CHAPTER

2

Determinants

Determinants

The determinant of a square matrix A is a single real number which contains an important amount of information about the matrix A. We denote the determinant of the matrix A by $\det(A)$ or $|A|$.

1×1 Matrices

If $A = [a]_{1\times 1}$ matrix, then the determinant of A is defined by

$$\det(A) = |A| = a$$

2×2 Matrices

If A is the 2×2 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

then the determinant of A is defined by

$$\det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

The following terminology and notation will help us to define and compute determinants of general $n \times n$ matrices.

Definition

Let $A = (a_{ij})$ be an $n \times n$ matrix and let M_{ij} denote the $(n-1) \times (n-1)$ matrix obtained from A by deleting the row and column containing a_{ij}. The determinant of M_{ij} is called the **minor** of a_{ij}. We define the **cofactor** A_{ij} of a_{ij} by

$$A_{ij} = (-1)^{i+j} \det(M_{ij})$$

Example. Let

$$A = \begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$$

(a) Find the minor and cofactor of a_{11}

(b) Find the minor and cofactor of a_{32}

Remark.

Note that a minor $\det(M_{ij})$ and its corresponding cofactor A_{ij} are either the same or negatives of each other and that the relating sign $(-1)^{i+j}$ is either $+1$ or -1 in accordance with the pattern in the "checkerboard" array

$$\begin{bmatrix} + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

The determinant of $n \times n$ matrices: Cofactor Expansion Method

If A is an $n \times n$ matrix, then regardless of which row or column of A is chosen, **the number** obtained by multiplying the entries in that row or column by the corresponding cofactors and adding the resulting products is always the same. **This number** is called the determinant of A, and the sums themselves are called cofactor expansions of A. That is,

$$\begin{aligned} \det(A) &= a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in} \\ &= a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj} \end{aligned}$$

EXAMPLE 1 If

$$A = \begin{bmatrix} 2 & 5 & 4 \\ 3 & 1 & 2 \\ 5 & 4 & 6 \end{bmatrix}$$

then

$$\begin{aligned} \det(A) &= a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} \\ &= (-1)^2 a_{11}\det(M_{11}) + (-1)^3 a_{12}\det(M_{12}) + (-1)^4 a_{13}\det(M_{13}) \\ &= 2\begin{vmatrix} 1 & 2 \\ 4 & 6 \end{vmatrix} - 5\begin{vmatrix} 3 & 2 \\ 5 & 6 \end{vmatrix} + 4\begin{vmatrix} 3 & 1 \\ 5 & 4 \end{vmatrix} \\ &= 2(6-8) - 5(18-10) + 4(12-5) \\ &= -16 \end{aligned}$$

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Example (Smart Choice of Row or Column).

Evaluate the determinant

$$\begin{vmatrix} 5 & 0 & 2 \\ -4 & 5 & 6 \\ 10 & 0 & 4 \end{vmatrix}$$

Example (Smart Choice of Row or Column).

Evaluate the determinant

$$\begin{vmatrix} 1 & 3 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & -2 & 0 \\ -1 & 2 & 1 & 1 \end{vmatrix}$$

Theorem 2.1.2 *If A is an $n \times n$ matrix, then* $\det(A^T) = \det(A)$.

Theorem 2.1.3 *If A is an $n \times n$ triangular matrix, then the determinant of A equals the product of the diagonal elements of A.*

Theorem 2.1.4 *Let A be an $n \times n$ matrix.*

(i) If A has a row or column consisting entirely of zeros, then $\det(A) = 0$.

(ii) If A has two identical rows or two identical columns, then $\det(A) = 0$.

4. Evaluate the following determinants by inspection:

(a) $\begin{vmatrix} 3 & 5 \\ 2 & 4 \end{vmatrix}$ **(b)** $\begin{vmatrix} 2 & 0 & 0 \\ 4 & 1 & 0 \\ 7 & 3 & -2 \end{vmatrix}$

(c) $\begin{vmatrix} 3 & 0 & 0 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{vmatrix}$ **(d)** $\begin{vmatrix} 4 & 0 & 2 & 1 \\ 5 & 0 & 4 & 2 \\ 2 & 0 & 3 & 4 \\ 1 & 0 & 2 & 3 \end{vmatrix}$

EXERCISES

6. Find all values of λ for which the following determinant will equal 0:

$$\begin{vmatrix} 2-\lambda & 4 \\ 3 & 3-\lambda \end{vmatrix}$$

11. Let A and B be 2×2 matrices.

(a) Does $\det(A+B) = \det(A) + \det(B)$?

(b) Does $\det(AB) = \det(A)\det(B)$?

(c) Does $\det(AB) = \det(BA)$?

Justify your answers.

(a) $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$\det(A+B) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

$\det A + \det B = 0 + 0 = 0$

$\therefore \det(A+B) \neq \det A + \det B$

(b) $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$

$|A|\,|B| = [a_{11}a_{22} - a_{12}a_{21}][b_{11}b_{22} - b_{12}b_{21}]$

$|AB| = \begin{vmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{vmatrix}$

$= (a_{11}b_{11} + a_{12}b_{21})(a_{21}b_{12} + a_{22}b_{22}) - (a_{11}b_{12} + a_{12}b_{22})(a_{21}b_{11} + a_{22}b_{21})$

$= a_{11} a_{21} b_{11} b_{12} + a_{11} a_{22} b_{11} b_{22} + a_{12} a_{21} b_{21} b_{12} + a_{12} a_{22} b_{21} b_{22}$

$- [a_{11} a_{21} b_{12} b_{11} + a_{11} a_{22} b_{12} b_{21} + a_{12} a_{21} b_{22} b_{11} + a_{12} a_{22} b_{22} b_{21}]$

$$|AB| = |A||B|$$

(c) $|AB| = |A||B| = |B||A| = |BA|$

