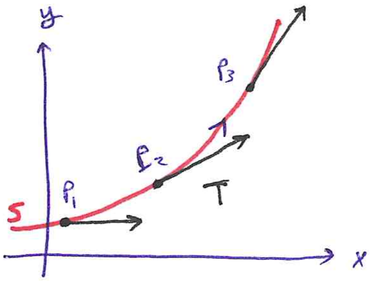
13.4 Curvature and Normal Vectors of a Curve

Def If $\vec{T}$ is the unit tangent vector of a smooth curve, then the curvature function of the curve is

$$K = \left|\frac{d\vec{T}}{ds}\right|$$



- Since $\vec{T}$ is a unit vector, its length is constant but its direction changes as the particle moves alonge the curve.

- Note that a smooth curve $\vec{r}(t)$ is given in terms of the parameter t other than the arc length parameter s, then

$$K = \left|\frac{d\vec{T}}{ds}\right| = \left|\frac{d\vec{T}}{dt}\,\frac{dt}{ds}\right| = \frac{1}{\left|\frac{ds}{dt}\right|}\left|\frac{d\vec{T}}{dt}\right| = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right|$$

Def If $\vec{r}(t)$ is a smooth curve, then the curvature is

$$\boxed{K = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right|}$$

where $\vec{T} = \frac{\vec{v}}{|\vec{v}|}$ is the unit tangent vector.

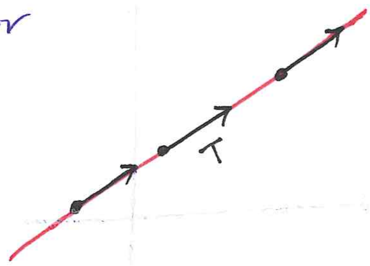
Exp If $\vec{r}(t) = t\vec{C_1} + \vec{C_2}$ is straight line for some constant vectors $\vec{C_1}$ and $\vec{C_2}$, then the curvature is zero.

- $\vec{v} = \vec{C_1} \Rightarrow |\vec{v}| = |\vec{C_1}|$ which is a constant

Thus, $\vec{T} = \frac{\vec{v}}{|\vec{v}|}$ is a constant vector

$\Rightarrow \frac{d\vec{T}}{dt} = \vec{0}$. Hence, $\left|\frac{d\vec{T}}{dt}\right| = 0$

$\Rightarrow K = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right| = \frac{1}{|\vec{v}|}(0) = 0$



Exp Find $\vec{T}$ and κ for the plane curve

$$\vec{r}(t) = (\ln \sec t)\,\vec{i} + t\,\vec{j}, \quad \boxed{-\frac{\pi}{2} < t < \frac{\pi}{2}}$$

- $\vec{V} = \left(\frac{\sec t \tan t}{\sec t}\right)\vec{i} + \vec{j} = (\tan t)\,\vec{i} + \vec{j}$

$$|\vec{V}| = \sqrt{\tan^2 t + 1} = \sqrt{\sec^2 t} = |\sec t| = \sec t$$

- $\vec{T} = \frac{\vec{V}}{|\vec{V}|} = \frac{1}{\sec t}\left((\tan t)\,\vec{i} + \vec{j}\right) = (\sin t)\,\vec{i} + (\cos t)\,\vec{j}$

- $\frac{d\vec{T}}{dt} = (\cos t)\,\vec{i} - (\sin t)\,\vec{j} \Rightarrow \left|\frac{d\vec{T}}{dt}\right| = \sqrt{\cos^2 t + \sin^2 t} = 1$

$$\kappa = \frac{1}{|\vec{V}|}\left|\frac{d\vec{T}}{dt}\right| = \frac{1}{\sec t}(1) = \cos t$$

Def At a point where $\kappa \neq 0$, the principle unit normal vector for a smooth curve in the plane is

$$\vec{N} = \frac{1}{\kappa}\frac{d\vec{T}}{ds}$$

$$= \frac{\frac{d\vec{T}}{dt}}{\left|\frac{d\vec{T}}{dt}\right|}$$

$$\frac{1}{\kappa}\frac{d\vec{T}}{ds} = \frac{1}{\left|\frac{d\vec{T}}{ds}\right|}\frac{d\vec{T}}{ds} = \frac{\frac{d\vec{T}}{dt}\frac{dt}{ds}}{\left|\frac{d\vec{T}}{dt}\frac{dt}{ds}\right|} = \frac{\frac{d\vec{T}}{dt}}{\left|\frac{d\vec{T}}{dt}\right|}\;\frac{1}{\frac{\frac{ds}{dt}}{\left|\frac{ds}{dt}\right|}}$$

since $\frac{ds}{dt} = |\vec{V}| > 0$

since $s(t) = \int_{t_0}^{t} |\vec{V}|\,dt$

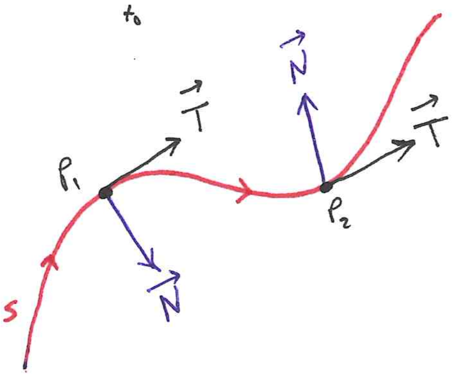
- Note that ① $\vec{N} \perp \vec{T}$ this because $|\vec{T}| = 1$ (constant length) and $\vec{N} = \frac{1}{\kappa}\frac{d\vec{T}}{ds}$

$\vec{T} \cdot \frac{d\vec{T}}{ds} = 0$ by Remark*

② $\vec{N}$ points the direction in which the curve is turning.

③ In Exp above $\Rightarrow \vec{N} = \frac{\frac{d\vec{T}}{dt}}{\left|\frac{d\vec{T}}{dt}\right|} \Rightarrow$

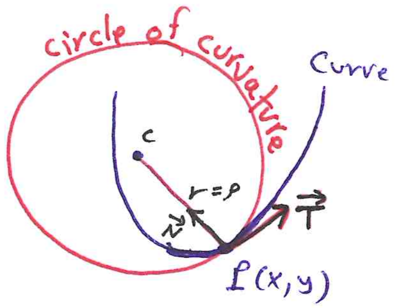
$$\vec{N} = (\cos t)\,\vec{i} - (\sin t)\,\vec{j}$$



Circle of Curvature for Plane Curves

The **circle of curvature** or osulating circle at point $P(x,y)$ on plane curve where $K \neq 0$ is the circle that

1. is tangent to the curve at P
2. has the same curvature the curve has at P.
3. lies toward the concave or inner side of the curve.



c : center of curvature

r : radius of curvature

- Note that the radius of the curve ture of the curve at P is the radius of the circle of the curvature

$$\text{Radius of curvature} = \rho = \frac{1}{K}$$

Exp Find the osculating circle of the parabola $y = x^2$ at origin.

- We parametrize the parabola $x = t$, $y = t^2$ $\Rightarrow$

$$\vec{r}(t) = t\,\vec{i} + t^2\,\vec{j}$$

- To find the curvature of the parabola:

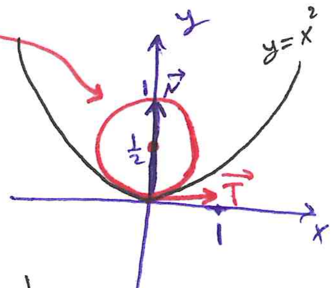
$$\vec{v} = \vec{i} + 2t\,\vec{j} \Rightarrow |\vec{v}| = \sqrt{1+4t^2}$$

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|} = \frac{1}{\sqrt{1+4t^2}}\,\vec{i} + \frac{2t}{\sqrt{1+4t^2}}\,\vec{j}$$

$$\frac{d\vec{T}}{dt} = \frac{-4t}{(1+4t^2)^{\frac{3}{2}}}\,\vec{i} + \left[\frac{2}{\sqrt{1+4t^2}} - \frac{8t^2}{(1+4t^2)^{\frac{3}{2}}}\right]\vec{j}$$

At origin $\Rightarrow$ the curvature is

$$K(0) = \frac{1}{|\vec{v}(0)|}\left|\frac{d\vec{T}}{dt}(0)\right| = \frac{1}{(1)}\left|2\,\vec{j}\right| = \sqrt{4} = 2$$



- $\rho = \frac{1}{K} = \frac{1}{2}$
- At origin $\Rightarrow$ $t = 0 \Rightarrow$
  $\vec{T} = \vec{i}$
  $\vec{N} = \vec{j}$
- center $(0, \frac{1}{2})$
- osculating circle $x^2 + (y - \frac{1}{2})^2 = \frac{1}{4}$