# Review of Differentiation and Integration (ch1-ch5)

## Chapter 1 : Functions: (1.1)

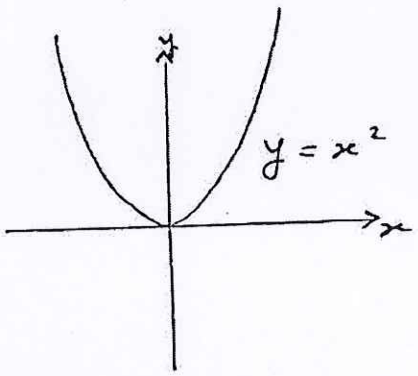
**Def:** a function $f$ is a rule that assigns to each point $x$ in the domain $D$ a unique point $y = f(x)$ in the range of $f$. $R$

$$f : D \longrightarrow R$$

D: Domain المجال

R: range المدى

**Example:** Find $D$ and $R$ for



ⓐ $f(x) = x^2$.

Domain $= (-\infty, \infty) = \mathbb{R}$

Range $= [0, \infty)$.

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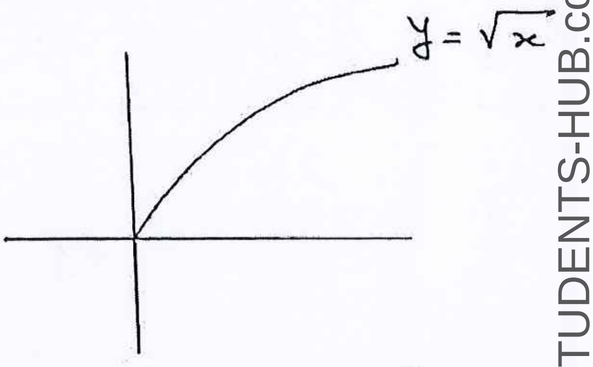


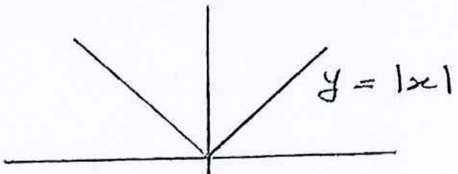
ⓑ $f(x) = \sqrt{x}$

Domain $= [0, \infty)$

Range $= [0, \infty)$

ⓒ $f(x) = |x|$ = absolute value of $x$.

$$f(x) = \begin{cases} x , & x \geq 0 \\ -x , & x < 0 \end{cases}$$



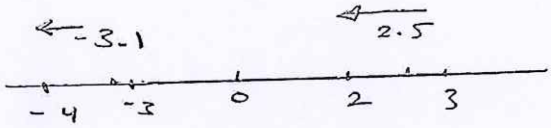
Domain = $(-\infty, \infty) = \mathbb{R}$ (The set of all real numbers).

Range = $[0, \infty)$

ⓓ The greatest integer function

$$\lfloor x \rfloor = \begin{cases} \vdots \\ 1 , & 1 \leq x < 2 \\ 0 , & 0 \leq x < 1 \\ -1 , & -1 \leq x < 0 \\ \vdots \end{cases}$$

$\lfloor 2.5 \rfloor = 2$

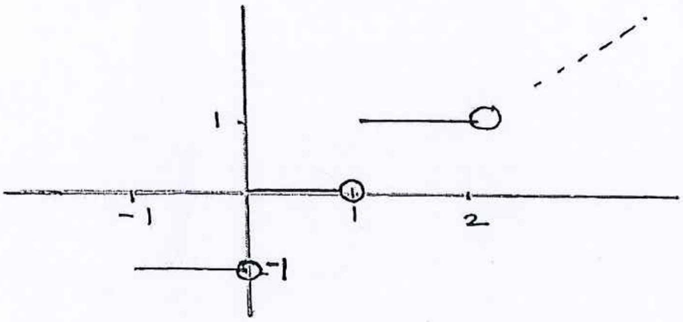


$\lfloor -3.1 \rfloor = -4$

Domain = $(-\infty, \infty) = \mathbb{R}$

Range = $\{0, \pm 1, \pm 2, \dots\}$

R = set of Integers.

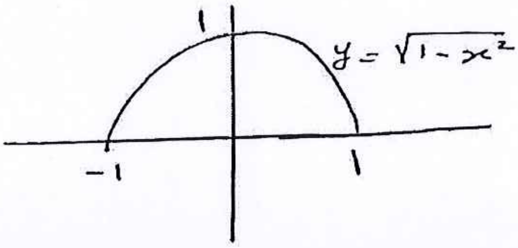


ⓔ $y = f(x) = \lfloor |x| \rfloor$

Domain $= (-\infty, \infty)$

Range $= \{0, 1, 2, 3, \dots\}$ = The set of positive Integers Union {Zero}.

ⓕ $y = f(x) = \sqrt{1 - x^2}$



$y = \sqrt{1 - x^2}$

$y^2 = 1 - x^2 \iff x^2 + y^2 = 1$

Domain $= [-1, 1]$

Range $= [0, 1]$

(For the domain, we need $1 - x^2 \geq 0 \iff 1 \geq x^2$

$\iff 1 \geq \sqrt{x^2} \iff 1 \geq |x| \iff 1 \geq x \geq -1$.).

ⓖ $y = f(x) = \dfrac{1}{\sqrt{1 - x^2}}$

Domain $= (-1, 1)$ [since $1 - x^2 > 0$]

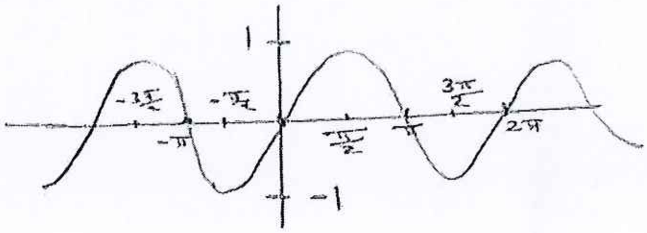
Range $= [1, \infty)$.

## 1.2 Trigonometric functions    الاقترانات المثلثية

① $y = \sin x$ ,    "ص = جاس"

Domain = $(-\infty, \infty)$

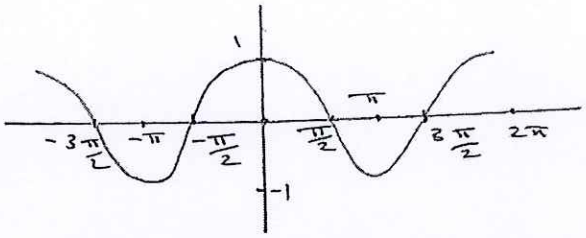
Range (R) = $[-1, 1]$



② $y = \cos x$ ,    "ص = جتاس"

Domain = $(-\infty, \infty)$

Range = $[-1, 1]$

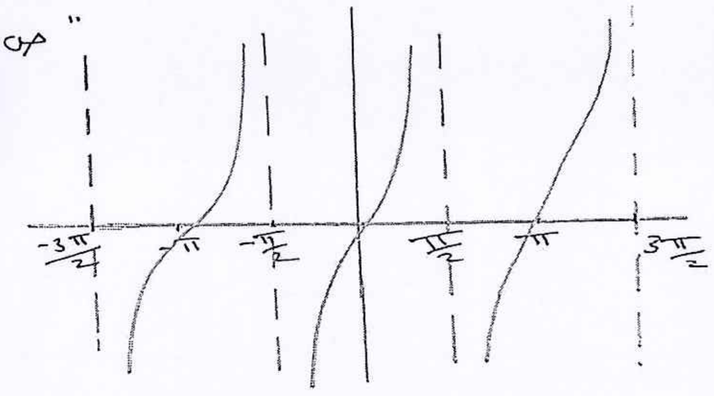


③ $y = \tan x$ ,    "ص = ظاس"

$= \frac{\sin x}{\cos x}$

Domain : $(-\infty, \infty) \setminus \{\frac{\pi}{2} \pm n\pi\}$ where $n = 0, 1, 2, \ldots$

Range : $(-\infty, \infty)$

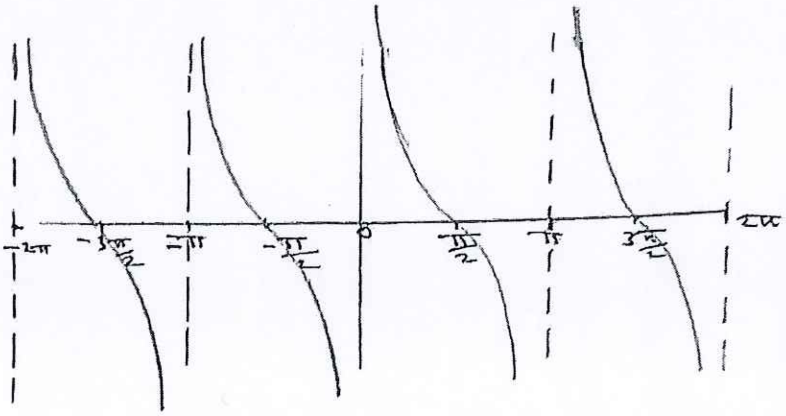


④ $y = \cot x$ , "ص = ظتا س"

$= \dfrac{\cos x}{\sin x}$

Domain = $(-\infty,\infty) \setminus \{\pm n\pi\}$
where $n = 0, 1, 2, \ldots$

Range = $(-\infty,\infty)$

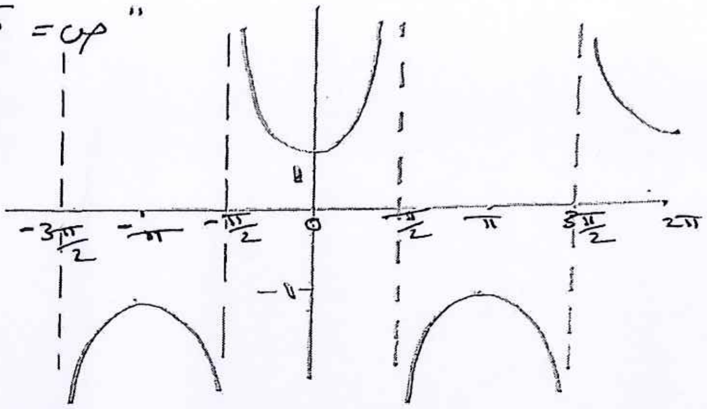


⑤ $y = \sec x$ , "ص = قا س"

$= \dfrac{1}{\cos x}$

Domain = $(-\infty,\infty) \setminus \{\frac{\pi}{2} \pm n\pi\}$
Where $n = 0, 1, 2, \ldots$

Range $= (-\infty, -1] \cup [1, \infty)$

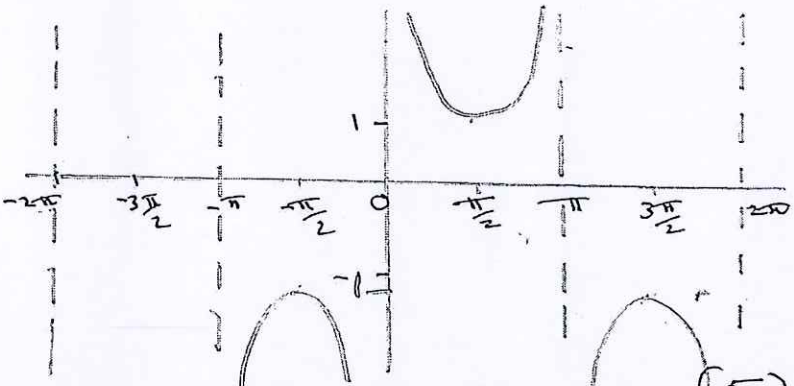


⑥ $y = \csc x$ "ص = قتا س"

$= \dfrac{1}{\sin x}$

Domain = $(-\infty,\infty) \setminus \{\pm n\pi\}$
when $n = 0, 1, 2, \ldots$

Range = $(-\infty, -1] \cup [1, \infty)$



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## Remark ①

$$\left.\begin{array}{ll} (1) & \sin(x+2\pi) = \sin x \\ (2) & \cos(x+2\pi) = \cos x \\ (3) & \sec(x+2\pi) = \sec x \\ (4) & \csc(x+2\pi) = \csc x \end{array}\right\}$$

periodic functions with period $2\pi$.

while:

$$\left.\begin{array}{ll} (5) & \tan(x+\pi) = \tan x \\ (6) & \cot(x+\pi) = \cot x \end{array}\right\}$$

period $\pi$

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## Trigonometric Identities:

(1) $\sin^2 x + \cos^2 x = 1$

(2) $\sin(2x) = 2\sin x\cos x$

(3) $\cos(2x) = \cos^2 x - \sin^2 x$

(4) $\cos^2 x = \dfrac{1+\cos(2x)}{2}$

(5) $\sin^2 x = \dfrac{1-\cos(2x)}{2}$

(6) $\sec^2 x = 1 + \tan^2 x$

(7) $\csc^2 x = 1 + \cot^2 x$

(8) $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

(9) $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

(10) $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

**Example:** (a) $\sin(x+\pi) = \sin x \overset{=-1}{\boxed{\cos\pi}} + \cos x \overset{=0}{\boxed{\sin\pi}}$

$= -\sin x$

(b) $\sin(2x) = \sin\overset{A+B}{(x+x)} =$ $(A = B = x)$

$= \sin x \cos x + \cos x \sin x = 2\sin x \cos x.$

(c) $\cos(2x) = \cos\overset{A+B}{(x+x)} = \cos x \cos x - \sin x \sin x$

$= \cos^2 x - \sin^2 x.$

(d) $\cos\left(x+\frac{\pi}{2}\right) = \cos x \overset{=0}{\boxed{\cos\frac{\pi}{2}}} - \sin x \overset{=1}{\boxed{\sin\frac{\pi}{2}}}$

$= \cos x(0) - \sin x(1) = -\sin x$

## 1.3 Even and Odd Functions:

**Def:** A function $y = f(x)$ is an:

(1) even function if $f(-x) = f(x)$

Its graph is symmetric about y-axis.

(2) Odd function if $f(-x) = -f(x)$

Its graph is symmetric about the origin.

**Example:** Determine whether the following functions are even, odd or neither.

(1) $f(x) = x^2+1$

$f(-x) = (-x)^2+1 = x^2+1 = f(x) \Rightarrow$ f is Even function

(2) $f(x) = x^3 + x$

$f(-x) = (-x)^3+(-x) = -x^3 - x$

$= -(x^3+x) = -f(x)$

Therefore f is an odd function.

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(3) $f(x) = x+1$

$f(-x) = -x+1 \neq f(x)$

and $\neq -f(x)$

There fore $f(x)$ is Neither

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(4) $f(x) = |x|$

$f(-x) = |-x| = |x| = f(x) \Rightarrow f$ is even

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(5) $f(x) = \sin x$

$f(-x) = \sin(-x) = \sin(0-x) = \overset{0}{\cancel{\sin} 0} \cos x - \overset{1}{\cancel{\cos} 0} \sin x$

$= -\sin x = -f(x) \Rightarrow f$ is odd.

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**Remark ②**

$$\left.\begin{aligned} y &= \sin x \\ y &= \tan x \\ y &= \cot x \\ y &= \csc x \end{aligned}\right\} \text{ Odd functions.}$$

while

$$\left.\begin{aligned} y &= \cos x \\ y &= \sec x \end{aligned}\right\} \text{ Even functions.}$$

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# Chapter 2 : Limits and Continuity :

## 2.1 Limits of functions :

$$\lim_{x \to a} f(x) = L$$

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means : $f(x) \xrightarrow{\text{goes}} L$ as $x \xrightarrow{\text{goes}} a$

" The function f gets arbitrarily close to L as x is sufficiently close to a "

**Note:** $\lim f(x) = L \overset{\text{Iff}}{\Longleftrightarrow} \underbrace{\lim_{x \to a^+} f(x)}_{\text{right hand limit}} = \underbrace{\lim_{x \to a^-} f(x)}_{\text{left hand limit}} = L$

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**Example:** Find the following limits :

① $\lim_{x \to 1} \frac{x-1}{x+1} = \frac{1-1}{1+1} = \frac{0}{2} = 0$ (exists)

This means that $f(x) = \frac{x-1}{x+1} \longrightarrow 0$

as $x \longrightarrow 1$.

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② $$\lim_{x \to 1} \frac{x^2-1}{x-1} = \frac{1-1}{1-1} = \frac{0}{0}$$

$$\Rightarrow \lim_{x \to 1} \frac{x^2-1}{x-1} = \lim_{x \to 1} \frac{(\cancel{x-1})(x+1)}{(\cancel{x-1})} = \lim_{x \to 1}(x+1) = 2$$

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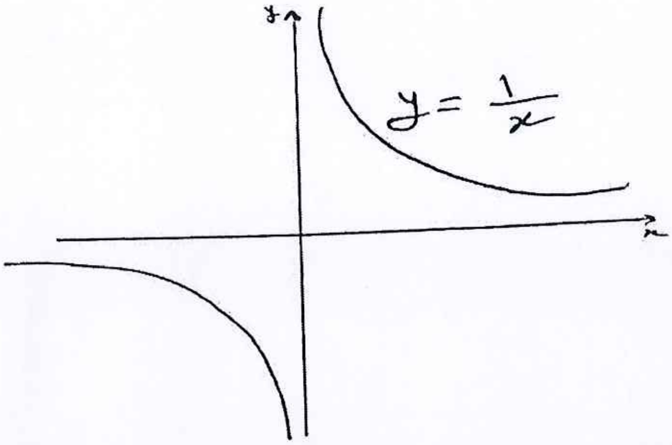
③ $$\lim_{x \to \infty} \frac{1}{x} = 0$$



④ $$\lim_{x \to -\infty} \frac{1}{x} = 0$$

⑤ $$\lim_{x \to 0^+} \frac{1}{x} = +\infty$$

⑥ $$\lim_{x \to 0^-} \frac{1}{x} = -\infty$$

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⑦ $$\lim_{x \to 1} \frac{x^2+x-2}{x^2-x} \overset{\frac{0}{0}}{=} \lim_{x \to 1} \frac{(\cancel{x-1})(x+2)}{x(\cancel{x-1})}$$

$$= \lim_{x \to 1} \frac{x+2}{x} = \frac{1+2}{1} = 3$$ (exists)

⑧ $$\lim_{x\to -1} \frac{\sqrt{x^2+8} - 3}{x+1} = \frac{\sqrt{9}-3}{-1+1} = \frac{0}{0}$$

"الضرب بالمرافق"

$$= \lim_{x\to -1} \left(\frac{\sqrt{x^2+8} \ominus 3}{x+1}\right)\cdot\left(\frac{\sqrt{x^2+8} \oplus 3}{\sqrt{x^2+8}+3}\right)$$

فرق بين مربعين

$$= \lim_{x\to -1} \frac{(\sqrt{x^2+8})^2 - (3)^2}{(x+1)(\sqrt{x^2+8}+3)} = \lim_{x\to -1} \frac{(x^2+8)-9}{(x+1)(\sqrt{x^2+8}+3)}$$

$$= \lim_{x\to -1} \frac{x^2-1}{(x+1)(\sqrt{x^2+8}+3)} \overset{\frac{0}{0}}{=} \lim_{x\to -1} \frac{(x-1)\cancel{(x+1)}}{\cancel{(x+1)}(\sqrt{x^2+8}+3)}$$

$$= \frac{-1-1}{\sqrt{1+8}+3} = \frac{-2}{\sqrt{9}+3} = \frac{-2}{3+3} = \frac{-2}{6} = \frac{-1}{3}$$

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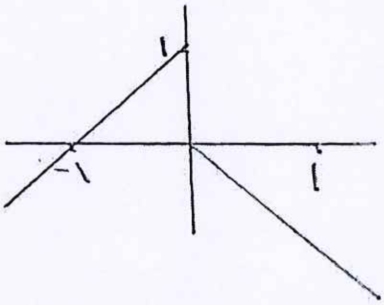
⑨ $\lim_{x\to 0} f(x)$ , where $f(x) = \begin{cases} x+1 , & x \le 0 \\ -x , & x > 0. \end{cases}$

- $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (-x) = \boxed{0}$.
- $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (x+1) = 0+1 = \boxed{1}$



$\Longrightarrow \lim_{x\to 0^+} f(x) \neq \lim_{x\to 0^-} f(x) \Longrightarrow \lim_{x\to 0} f(x)$ Does NOT Exist (DNE)

⑩ $\lim_{x \to -1} g(x)$ and $\lim_{x \to 1} g(x)$

where $g(x) = \begin{cases} x+2 & , \ x \le -1 \\ x^2 & ; \ -1 < x \le 1 \\ x-1 & , \ x > 1 \end{cases}$

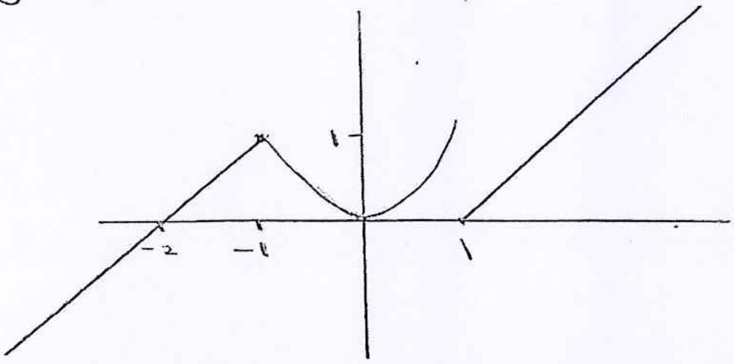
$\lim_{x \to -1^+} g(x) = (-1)^2 = 1$ , $\lim_{x \to -1^-} g(x) = -1+2 = 1$

So $\lim_{x \to -1} g(x) = \boxed{1}$

while: $\lim_{x \to 1^+} g(x) = 1-1 = 0$, $\lim_{x \to 1^-} g(x) = (1)^2 = 1$

$\Longrightarrow \lim_{x \to 1^+} g(x) \neq \lim_{x \to 1^-} g(x)$

$\Rightarrow \lim_{x \to 1} g(x)$ Does Not Exist. (DNE).



graph of $g(x)$.

**Theorem: (The Sandwich theorem): (squeeze thm.)**

Suppose that $g(x) \leq f(x) \leq h(x)$ for all $x$ in some open interval containing $C$, except at $x = C$, and

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L \text{ , then:}$$

$$\lim_{x \to c} f(x) = L.$$

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**Example:** Suppose that $\underbrace{1-x^2}_{g(x)} \leq f(x) \leq \underbrace{1+x^2}_{h(x)}$.

Find $\lim_{x \to 0} f(x)$.

$$\lim_{x \to 0} (\overset{g(x)}{1-x^2}) = 1-0 = 0 \text{ , } \lim_{x \to 0} (\overset{h(x)}{1+x^2}) = 1+0 = 0$$

$$\Rightarrow \lim_{x \to 0} f(x) = 1 \text{ by Sandwich theorem.}$$

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**Example:** If $2 - x^2 \leqslant g(x) \leqslant 2\cos x \quad , \forall x$

Find $\lim\limits_{x \to 0} g(x)$

$\lim\limits_{x \to 0} (2 - x^2) = 2 \quad , \quad \lim\limits_{x \to 0} 2\cos x = 2\cos(0) = 2 \cdot 1 = 2$

$\Longrightarrow \lim\limits_{x \to 0} g(x) = 2$ by Sandwich theorem.

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**Example:** Find $\lim\limits_{x \to +\infty} \frac{\sin x}{x}$ .

We know that $\quad -1 \leqslant \sin x \leqslant 1$

$$\Longrightarrow \frac{-1}{x} \leqslant \frac{\sin x}{x} \leqslant \frac{1}{x}$$

$$\lim_{x \to +\infty} \frac{-1}{x} \leqslant \lim_{x \to +\infty} \frac{\sin x}{x} \leqslant \lim_{x \to +\infty} \frac{1}{x}$$

$$0 \leqslant \lim_{x \to +\infty} \frac{\sin x}{x} \leqslant 0$$

Therefore : $\lim\limits_{x \to +\infty} \frac{\sin x}{x} = 0$

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Similarly: $\lim\limits_{x \to \infty} \frac{\cos x}{x} = 0 \quad$ (by the same reason) above

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**Remark ①:** $\lim_{x \to 0} \frac{\sin x}{x} = 1$

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**Example:** Find $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right)$

We know: $-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$

$$-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2$$

$$\lim_{x \to 0} (-x^2) = \lim_{x \to 0} (x^2) = 0 \Rightarrow \lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$$

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**Example:** $\lim_{x \to 0} \sqrt{x^4 + x^2} \, \sin\left(\frac{\pi}{x}\right)$

$$-1 \leq \sin\left(\frac{\pi}{x}\right) \leq 1$$

$$-\sqrt{x^2 + x^4} \leq \sqrt{x^4 + x^2} \, \sin\left(\frac{\pi}{x}\right) \leq \sqrt{x^4 + x^2}$$

$$\lim_{x \to 0} \left(-\sqrt{x^4 + x^2}\right) = \lim_{x \to 0} \left(\sqrt{x^4 + x^2}\right) = \sqrt{0+0} = \sqrt{0} = 0.$$

$\Rightarrow \lim_{x \to 0} \sqrt{x^4 + x^2} \, \sin\left(\frac{\pi}{x}\right) = 0$ by Sandwich thm.

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## 2.2 Continuity

**Def:** A function $f$ is continuous at $x = a$ if

(1) $f(a)$ exists (معرفة)

(2) $\lim_{x \to a} f(x)$ exists

(3) $\lim_{x \to a} f(x) = f(a)$

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**Examples:** ① $y = \sin x$, $y = \cos x$, $y = |x|$ are continuous on $(-\infty, \infty)$.

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② polynomials (كثيرات الحدود) are continuous on $(-\infty, \infty)$

ex: $f(x) = x^4 + 5x^3 - 2x + 1$

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③ $g(x) = \dfrac{x^3 + x - 5}{x^2 - 1}$ is continuous on $\mathbb{R} \setminus \{-1, 1\}$

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④ $h(x) = \dfrac{x^2 - 5}{x^2 + 3x}$ is continuous on $\mathbb{R} \setminus \{0, -3\}$

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**Example:** (A function with removable discontinuity).

$f(x) = \dfrac{x^2+2x-3}{x^2-1}$ is Continuous on $\mathbb{R}\setminus\{1,-1\}$

$$\frac{x^2+2x-3}{x^2-1} = \frac{(x-1)(x+3)}{(x-1)(x+1)}$$

قبل الاختصار

Note that $f(1)$ is undefined "غير معرف"

but at $x \to 1$, $\displaystyle\lim_{x\to1} f(x) = \lim_{\substack{x\to1\\x\neq1}} \frac{(\cancel{x-1})(x+3)}{(\cancel{x-1})(x+1)} = \frac{4}{2} = \boxed{2}$

In this case $x=1$ is called a removable discontinuity

• We can define a Continuous extension of $f$ at $x=1$

as : $F(x) = \begin{cases} f(x) & , x\neq1 \\ \displaystyle\lim_{x\to1} f(x) & , x=1 \end{cases} = \begin{cases} f(x) & , x\neq1 \\ 2 & , x=1 \end{cases}$

$$= \begin{cases} \dfrac{x^2+2x-3}{x^2-1} & , x\neq1 \\ 2 & , x=1 . \end{cases}$$

**Example:** Find the Continuous extension $F(x)$ of

$$f(x) = \frac{x^2+3x-10}{x-2}.$$

$$f(x) = \frac{x^2+3x-10}{x-2} = \frac{(x-2)(x+5)}{(x-2)}$$

$$\Rightarrow \lim_{x\to 2} f(x) = \lim_{x \to 2} (x+5) = 7$$

$$\Rightarrow F(x) = \begin{cases} \dfrac{x^2+3x-10}{x-2}, & x \neq 2 \\ 7, & x = 2. \end{cases}$$

$x = 2$ is called a removable discontinuity.

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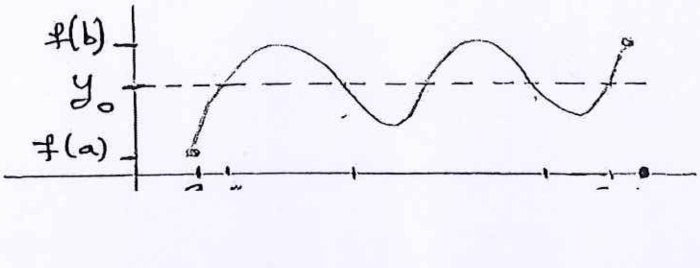
**Theorem:** The Intermidate Value Theorem. (IMVT)

If is continuous function on $[a,b]$, and if $y_0$ is any value between $f(a)$ and $f(b)$, then $\exists\ c \in [a,b]$ such that $f(c) = y_0$.



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**Note:** If $y_0 = 0$ in IMVT ,

$f(c) = y_0 = 0$ , then $C$ is called a root of $f$. (جذر الاقتران / صفر الاقتران)

(i.e) : since $y_0 = 0$ is between $f(a)$ and $f(b)$ ,

then $f(a) < 0 < f(b)$ (أو عكس المتباينة)

this means $f(a) . f(b) < 0$ . (مختلفين بالإشارة)

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**Theorem:** Bolzano Theorem

If $f$ is continuous on $[a,b]$ and if $f(a).f(b) < 0$ then there exists $C \in [a,b]$ such that $f(c) = 0$.

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**Example :** show that $f(x) = x^3 - x - 1$ has a root in $[1,2]$.

Using IMVT ( in this case Bolzano theorem):

1) $f(x)$ is continuous on $[1,2]$ since its polynomial.

2) $f(1) = 1^3 - 1 - 1 = -1 < 0$ , $f(2) = 2^3 - 2 - 1 = 5 > 0$

then: $f(1) < 0 < f(2)$ , that is $f(1) f(2) < 0$,

so by IMVT (Bolzano theorem): $\exists\ C \in [1,2]$ such that $f(c) = 0$ , (i.e) $f$ has a root on $[1,2]$. (20)

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## Asymptotes: "الخطوط التقاربية"

There are three types of Asymptotes:

1) Horizontal Asymptote. (H.A) "أفقي"

2) Vertical Asymptote. (V.A) "عمودي"

3) Oblique (slant) Asymptote. (O.A) "مائل"

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**Def:** A line $y = b$ is a Horizontal Asymptote of $y = f(x)$ if either:

$$\lim_{x \to \infty} f(x) = b \quad \text{or} \quad \lim_{x \to -\infty} f(x) = b$$

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**Note:** We are dealing with rational functions:

$$f(x) = \frac{P(x)}{q(x)} = \frac{a_m x^m + a_{m-1} x^{m-1} + \cdots + a_1 x + a_0}{b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0}$$

$$= \frac{\text{polynomial of degree } m}{\text{polynomial of degree } n}$$

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(a) If $m = n$, then $\lim\limits_{x \to \pm\infty} f(x) = \frac{a_m}{b_n}$

(b) If $m < n$, then $\lim\limits_{x \to \pm\infty} f(x) = 0$

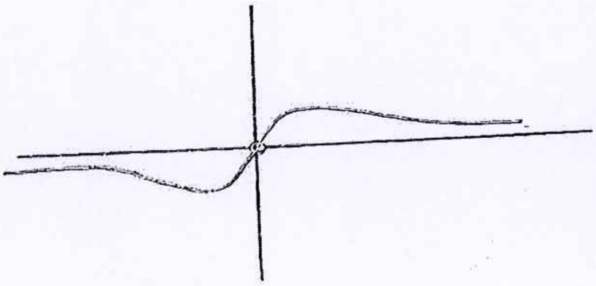
(c) If $m > n$, then $\lim\limits_{x \to \pm\infty} f(x) = \pm\infty$.

---

**Example:** Find H. A of the graph of $f(x) = \frac{x}{x^2+1}$

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{x}{x^2+1} = \lim_{x \to \infty} \frac{x}{x^2(1+\frac{1}{x})}$$

$$= \lim_{x \to \infty} \frac{\frac{1}{x}}{(1+\frac{1}{x})} = 0$$



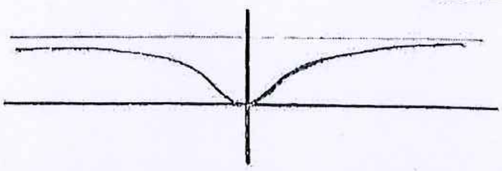
Similarly : $\lim\limits_{x \to -\infty} f(x) = 0$

$\Rightarrow$ $\boxed{y = 0}$ is a Horizontal Asymbtote.

---

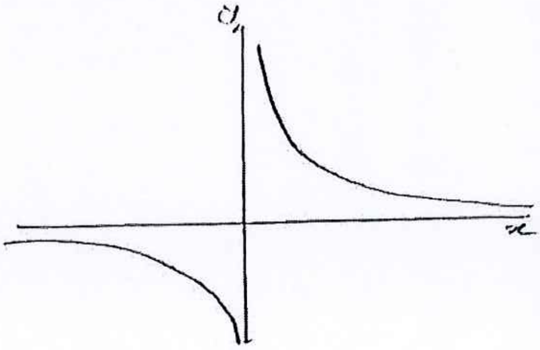
**Example** Find H. A for $f(x) = \frac{2x^2}{x^2+1}$

$\lim\limits_{x \to \pm\infty} f(x) = \frac{2}{1} = 2$ $\Rightarrow$ $\boxed{y = 2}$ is a H.A



**Example:** Find H.A for $f(x) = \frac{\sin x}{x^2}$.

$$-1 \le \sin x \le 1$$



$$\Rightarrow \quad \frac{-1}{x^2} \le \frac{\sin x}{x^2} \le \frac{1}{x^2}$$

Since $\lim\limits_{x\to\pm\infty} \frac{-1}{x^2} = \lim\limits_{x\to\pm\infty} \frac{1}{x^2} = 0$, then

by Sandwich theorem $\lim\limits_{x\to\pm\infty} \frac{\sin x}{x^2} = 0$.

$\Rightarrow$ $\boxed{y = 0}$ is a H.A for $f(x)$

**Example:** Find H.A for $f(x) = \frac{\sqrt{x^2+1}}{x}$.

$$\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x} = \lim_{x\to\infty} \frac{\sqrt{x^2\left(1+\frac{1}{x^2}\right)}}{x}$$

$|x| = x$
since $x \to \infty$

$$= \lim_{x\to\infty} \frac{\sqrt{x^2}\sqrt{1+\frac{1}{x^2}}}{x} = \lim_{x\to\infty} \frac{|x|\sqrt{1+\frac{1}{x^2}}}{x}$$

$$= \lim_{x\to\infty} \frac{\cancel{x}\sqrt{1+\frac{1}{x^2}}}{\cancel{x}} = \lim_{x\to\infty} \sqrt{1+\frac{1}{x^2}} = \boxed{1}$$

$\Rightarrow$ $\boxed{y = 1}$ is H.A

Now: $\lim\limits_{x\to-\infty} \dfrac{\sqrt{x^2+1}}{x} = \lim\limits_{x\to-\infty} \dfrac{\sqrt{x^2\left(1+\frac{1}{x^2}\right)}}{x}$

$= \lim\limits_{x\to-\infty} \dfrac{\sqrt{x^2}\,\sqrt{\left(1+\frac{1}{x^2}\right)}}{x} = \lim\limits_{x\to-\infty} \dfrac{|x|\sqrt{1+\frac{1}{x^2}}}{x}$

$|x| = -x$

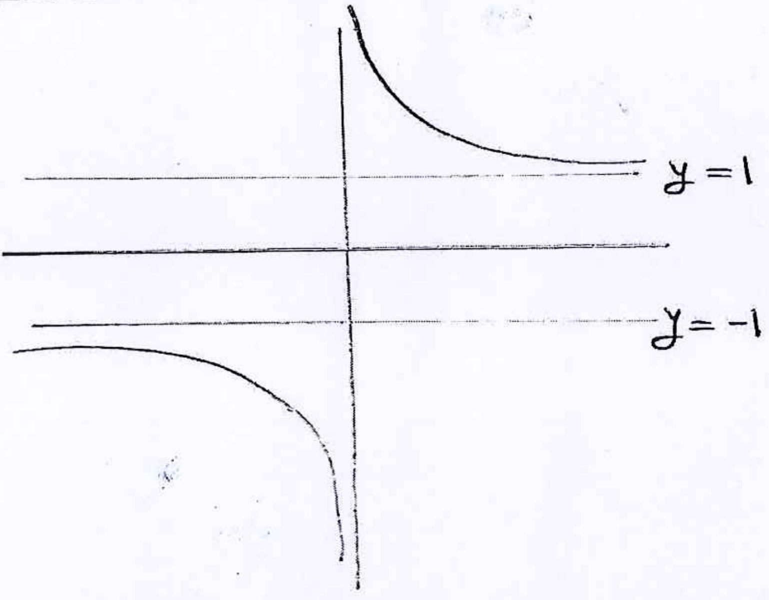
since $x\to-\infty$

$= \lim\limits_{x\to-\infty} \dfrac{-\cancel{x}\sqrt{1+\frac{1}{x^2}}}{\cancel{x}} = \lim\limits_{x\to-\infty} -\sqrt{1+\frac{1}{x^2}} = \boxed{-1}$

then $\boxed{y=-1}$ is a H.A.

Graph of

$f(x) = \dfrac{\sqrt{x^2+1}}{x}$



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**Def:** A Line $x = a$ is a Vertical Asymptote.

of the graph of $y = f(x)$ if either:

$$\lim_{x \to a^+} f(x) = \pm\infty \quad \boxed{\text{or}} \quad \lim_{x \to a^-} f(x) = \pm\infty$$

---

**Example:** Find the V.A for $y = f(x) = \frac{1}{x}$.

$$\lim_{x \to 0^+} \frac{1}{x} = +\infty \quad \Longrightarrow \quad \boxed{x = 0} \text{ is a V.A.}$$

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty$$

---

**Example:** Find the V.A for $y = \frac{2x+1}{x-1}$

$$\lim_{x \to 1^+} \frac{2x+1}{x-1} = +\infty \quad \Longrightarrow \quad \boxed{x = 1} \text{ is a V.A}$$

We can Also show that

$$\lim_{x \to 1^-} \frac{2x+1}{x-1} = -\infty. \quad \Longrightarrow \boxed{x = 1} \text{ is a V.A.}$$

**Example:** Find the V.A for $f(x) = \frac{x^2+2x-3}{x^2-1}$

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{\cancel{(x-1)}(x+3)}{\cancel{(x-1)}(x+1)} = \frac{4}{2} = \boxed{2} \neq \pm\infty$$

There fore $\boxed{x=1}$ is NOT a V.A

we call it removabl dis continuity.

While :

$$\lim_{x\to -1^-} \frac{\cancel{(x-1)}(x+3)}{\cancel{(x-1)}(x+1)} = -\infty \Rightarrow \boxed{x=-1} \text{ is a V.A}$$

---

**Example:** Find the H.A & V.A for $f(x) = \frac{2x+1}{x-1}$

H.A : $\lim_{x\to \pm 2} \frac{2x+1}{x-1} = \frac{2}{1} = 2 \Rightarrow \boxed{y=2}$ is H.A.

V.A : $\lim_{x\to 1^-} \frac{2x+1}{x-1} = -\infty \Rightarrow \boxed{x=1}$ is V.A

We need to sketch $f(x)$ :

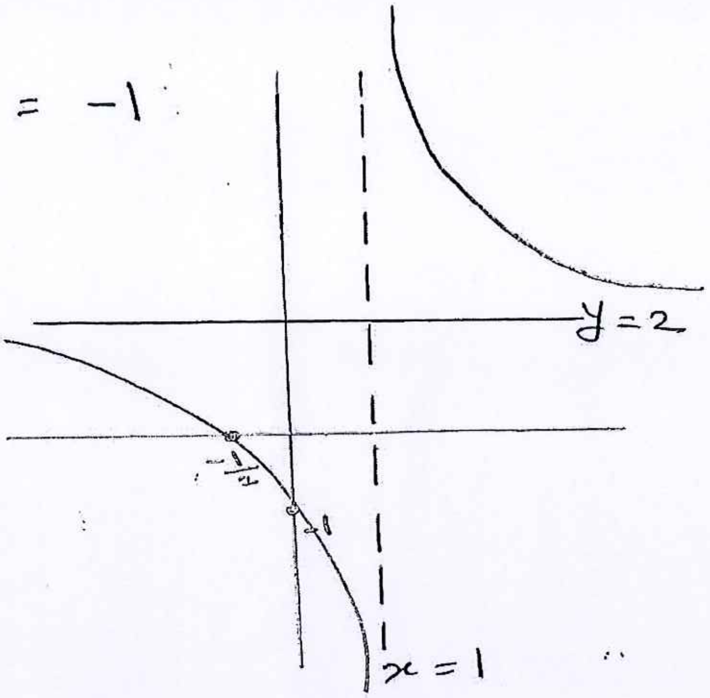
---

x_intercept : Let $y = 0$, then

$$\frac{2x+1}{x-1} = 0 \iff x = -\frac{1}{2}$$

y - intercept : Let $x = 0$, then

$$y = \frac{0+1}{0-1} = -1$$



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**Remark :** Suppose that $f(x)$ is a rational function

1) The graph of $f(x)$ Can intersect its H.A.

2) The graph of $f(x)$ Can have both H.A & V.A.

3) The graph of $f(x)$ have only one H.A.

---

**Remark:** $f(x) = \frac{\sin x}{x}$ has NO V. A

Since $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$.

---

**Def: Oblique Asymptote (O.A):**

If the degree of the numerator (بسط) is 1 greater than the degree of the denominator (مقام), then f has an Oblique Asymptote.

---

**Example:** Find the O. A for $f(x) = \frac{x^2}{x-1}$

$$\Rightarrow \frac{x^2}{x-1} = \underbrace{(x+1)}_{\swarrow \; O \cdot A} + \underbrace{\frac{1}{x-1}}_{\downarrow \; 0 \text{ as } x \to \infty}$$

$$\begin{array}{r|l} & x+1 \\ \hline x-1 & \cancel{x^2} \\ & -\cancel{x^2} \overset{+}{-} x \\ \hline & x \\ & -x \overset{+}{-} 1 \\ \hline & 1 \end{array}$$

Therefore: $\boxed{y = x+1}$ is O. A

Note that: $\lim\limits_{x \to 1^-} f(x) = -\infty \Rightarrow \boxed{x=1}$ is V.A

**Remark:** The rational function Cannot have a Horizontal and an Oblique Asymptote at the same time.

**Example:** Q2) (c) Find the asymptotes of:

$$f(x) = \frac{x^2+1}{x-1}$$

$$\begin{array}{r|l} & x+1 \\ \hline x-1 & \cancel{x^2}+1 \\ & -\left(\cancel{x^2} \overset{+}{-} x\right) \\ \hline & \cancel{x}+1 \\ & -\left(\cancel{x} \overset{+}{-} 1\right) \\ \hline & 2 \end{array}$$

$$\Rightarrow \quad f(x) = \underbrace{(x+1)}_{O.A} + \frac{2}{x-1}$$

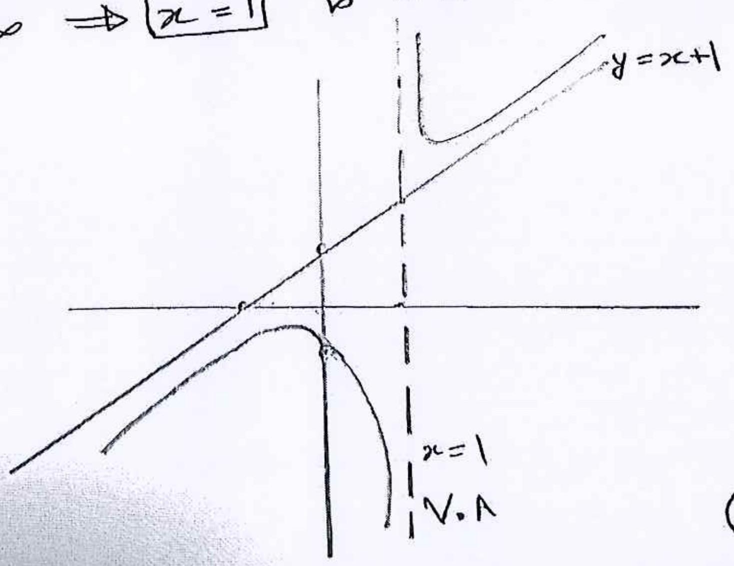
$\Rightarrow \quad \boxed{y = x+1}$ is the Oblique Asym.

- $\lim\limits_{x \to \pm\infty} f(x) = \pm\infty \Rightarrow$ There is <u>NO</u> H.A

- $\lim\limits_{x \to 1^+} f(x) = +\infty \Rightarrow \boxed{x=1}$ is V.A

- $\lim\limits_{x \to 1^-} f(x) = -\infty$

y-intercept : $y = -1$



Example Q2) (d)

$$f(x) = \frac{x^3+1}{x^2-1}$$

$$\begin{array}{r|l} & x \\ \hline x^2-1 & x^3+1 \\ & -x^3 + x \\ \hline & x+1 \end{array}$$

$$f(x) = x + \frac{x+1}{x^2-1} \xrightarrow[x\to\infty]{} 0$$

$\Rightarrow \boxed{y = x}$ is Oblique Asy.

Now:

$\lim\limits_{x\to+\infty} f(x) = +\infty$ & $\lim\limits_{x\to-\infty} f(x) = -\infty$

Therefore (NO) H.A

Now: $f(x) = \frac{x^3+1}{x^2-1} = \frac{\cancel{(x+1)}(x^2-x+1)}{\cancel{(x+1)}(x-1)}$

$$\lim_{x\to-1} f(x) = \frac{(-1)^2-(-1)+1}{(-1-1)} = \frac{3}{-2} \neq \infty \quad \neq -\infty$$

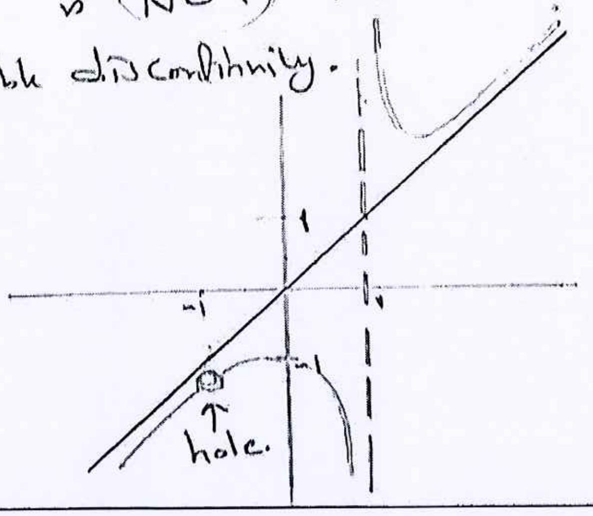
Therefore $x = -1$ is (NOT) V.A

We call it removable discontinuity.

While

$\lim\limits_{x\to1^+} f(x) = +\infty$

$\Rightarrow \boxed{x = 1}$ is V.A



## Chapter 3 : Differentiation :

### 3.1 Definition of derivative: تعريف المشتقة

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

(we read it the derivative of f at $x = a$).

- If $f'(a)$ exists, then f is differentiable at $x = a$.

---

**Note:** f is differentiable on $[a,b]$ if

(a) f is diff in $(a,b)$

(b) $f'_{+}(a) = \lim_{h \to 0^{+}} \frac{f(a+h) - f(a)}{h}$ exists. (right hand derivative of f at $x=a$)

(c) $f'_{-}(b) = \lim_{h \to 0^{-}} \frac{f(b+h) - f(b)}{h}$ exists. (Left hand derivative of f at $x = b$)

**Remark:** $f$ is diff. at $x = c$ if and only if $f'_{+}(c) = f'_{-}(c)$ [exist and equal].

**Example:** Use the definition of derivative to find $f'(x)$ for $f(x) = \sqrt{x}$.

$$f'(x) = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \lim_{h \to 0} \left(\frac{\sqrt{x+h} - \sqrt{x}}{h}\right)\left(\frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}\right)$$

$$= \lim_{h \to 0} \frac{(x+h) - (x)}{h\left(\sqrt{x+h} + \sqrt{x}\right)} =$$

$$= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}\left(\sqrt{x+h} + \sqrt{x}\right)} = \frac{1}{2\sqrt{x}}.$$

**Theorem:** If f is differentiable at $x = C$ then f is Continuous at $x = C$

---

**Note:** The Converse of the theorem is NOT true.

(i-e) If f is continuous at $x = C$, then f need not be differentiable at $x = C$.

---

**Example:** Let $f(x) = |x|$. $f(x)$ is Continuous function at $x=0$

$$f'_+(0) = \lim_{h \to 0^+} \frac{|0+h| - |0|}{h} = \lim_{h \to 0^+} \frac{h}{h} = 1$$

while:

$$f'_-(0) = \lim_{h \to 0^-} \frac{|h+0| - |0|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = -1$$

then $f'_+(0) \neq f'_-(0)$

so f is not differentiable at $x = 0$

---

---

Example: Determine whether the following function is differentiable at $x = 0$ :

$$f(x) = \begin{cases} x^{2/3} & , x \geq 0 \\ x^{1/3} & , x < 0. \end{cases}$$

- $f'_{+}(0) = \lim\limits_{h \to 0^{+}} \frac{f(0+h) - f(0)}{h} = \lim\limits_{h \to 0^{+}} \frac{h^{2/3}}{h} = \lim\limits_{h \to 0^{+}} \frac{1}{h^{\frac{1}{3}}}$

  $= +\infty$ .

- $f'_{-}(0) = \lim\limits_{h \to 0^{-}} \frac{f(0+h) - f(0)}{h} = \lim\limits_{h \to 0^{-}} \frac{h^{\frac{1}{3}}}{h} = \lim\limits_{h \to 0^{-}} \frac{1}{h^{2/3}} = +\infty$

So $f$ is Not differentiable at $x = 0$

In this Case we say that $f$ has a vertical tangent at $x = 0$ .

---

## 3.2 Differentiation rules:

Theorem : suppose $f(x)$ and $g(x)$ are diff. at $x$, and $c$ is constant, then:

1) $\frac{d}{dx}(c) = 0$.

2) $\frac{d}{dx} x^n = n x^{n-1}$, where $n \in \mathbb{Z}^+$.

3) $\frac{d}{dx}\left(c f(x)\right) = c \frac{d}{dx}(f(x))$.

4) $\frac{d}{dx}\left(f(x) \pm g(x)\right) = \frac{df}{dx} \pm \frac{dg}{dx}$.

5) $\frac{d}{dx}\left(f(x) \cdot g(x)\right) = \frac{df}{dx} g(x) + f(x) \frac{dg}{dx}$.

6) $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x) \frac{df}{dx} - f(x) \frac{dg}{dx}}{g^2(x)}$.

7) $\frac{d}{dx}(f \circ g)(x) = \frac{d}{dx} f(g(x)) \cdot \frac{dg(x)}{dx}$. (Chain Rule)