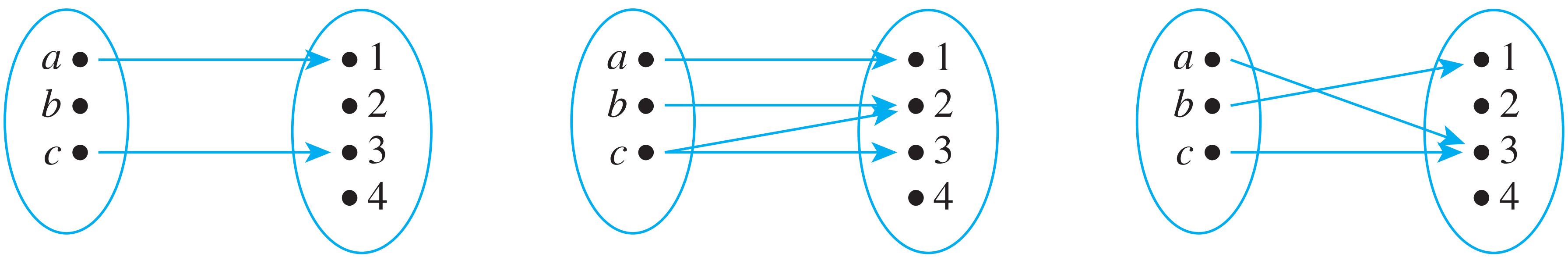
Which are functions?



(a) (b) (c)

Example

Which are functions?



(a) (b) (c)

(a) There is an element x, namely b, that is not sent to any element in of Y (i.e., there is no arrow coming out of Y)
(b) The element c isn't sent to a unique element of Y: that is, there are two arrows coming out of c; one pointing to 2 and the other is pointing to 3

Functions

7.1 Introduction to Functions

In this lecture:

- ❑ Part 1: What is a function
- ❑ Part 2: **Equality of Functions**
- ❑ Part 3: Examples of Functions
- ❑ Part 3: Checking Well Defined Functions

Equality of Functions

Theorem 7.1.1 A Test for Function Equality

If $F: X \to Y$ and $G: X \to Y$ are functions, then $F = G$ if, and only if, $F(x) = G(x)$ for all $x \in X$.

Example:

Let $L = \{0, 1, 2\}$, and define functions f and g:
For all x *in* L

$$f(x) = (x^2 + x + 1) \; mod \; 3 \quad \text{and} \quad g(x) = (x + 2)^2 \; mod \; 3.$$

Does *f*= *g* ?

Equality of Functions

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Example:

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For all x *in* L

$$f(x) = (x^2 + x + 1) \bmod 3 \quad \text{and } g(x) = (x + 2)^2 \bmod 3.$$

Does $f = g$?

x	$x^2 + x + 1$	$f(x) = (x^2 + x + 1) \bmod 3$	$(x + 2)^2$	$g(x) = (x + 2)^2 \bmod 3$
0	1	$1 \bmod 3 = 1$	4	$4 \bmod 3 = 1$
1	3	$3 \bmod 3 = 0$	9	$9 \bmod 3 = 0$
2	7	$7 \bmod 3 = 1$	16	$16 \bmod 3 = 1$

Equal functions in reality?

Equality of Functions

Theorem 7.1.1 A Test for Function Equality

If $F: X \to Y$ and $G: X \to Y$ are functions, then $F = G$ if, and only if, $F(x) = G(x)$ for all $x \in X$.

Example:

Let $F: \mathbf{R} \to \mathbf{R}$ and $G: \mathbf{R} \to \mathbf{R}$ be functions. Define new functions $F + G: \mathbf{R} \to \mathbf{R}$ and $G + F: \mathbf{R} \to \mathbf{R}$ as follows: For all $x \in \mathbf{R}$,

$$(F + G)(x) = F(x) + G(x) \quad \text{and} \quad (G + F)(x) = G(x) + F(x).$$

Does $F + G = G + F$?

Equality of Functions

Theorem 7.1.1 A Test for Function Equality

If $F: X \to Y$ and $G: X \to Y$ are functions, then $F = G$ if, and only if, $F(x) = G(x)$ for all $x \in X$.

Example:

Let $F: \mathbf{R} \to \mathbf{R}$ and $G: \mathbf{R} \to \mathbf{R}$ be functions. Define new functions $F + G: \mathbf{R} \to \mathbf{R}$ and $G + F: \mathbf{R} \to \mathbf{R}$ as follows: For all $x \in \mathbf{R}$,

$$(F + G)(x) = F(x) + G(x) \quad \text{and} \quad (G + F)(x) = G(x) + F(x).$$

Does $F + G = G + F$?

$$\begin{aligned} (F+G)(x) &= F(x) + G(x) && \text{by definition of } F+G \\ &= G(x) + F(x) && \text{by the commutative law for addition of real numbers} \\ &= (G+F)(x) && \text{by definition of } G+F \end{aligned}$$

Hence $F + G = G + F$.

Functions

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- Part 3: **Examples of Functions**
- Part 3: Checking Well Defined Functions

Examples of Functions

Identity Function

$$I_X(x) = x \text{ for all } x \text{ in } X.$$

Identity function send each element of X to the element that is identical to it

E.g., $I_x(y) = y$

Examples of Functions

Sequences

An infinite sequence is a function defined on set of integers that are greater than or equal to a particular integer.

E.g., Define the following sequence as a function from the set of positive integers to the set of real numbers

$$1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, \ldots, \frac{(-1)^n}{n+1}, \ldots$$

can be thought as a function *f* from the nonnegative integers to the real numbers that associate 0 → 1, 1 → -1/2, 2 → 1/3, …

$$\text{Send each integer } n \geq 0 \text{ to } f(n) = \frac{(-1)^n}{n+1}.$$

$$g(n+1) = \frac{(-1)^{n+2}}{n+1}$$

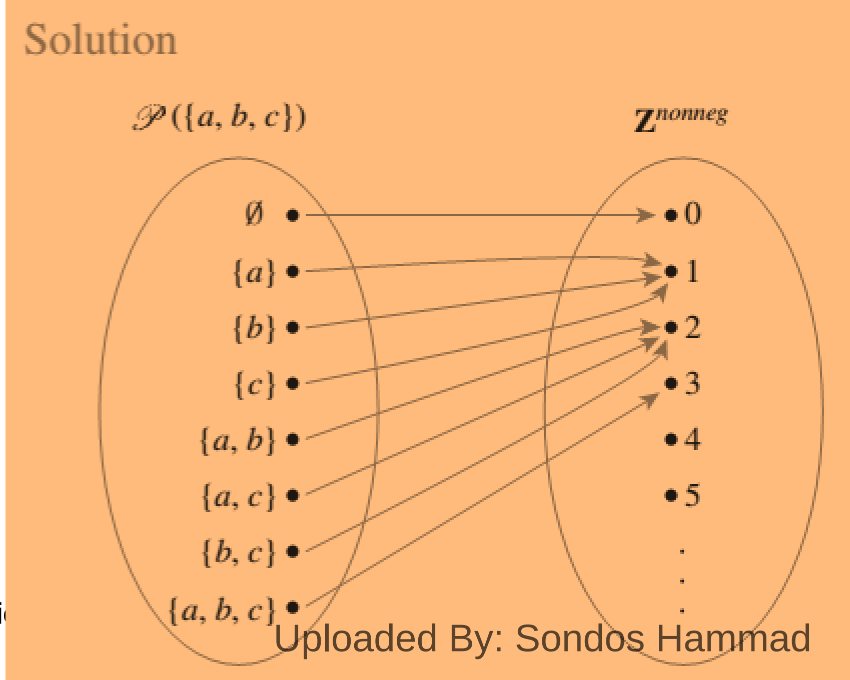
Examples of Functions

Function Defined on a Power Set

Recall from Section 6.1 that P(A) denotes the set of all subsets of the set A.

Define a function F: P({a, b, c}) → $\mathbf{Z}^{nonneg}$
as follows: For each *X* ∈ P({a, b, c}),

F(X) = the number of elements in *X.*
Draw an arrow diagram for *F.*



Examples of Functions

Cartesian product

Define functions $M: \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ and $R: \mathbf{R} \times \mathbf{R} \to \mathbf{R} \times \mathbf{R}$ as follows: For all ordered pairs (a, b) of integers,

$$M(a, b) = ab \quad \text{and} \quad R(a, b) = (-a, b).$$

M is the multiplication function that sends each pair of real numbers to the product of the two.
R is the reflection function that sends each point in the plane that corresponds to a pair of real numbers to the mirror image of the point across the vertical axis.

Find the following

a. $M(-1, -1)$ b. $M\left(\frac{1}{2}, \frac{1}{2}\right)$ c. $M(\sqrt{2}, \sqrt{2})$
d. $R(2, 5)$ e. $R(-2, 5)$ f. $R(3, -4)$

a. $(-1)(-1) = 1$ b. $(1/2)(1/2) = 1/4$ c. $\sqrt{2} \cdot \sqrt{2} = 2$
d. $(-2, 5)$ e. $(-(-2), 5) = (2, 5)$ f. $(-3, -4)$

Examples of Functions

Logarithmic functions

- **Definition Logarithms and Logarithmic Functions**

Let b be a positive real number with $b \neq 1$. For each positive real number x, the **logarithm with base b of x**, written $\log_b x$, is the exponent to which b must be raised to obtain x. Symbolically,

$$\log_b x = y \quad \Leftrightarrow \quad b^y = x.$$

The **logarithmic function with base b** is the function from $\mathbf{R}^+$ to $\mathbf{R}$ that takes each positive real number x to $\log_b x$.

- $\log_3 9 = 2$ because $3^2 = 9$.
- $\log_2 (1/2) = -1$ because $2^{-1} = ½$.
- $\log_{10}(1) = 0$ because $10^0 = 1$.
- $\log_2(2^m) = m$ because the exponent to which 2 must be raised to obtain 2^m is m.
- $2^{\log_2 m} = m$ because $\log_2 m$ is the exponent to which 2 must be raised to obtain m.

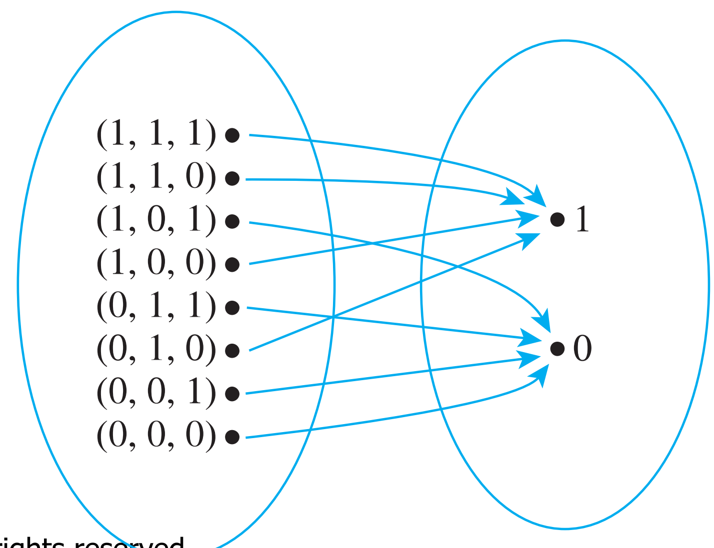
Examples of Functions

Boolean Functions

• Definition

An **(*n*-place) Boolean function** f is a function whose domain is the set of all ordered n-tuples of 0's and 1's and whose co-domain is the set {0, 1}. More formally, the domain of a Boolean function can be described as the Cartesian product of n copies of the set {0, 1}, which is denoted $\{0, 1\}^n$. Thus $f: \{0, 1\}^n \to \{0, 1\}$.

Input			Output
P	***Q***	***R***	***S***
1	1	1	1
1	1	0	1
1	0	1	0
1	0	0	1
0	1	1	0
0	1	0	1
0	0	1	0
0	0	0	0



Examples of Functions

Boolean Functions

Consider the three-place Boolean function defined from the set of all 3-tuples of 0's and 1's to {0, 1} as follows: For each triple (x_1, x_2, x_3) of 0's and 1's,

$$f(x_1, x_2, x_3) = (x_1 + x_2 + x_3)\ mod\ 2.$$

Describe f using an input/output table.

$f(1, 1, 1) = (1 + 1 + 1)\ mod\, 2 = 3\ mod\, 2 = 1$
$f(1, 1, 0) = (1 + 1 + 0)\ mod\, 2 = 2\ mod\, 2 = 0$
and so on to calculate the other values

Input			Output
x_1	x_2	x_3	$(x_1 + x_2 + x_3)\ mod\ 2$
1	1	1	1
1	1	0	0
1	0	1	0
1	0	0	1
0	1	1	0
0	1	0	1
0	0	1	1
0	0	0	0

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Well-defined Functions

Checking Whether a Function Is Well Defined

A function is **NOT** well defined if it fails to satisfy at least one of the requirements of being a function

E.g., Define a function f: **R** → **R** by specifying that for all real numbers x, $f(x)$ is the real number y such that $x^2+y^2=1$.

There are two reasons why this function is not well defined:
For almost all values of x either (1) there is no y that satisfies the given equation or (2) there are two different values of y that satisfy the equation

Consider when x=2
Consider when x=0

Well-defined Functions

Checking Whether a Function Is Well Defined

$f: \mathbf{Q} \rightarrow \mathbf{Z}$ defines this formula:

$$f\left(\frac{m}{n}\right) = m \quad \text{for all integers } m \text{ and } n \text{ with } n \neq 0.$$

Is f a well defined function?

It is not a well defined function since fractions have more than one representation as quotients of integers.

$$f\left(\frac{1}{2}\right) = 1 \quad \text{and} \quad f\left(\frac{3}{6}\right) = 3,$$

$$f\left(\frac{1}{2}\right) \neq f\left(\frac{3}{6}\right).$$

Well-defined Functions

Checking Whether a Function or not

Y= BortherOf(x)

Y= SonOf(x)

Y= FatherOf(x)

Y= Wife Of(x)

.

.

.

Functions

7.2 Properties of Functions

In this lecture:

- ❑ Part 1: **One-to-one Functions**
- ❑ Part 2: Onto Functions
- ❑ Part 3: one-to-one Correspondence Functions
- ❑ Part 4: Inverse Functions
- ❑ Part 5: Applications: Hash and Logarithmic Functions

One-to-One Functions

- **Definition**

Let F be a function from a set X to a set Y. F is **one-to-one** (or **injective**) if, and only if, for all elements x_1 and x_2 in X,

$$\text{if } F(x_1) = F(x_2), \text{ then } x_1 = x_2,$$

or, equivalently,

$$\text{if } x_1 \neq x_2, \text{ then } F(x_1) \neq F(x_2).$$

Symbolically,

$$F\colon X \to Y \text{ is one-to-one} \quad \Leftrightarrow \quad \forall x_1, x_2 \in X, \text{if } F(x_1) = F(x_2) \text{ then } x_1 = x_2.$$

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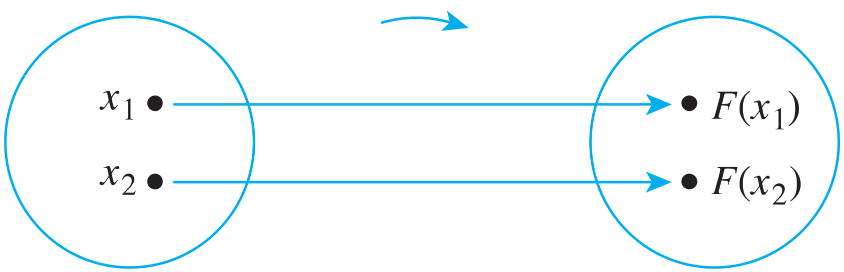
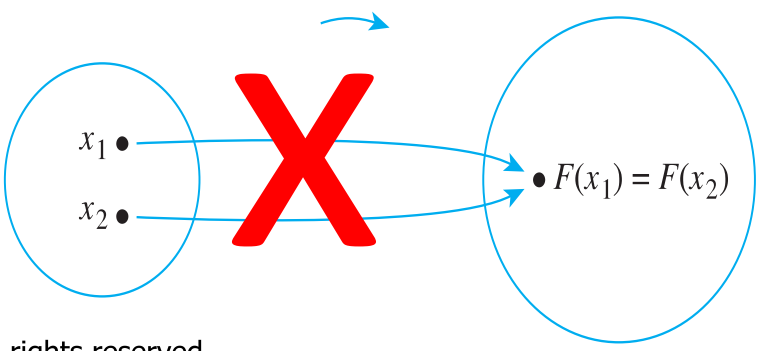
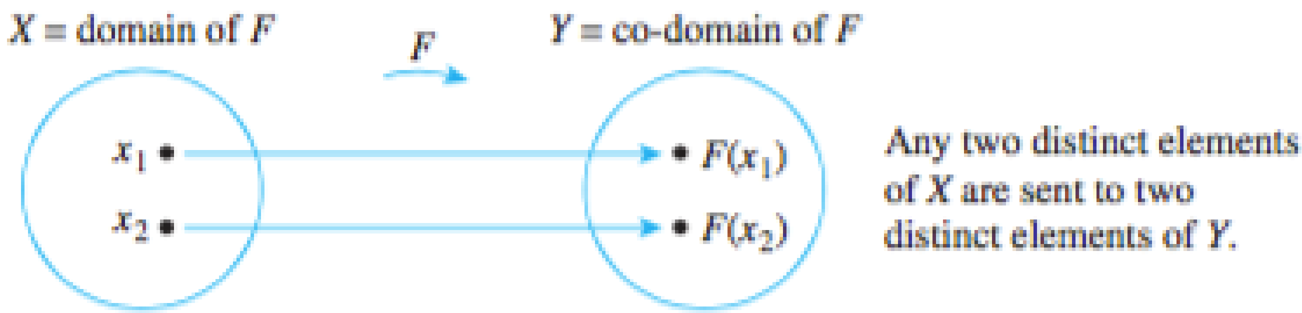
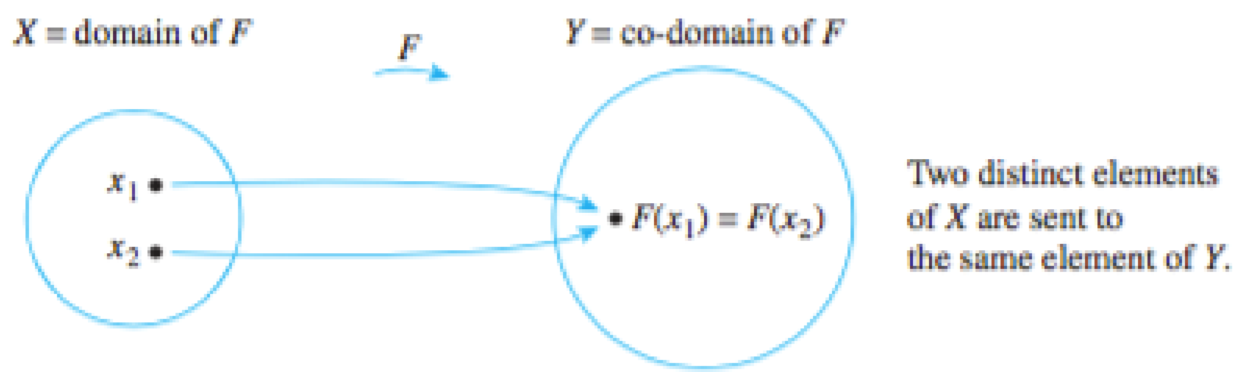




One-to-One Functions



A One-to-One Function Separates Points



A Function That Is Not One-to-One

One-to-One Functions

a. Do either of the arrow diagrams in Figure 7.2.2 define one-to-one functions?

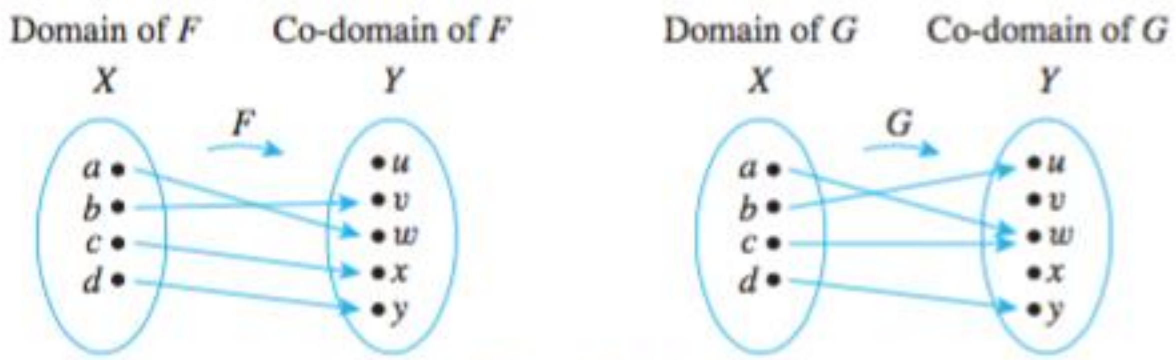


Figure 7.2.2

b. Let $X = \{1, 2, 3\}$ and $Y = \{a, b, c, d\}$. Define $H: X \to Y$ as follows: $H(1) = c$, $H(2) = a$, and $H(3) = d$. Define $K: X \to Y$ as follows: $K(1) = d$, $K(2) = b$, and $K(3) = d$. Is either H or K one-to-one?

One-to-One Functions

a. Do either of the arrow diagrams in Figure 7.2.2 define one-to-one functions?

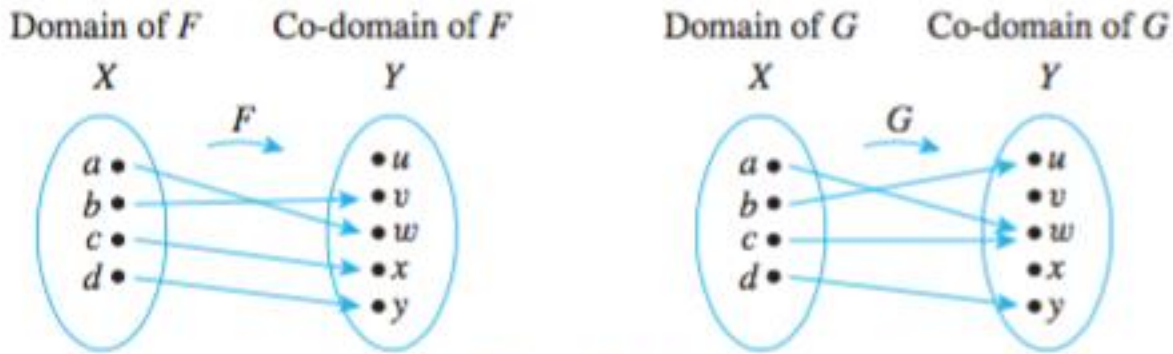


Figure 7.2.2

b. Let $X = \{1, 2, 3\}$ and $Y = \{a, b, c, d\}$. Define $H: X \to Y$ as follows: $H(1) = c$, $H(2) = a$, and $H(3) = d$. Define $K: X \to Y$ as follows: $K(1) = d$, $K(2) = b$, and $K(3) = d$. Is either H or K one-to-one?

(a) F is one-to-one but G is not. F is one-to-one because no two different elements of X are sent by F to the same element of Y. G is not one-to-one because the elements a and c are both sent by G to the same element of Y: $G(a) = G(c) = w$ but $a \neq c$.

(b) H is one-to-one but K is not. H is one-to-one because each of the three elements of the domain of H is sent by H to a different element of the co-domain: $H(1) \neq H(2)$, $H(1) \neq H(3)$, and $H(2) \neq H(3)$. K, however, is not one-to-one because $K(1) = K(3) = d$ but $1 \neq 3$.

Proving/Disproving Functions are One-to-One

To prove f is one-to-one (Direct Method):

suppose x_1 and x_2 are elements of $X \mid f(x_1) = f(x_2)$, and **show** that $x_1 = x_2$.

To show that f is *not* one-to-one:

Find elements x_1 and x_2 in X so $f(x_1) = f(x_2)$ but $x_1 \neq x_2$.

Proving/Disproving Functions are One-to-One

Example 1

Define f: $\mathbf{R} \to \mathbf{R}$ by the rule

$$f(x) = 4x - 1 \quad \text{for all } x \in \mathbf{R}$$

Is f one-to-one? Prove or give a counterexample.

Proving/Disproving Functions are One-to-One

Example 1

Define f: $\mathbf{R} \to \mathbf{R}$ by the rule

$$f(x) = 4x - 1 \quad \text{for all } x \in \mathbf{R}$$

Is f one-to-one? Prove or give a counterexample.

Suppose x_1 and x_2 are real numbers such that $f(x_1) = f(x_2)$. *[We must show that $x_1 = x_2$.]* By definition of f,

$4x_1 - 1 = 4x_2 - 1$. Adding 1 to both sides gives

$4x_1 = 4x_2$, and dividing both sides by 4 gives

$x_1 = x_2$, which is what was to be shown.

Proving/Disproving Functions are One-to-One

Example 2

Define $g : \mathbf{Z} \to \mathbf{Z}$ by the rule

$$g(n) = n^2 \qquad \text{for all } n \in \mathbf{Z}.$$

Is g one-to-one? Prove or give a counterexample.

Proving/Disproving Functions are One-to-One

Example 2

Define $g: \mathbf{Z} \to \mathbf{Z}$ by the rule

$$g(n) = n^2 \qquad \text{for all } n \in \mathbf{Z}.$$

Is g one-to-one? Prove or give a counterexample.

Counterexample:

Let $n_1 = 2$ and $n_2 = -2$. Then by definition of g,

$$g(n_1) = g(2) = 2^2 = 4 \text{ and also}$$

$$g(n_2) = g(-2) = (-2)^2 = 4.$$

Hence $\quad g(n_1) = g(n_2)$ but $n_1 \neq n_2$,

and so g is not one-to-one.

Proving/Disproving Functions are One-to-One

Example 3

Define g : **MobileNumber** → **People** by the rule

$g(x) = Person$ for all $x \in$ **MobileNumber**

Is g one-to-one? Prove or give a counterexample.

Counter example:

0599123456 and 0569123456 are both for Sami

Proving/Disproving Functions are One-to-One

Example 4

Define g : **Fingerprints** → **People** by the rule

$g(x) = Person$ for all $x \in$ **R** **Fingerprint**

Is g one-to-one? Prove or give a counterexample.

Prove:

In biology and forensic science: "The flexibility of friction ridge skin means that no two finger or palm prints are ever exactly alike in every detail" [w].

Functions

7.2 Properties of Functions

In this lecture:

- Part 1: One-to-one Functions
- Part 2: **Onto Functions**
- Part 3: one-to-one Correspondence Functions
- Part 4: Inverse Functions
- Part 5: Applications: Hash and Logarithmic Functions

Onto Functions

- **Definition**

Let F be a function from a set X to a set Y. F is **onto** (or **surjective**) if, and only if, given any element y in Y, it is possible to find an element x in X with the property that $y = F(x)$.

Symbolically:

$$F: X \to Y \text{ is onto} \quad \Leftrightarrow \quad \forall y \in Y, \exists x \in X \text{ such that } F(x) = y.$$

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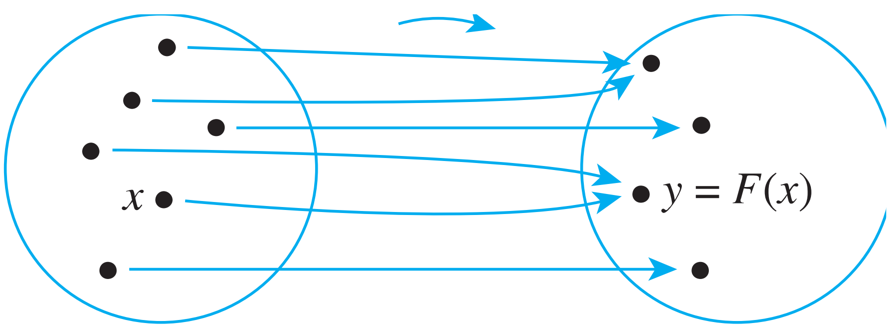

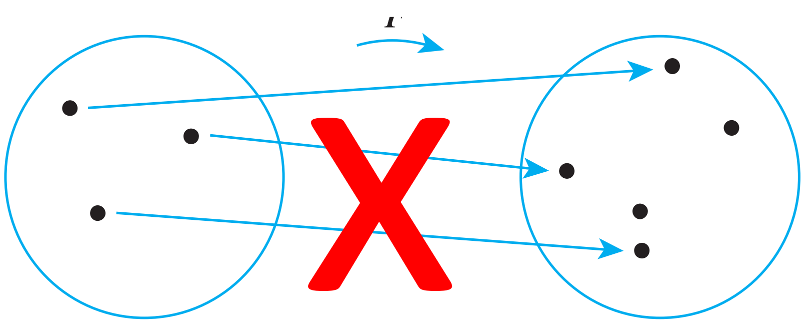

Onto Functions

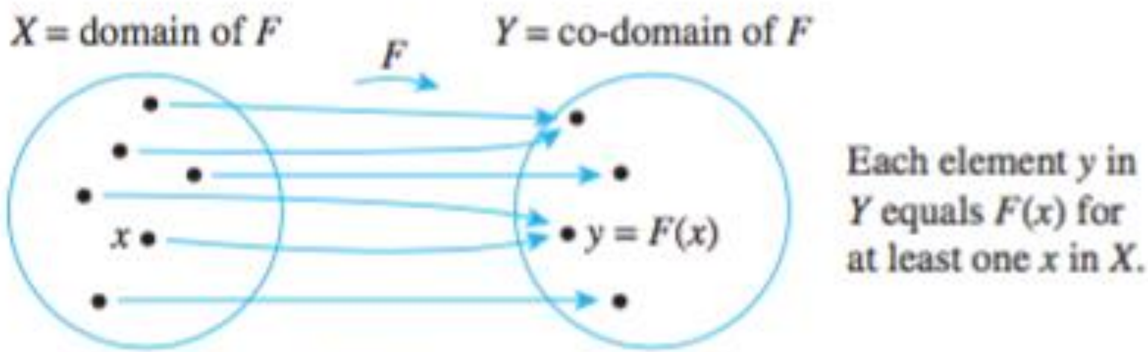

Figure 7.2.3(a) A Function That Is Onto

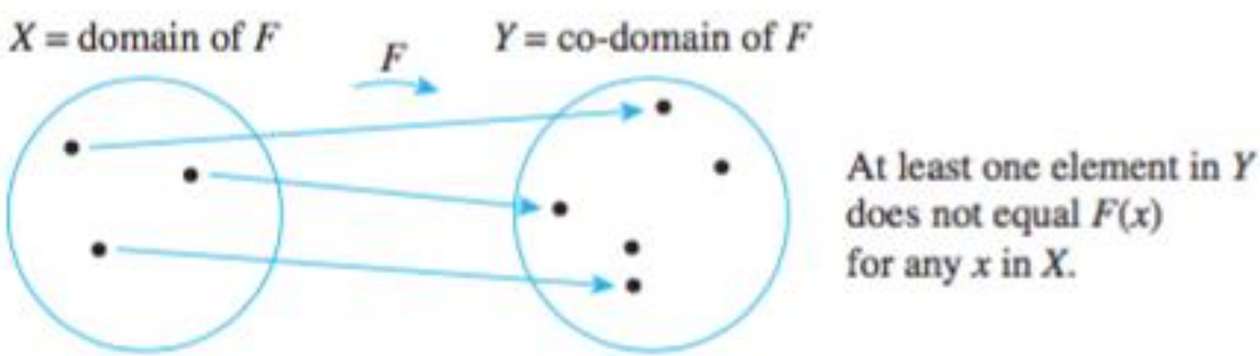

Figure 7.2.3(b) A Function That Is Not Onto

Onto Functions

a. Do either of the arrow diagrams in Figure 7.2.4 define onto functions?

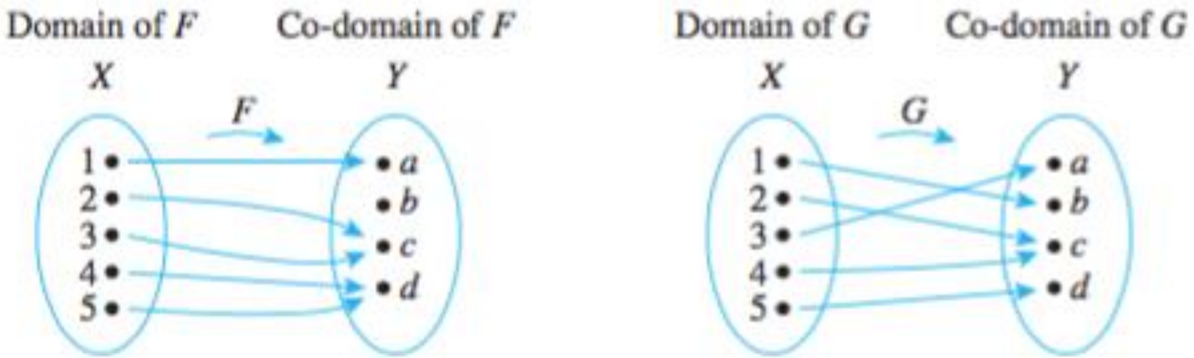

Figure 7.2.4

b. Let $X = \{1, 2, 3, 4\}$ and $Y = \{a, b, c\}$. Define $H: X \to Y$ as follows: $H(1) = c$, $H(2) = a$, $H(3) = c$, $H(4) = b$. Define $K: X \to Y$ as follows: $K(1) = c$, $K(2) = b$, $K(3) = b$, and $K(4) = c$. Is either H or K onto?

Onto Functions

a. Do either of the arrow diagrams in Figure 7.2.4 define onto functions?

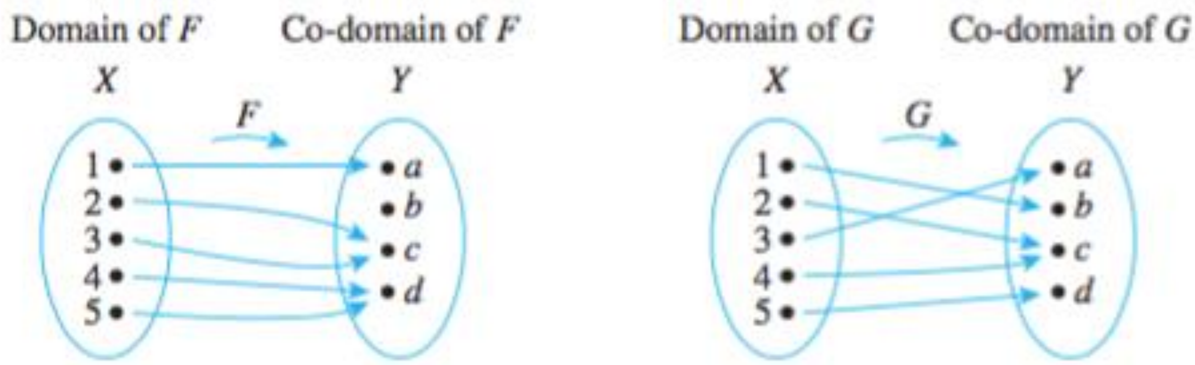

Figure 7.2.4

b. Let $X = \{1, 2, 3, 4\}$ and $Y = \{a, b, c\}$. Define $H: X \to Y$ as follows: $H(1) = c$, $H(2) = a$, $H(3) = c$, $H(4) = b$. Define $K: X \to Y$ as follows: $K(1) = c$, $K(2) = b$, $K(3) = b$, and $K(4) = c$. Is either H or K onto?

***(a) F* is not onto** because $b \neq F(x)$ for any x in X.

***G* is onto** because each element of Y equals $G(x)$ for some x in X:

$a = G(3)$, $b = G(1)$, $c = G(2) = G(4)$, and $d = G(5)$.

***(b) H* is onto but *K* is not.**

H is onto because each of the three elements of the co-domain of H is the image of some element of the domain of H: $a = H(2), b = H(4)$, and $c=H(1)=H(3)$.

K, however, is not onto because $a \neq K(x)$ for any x in $\{1,2,3,4\}$.

Proving/Disproving Functions are Onto

To prove F is onto, (**method of generalizing from the generic particular**)

suppose that y is any element of Y

show that there is an element x of X with $F(x) = y$.

To prove F is *not* onto, you will usually

find an element y of Y | $y \neq F(x)$ for *any* x in X.

Proving/Disproving Functions are Onto

Example 1

Define f: $\mathbf{R} \to \mathbf{R}$

$$f(x) = 4x - 1 \qquad \text{for all } x \in \mathbf{R}$$

Is f onto? Prove or give a counterexample.

Proving/Disproving Functions are Onto

Example 1

Define f: $\mathbf{R} \to \mathbf{R}$

$$f(x) = 4x - 1 \qquad \text{for all } x \in \mathbf{R}$$

Is f onto? Prove or give a counterexample.

Let $y \in \mathbf{R}$. *[We must show that $\exists x$ in* $\mathbf{R}$ *such that $f(x) = y$.]* Let $x = (y+1)/4$. Then x is a real number since sums and quotients (other than by 0) of real numbers are real numbers. It follows that

$$\begin{aligned} f(x) &= f\left(\frac{y+1}{4}\right) && \text{by substitution} \\ &= 4 \cdot \left(\frac{y+1}{4}\right) - 1 && \text{by definition of } f \\ &= (y+1) - 1 = y && \text{by basic algebra.} \end{aligned}$$

[This is what was to be shown.]

Proving/Disproving Functions are Onto

Example 2

Define $h : \mathbf{Z} \rightarrow \mathbf{Z}$ by the rules

$$h(n) = 4n - 1 \qquad \text{for all } n \in \mathbf{Z}.$$

Is h onto? Prove or give a counterexample.

Proving/Disproving Functions are Onto

Example 2

Define $h: \mathbf{Z} \rightarrow \mathbf{Z}$ by the rules

$$h(n) = 4n - 1 \qquad \text{for all } n \in \mathbf{Z}.$$

Is h onto? Prove or give a counterexample.

Counterexample:

The co-domain of h is $\mathbf{Z}$ and $0 \in \mathbf{Z}$. But $h(n) \neq 0$ for any integer n. For if $h(n) = 0$, then

$$4n - 1 = 0 \qquad \text{by definition of } h$$

which implies that

$$4n = 1 \qquad \text{by adding 1 to both sides}$$

and so

$$n = \frac{1}{4} \qquad \text{by dividing both sides by 4.}$$

But $1/4$ is not an integer. Hence there is no integer n for which $f(n) = 0$, and thus f is not onto.

Proving/Disproving Functions are Onto

Example 3

Define g : **MobileNumber** → **People** by the rule

$g(x) = Person$ for all $x \in$ **MobileNumber**

Is g onto? Prove or give a counterexample.

Counter example:

Sami does not have a mobile number

Proving/Disproving Functions are Onto

Example 4

Define g : **Fingerprints** → **People** by the rule

$g(x) = \mathit{Person}$ for all $x \in$ **Fingerprint**

Is g onto? Prove or give a counterexample.

Prove:

In biology and forensic science: there is no person without fingerprint

Functions

7.2 Properties of Functions

In this lecture:

- ❑ Part 1: One-to-one Functions
- ❑ Part 2: Onto Functions
- ❑ Part 3: **one-to-one Correspondence Functions**
- ❑ Part 4: Inverse Functions
- ❑ Part 5: Applications: Hash and Logarithmic Functions

One-to-One Correspondences

- **Definition**

A **one-to-one correspondence** (or **bijection**) from a set X to a set Y is a function $F: X \to Y$ that is both one-to-one and onto.

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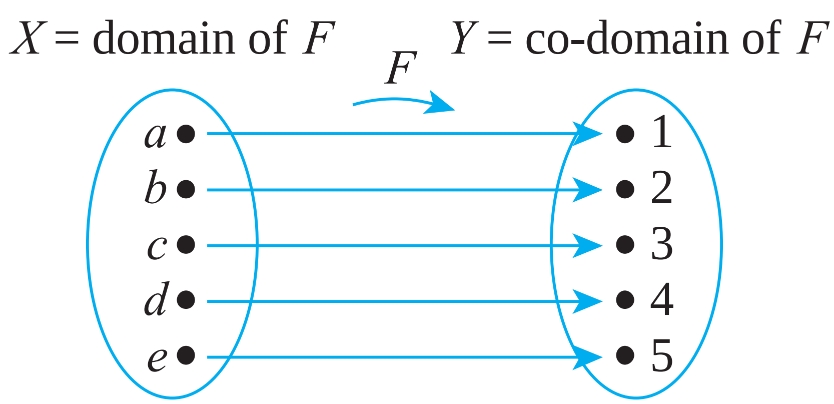

String-Reversing Function

Let T be the set of all finite strings of x's and y's.

Define

$g: T \rightarrow T$ by the rule: For all strings $s \in T$,

$g(s)$ = the string obtained by writing the characters of ***s*** in reverse order.

E.g., g("Ali") = "ilA"

Is g a one-to-one correspondence from T to itself?

(a)one-to-one:

(b)onto

String-Reversing Function

Let T be the set of all finite strings of x's and y's. Define

$g: T \rightarrow T$ by the rule: For all strings $s \in T$,

$g(s)$ = the string obtained by writing the characters of s in reverse order. E.g., g("Ali") = "ilA"

(a) one-to-one:

- suppose that for some strings $s1$ and $s2$ in T, $g(s1) = g(s2)$. *[We must show that $s1 = s2$.]*
- Now to say that $g(s1) = g(s2)$ is the same as saying that the string obtained by writing the characters of ***s*1** in reverse order equals the string obtained by writing the characters of ***s*2** in reverse order.
- But if ***s*1** and ***s*2** are equal when written in reverse order, then they must be equal to original.

  In other words, $s1 = s2$ *[as was to be shown].*

String-Reversing Function

(b) onto: suppose $\boldsymbol{t}$ is a string in T.

- *[We must find a string* $\boldsymbol{s}$ *in* T *such that* $g(s) = t$*.]*
- Let $\boldsymbol{s} = \boldsymbol{g}(\boldsymbol{t})$.
- By definition of g, $s = g(t)$ is the string in T obtained by writing the characters of $\boldsymbol{t}$ in reverse order.
- But when the order of the characters of a string is reversed once and then reversed again, the original string is recovered.
- $g(\boldsymbol{s}) = g(g(\boldsymbol{t}))$

  = the string obtained by writing the characters of $\boldsymbol{t}$ in reverse order and then writing those characters in reverse order again

  $= \boldsymbol{t}$

This is what was to be shown.

A Function of Two Variables

Define a function $F: \mathbf{R} \times \mathbf{R} \to \mathbf{R} \times \mathbf{R}$ as follows: For all $(x, y) \in \mathbf{R} \times \mathbf{R}$,

$$F(x, y) = (x + y, x - y).$$

Is F a one-to-one correspondence from $\mathbf{R} \times \mathbf{R}$ to itself?

A Function of Two Variables

Define a function $F: \mathbf{R} \times \mathbf{R} \to \mathbf{R} \times \mathbf{R}$ as follows: For all $(x, y) \in \mathbf{R} \times \mathbf{R}$,

$$F(x, y) = (x + y, x - y).$$

Is F a one-to-one correspondence from $\mathbf{R} \times \mathbf{R}$ to itself?

Functions

7.2 Properties of Functions

In this lecture:

- ❑ Part 1: One-to-one Functions
- ❑ Part 2: Onto Functions
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- ❑ Part 4: **Inverse Functions**
- ❑ Part 5: Applications: Hash and Logarithmic Functions

Inverse Functions

Theorem 7.2.2

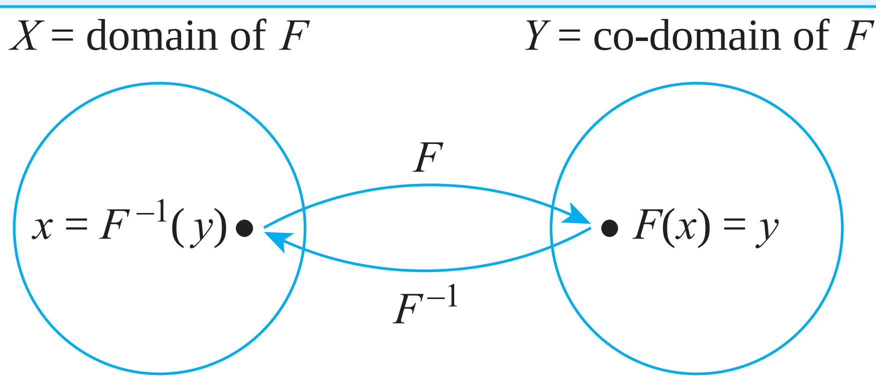
Suppose $F: X \to Y$ is a one-to-one correspondence; that is, suppose F is one-to-one and onto. Then there is a function $\boldsymbol{F^{-1}: Y \to X}$ that is defined as follows:

Given any element y in Y,

$$F^{-1}(y) = \text{ that unique element } x \text{ in } X \text{ such that } F(x) \text{ equals } y.$$

In other words,

$$F^{-1}(y) = x \quad \Leftrightarrow \quad y = F(x).$$



➔ Is it always that the inverse of a function is a function?

Finding Inverse Functions

The function f: $\mathbf{R} \to \mathbf{R}$ defined by the formula

$$f(x) = 4x - 1 \text{ for all real numbers } x$$

(was shown one-to-one and onto)

Find its inverse function?

Finding Inverse Functions

The function f: $\mathbf{R} \to \mathbf{R}$ defined by the formula

$$f(x) = 4x - 1 \text{ for all real numbers } x$$

(was shown one-to-one and onto)

Find its inverse function?

Solution For any *[particular but arbitrarily chosen]* y in $\mathbf{R}$, by definition of f^{-1},

$$f^{-1}(y) = \text{that unique real number } x \text{ such that } f(x) = y.$$

But

$$\begin{aligned} & f(x) = y \\ \Leftrightarrow \quad & 4x - 1 = y && \text{by definition of } f \\ \Leftrightarrow \quad & x = \frac{y+1}{4} && \text{by algebra.} \end{aligned}$$

Hence $f^{-1}(y) = \dfrac{y+1}{4}$.

Functions

7.2 Properties of Functions

In this lecture:

- ❑ Part 1: One-to-one Functions
- ❑ Part 2: Onto Functions
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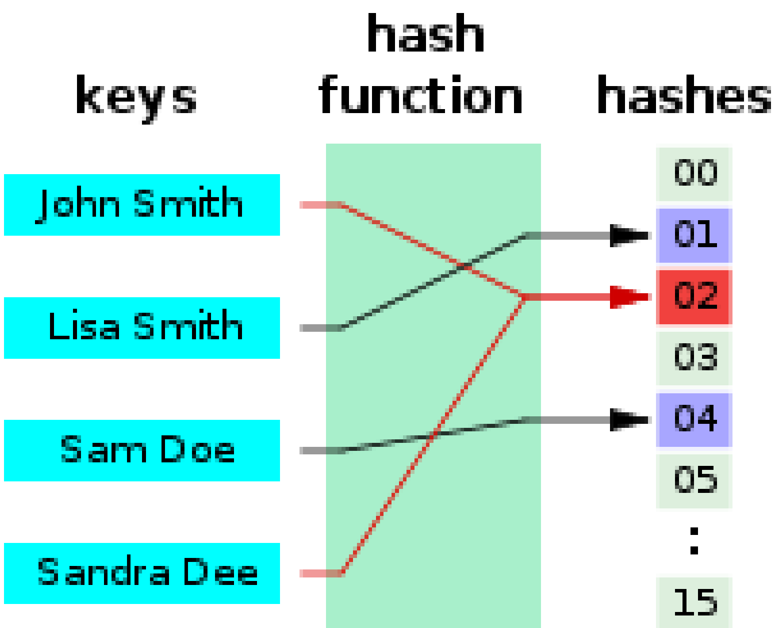
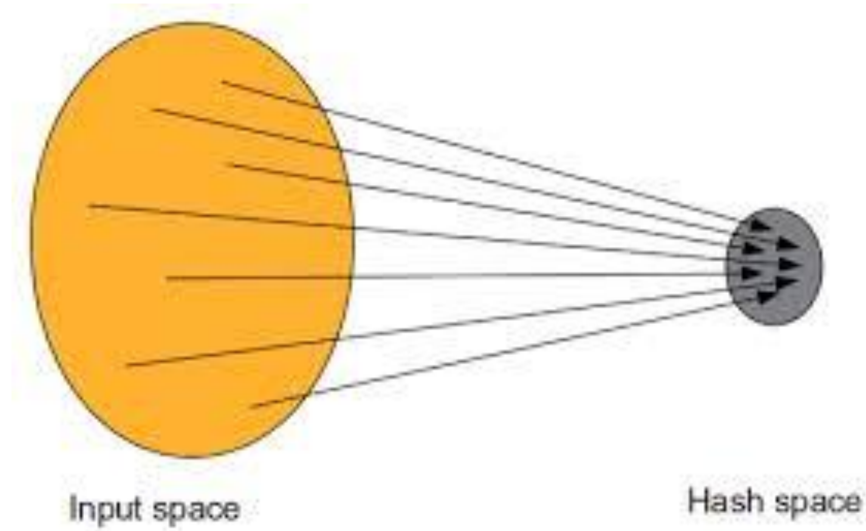
- ❑ Part 4: Inverse Functions
- ❑ Part 5: **Applications: Hash and Logarithmic Functions**

Hash Functions

- Maps data of arbitrary length to data of a fixed length.
- Very much used in databases and security





Hash Functions

How to store long (ID numbers) for a small set of people

For example: ***n*** is an ID number, and ***m*** is number of people we have

$$\text{Hash}(n) = n \bmod m$$

$$Hash(n) = n \bmod 7 \quad \text{for all numbers } n.$$

0	356-63-3102
1	
2	513-40-8716
3	223-79-9061
4	
5	328-34-3419
6	

collision?

Exponential and Logarithmic Functions

$$\text{Log}_b\, x = y \iff b^y = x$$

Relations between Exponential and Logarithmic Functions

Laws of Exponents

If b and c are any positive real numbers and u and v are any real numbers, the following laws of exponents hold true:

$$b^u b^v = b^{u+v} \quad 7.2.1$$

$$(b^u)^v = b^{uv} \quad 7.2.2$$

$$\frac{b^u}{b^v} = b^{u-v} \quad 7.2.3$$

$$(bc)^u = b^u c^u \quad 7.2.4$$

The exponential and logarithmic functions are one-to-one and onto. Thus the following properties hold:

For any positive real number b with $b \neq 1$,

$$\text{if } b^u = b^v \text{ then } u = v \quad \text{for all real numbers } u \text{ and } v, \quad 7.2.5$$

and

$$\text{if } \log_b u = \log_b v \text{ then } u = v \quad \text{for all positive real numbers } u \text{ and } v. \quad 7.2.6$$

Relations between Exponential and Logarithmic Functions

We can derive additional facts about exponents and logarithms, e.g.:

Theorem 7.2.1 Properties of Logarithms

For any positive real numbers b, c and x with $b \neq 1$ and $c \neq 1$:

a. $\log_b(xy) = \log_b x + \log_b y$

b. $\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$

c. $\log_b(x^a) = a \log_b x$

d. $\log_c x = \frac{\log_b x}{\log_b c}$

How to prove this?

Using the One-to-Oneness of the Exponential Function

Prove that:

$$\log_c x = \frac{\log_b x}{\log_b c}.$$

Solution Suppose positive real numbers b, c, and x are given. Let

$$(1)\ \ u = \log_b c \qquad (2)\ \ v = \log_c x \qquad (3)\ \ w = \log_b x.$$

Then, by definition of logarithm,

$$(1')\ \ c = b^u \qquad (2')\ \ x = c^v \qquad (3')\ \ x = b^w.$$

Substituting (1′) into (2′) and using one of the laws of exponents gives

$$x = c^v = (b^u)^v = b^{uv}$$ by 7.2.2

But by (3), $x = b^w$ also. Hence

$$b^{uv} = b^w,$$

and so by the one-to-oneness of the exponential function (property 7.2.5),

$$uv = w.$$

Substituting from (1), (2), and (3) gives that

$$(\log_b c)(\log_c x) = \log_b x.$$

And dividing both sides by $\log_b c$ (which is nonzero because $c \neq 1$) results in

$$\log_c x = \frac{\log_b x}{\log_b c}.$$