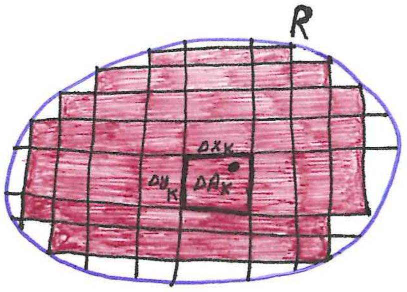
15.2 Double Integrals over General Regions

- we define the double integral of a function $f(x,y)$ over bounded general region R similarly but we approximately cover R by a grid of small rectangulars that are completely inside R. That is,



$$\iint_R f(x,y)\,dA = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k$$

- If $z=f(x,y)$ is positive continuous, then the volume is

$$V = \iint_R f(x,y)\,dA.$$

Th (Fubini's Theorem - stronger form)

Let $f(x,y)$ be a continuous on region R. If R is defined by

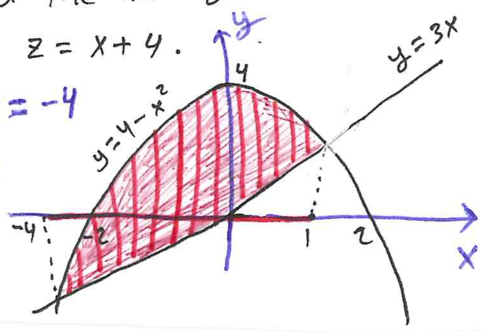
1) $a \le x \le b$, $g_1(x) \le y \le g_2(x)$ with g_1 and g_2 continuous on $[a,b]$, then $$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$ "Vertical cross Sec."

2) $c \le y \le d$, $h_1(y_1) \le x \le h_2(y)$ with h_1 and h_2 continuous on $[a,b]$, then $$\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$ "Horizontal Cros Sc"

Exp Find the volume of the solid whose base is the region in the xy-plane that is bounded by the parabola $y=4-x^2$ and the line $y=3x$, while the top of the solid is bounded by the plane $z=x+4$.

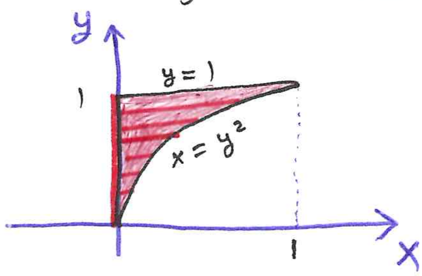
- $4-x^2 = 3x \Leftrightarrow x^2+3x-4=0 \Leftrightarrow x=1$ or $x=-4$

$$V = \int_{-4}^{1}\int_{3x}^{4-x^2} (x+4)\,dy\,dx = \int_{-4}^{1} (x+4)(4-3x-x^2)\,dx = \frac{625}{12}$$



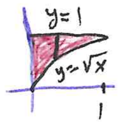
Exp [1] Sketch the region of integration and evaluate the integral

$$\int_0^1 \int_0^{y^2} 3y^3 e^{xy}\,dx\,dy = \int_0^1 3y^2 e^{xy}\Big|_{x=0}^{x=y^2} dy$$



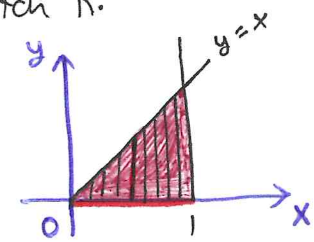
$$= \int_0^1 \left(3y^2 e^{y^3} - 3y^2\right) dy = e^{y^3} - y^3 \Big|_0^1 = e - 2$$

[2] reverse the order of integration $\displaystyle\int_0^1 \int_{\sqrt{x}}^1 3y^3 e^{xy}\,dy\,dx$



Exp [1] Evaluate $\displaystyle\iint_R \frac{\sin x}{x}\,dA$ where R is triangle in the xy-plane bounded by the x-axis, the line $y=x$ and the line $x=1$. Sketch R.

$$\int_0^1 \int_0^x \frac{\sin x}{x}\,dy\,dx = \int_0^1 \sin x\,dx = 1 - \cos(1)$$



[2] reverse the order of integration $\displaystyle\int_0^1 \int_y^1 \frac{\sin x}{x}\,dx\,dy$ difficult to integrate.

*** Properties of Double Integrals:**

If $f(x,y)$ and $g(x,y)$ are continuous on the bounded region R, then the following is true:

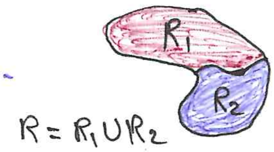
[1] Constant Multiple: $\displaystyle\iint_R c\,f(x,y)\,dA = c\iint_R f(x,y)\,dA$, $c \in \mathbb{R}$.

[2] Sum and Difference: $\displaystyle\iint_R \left(f(x,y) \pm g(x,y)\right)dA = \iint_R f(x,y)\,dA \pm \iint_R g(x,y)\,dA$

[3] Domination: [a] If $f(x,y) \geq 0$ on R, then $\displaystyle\iint_R f(x,y)\,dA \geq 0$.

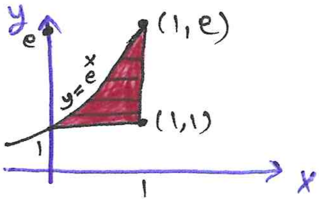
[b] If $f(x,y) \geq g(x,y)$ on R, then $\displaystyle\iint_R f(x,y)\,dA \geq \iint_R g(x,y)\,dA$

[4] Additivity: $\displaystyle\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$ if R is the union of two nonoverlapping regions R_1 and R_2.

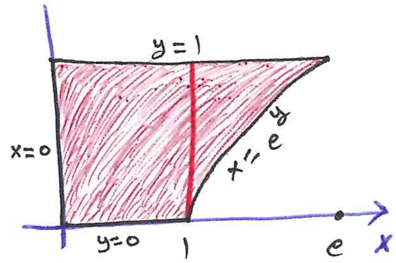


Exp sketch the region of integration and write an equivalent double integral with the order of integration reversed for

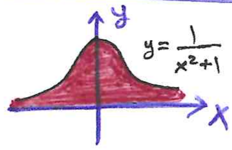
1. $$\int_0^1 \int_1^{e^x} dy\, dx = \int_1^e \int_{\ln y}^1 dx\, dy$$



2. $$\int_0^1 \int_0^{e^y} dx\, dy = \int_0^1 \int_0^1 dy\, dx + \int_1^e \int_{\ln x}^1 dy\, dx$$



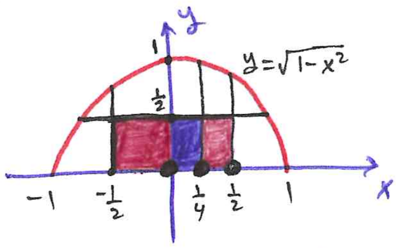
Exp Evaluate the improper integral over the unbounded region R: $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dx\, dy}{(x^2+1)(y^2+1)}$



$$= \int_{-\infty}^{\infty} 2 \int_0^{\infty} \frac{dx\, dy}{(x^2+1)(y^2+1)} = 2\int_{-\infty}^{\infty} \left(\lim_{b \to \infty} \tan^{-1} x \Big|_{x=0}^{x=b} \right) \frac{dy}{y^2+1}$$

$$= \cancel{2} \int_{-\infty}^{\infty} \frac{\pi}{\cancel{2}} \frac{dy}{y^2+1} = 2\pi \int_0^{\infty} \frac{dy}{y^2+1} = 2\pi \lim_{b \to \infty} \tan^{-1} y \Big|_0^{\infty} = 2\pi\left(\frac{\pi}{2}\right) = \pi^2$$

Exp Approximate the double integral of $f(x,y) = x + y$ over the region R bounded above by the semicircle $y = \sqrt{1-x^2}$ and below by the x-axis, using the partition $x = -1, -\frac{1}{2}, 0, \frac{1}{4}, \frac{1}{2}, 1$ and $y = 0, \frac{1}{2}, 1$ with (x_k, y_k) the right lower corner in the k^{th} subrectangle.



$$\iint_R f(x,y)\, dA \approx \sum_{k=1}^{3} f(x_k, y_k) \Delta A_k$$

$$= f(0,0)\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right) + f\left(\tfrac{1}{4}, 0\right)\left(\tfrac{1}{4}\right)\left(\tfrac{1}{2}\right) + f\left(\tfrac{1}{2}, 0\right)\left(\tfrac{1}{4}\right)\left(\tfrac{1}{2}\right)$$

$$= (0)\left(\tfrac{1}{4}\right) + \left(\tfrac{1}{4}\right)\left(\tfrac{1}{8}\right) + \left(\tfrac{1}{2}\right)\left(\tfrac{1}{8}\right)$$

$$= \left(\tfrac{1}{8}\right)\left(\tfrac{3}{4}\right) = \frac{3}{32}$$